Reconstructing structures from their abstract clones

Michael Pinsker

Vienna University of Technology / Charles University Prague

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Set Theory, Model Theory and Applications

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Outline

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- Reconstructing structures from their automorphism groups and polymorphism clones

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- The topology of algebras

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- The topology of algebras
- Reconstruction notions, results, problems

Part I

Reconstructing structures from their automorphism groups and polymorphism clones

Reconstructing structures up to first-order . . .

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countable

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countable, ω-categorical

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Aut()

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Let Δ, Γ be ω-categorical structures on the same domain.
Then $\mathrm{Aut}(\Delta) = \mathrm{Aut}(\Gamma) \;\Leftrightarrow\; \Delta, \Gamma$ are first-order interdefinable.

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Observe: $\mathrm{Pol}(\Delta) \supseteq \mathrm{End}(\Delta) \supseteq \mathrm{Aut}(\Delta)$.

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Let Δ, Γ be ω-categorical structures on the same domain.
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Definition (Constraint Satisfaction Problem)

CSP(Δ) is the computational problem to decide whether a given primitive positive τ-sentence holds in Δ.

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Theorem (Bodirsky + MP '12)

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Part II

The topology of algebras

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Universal Algebra: Structure of $\mathfrak{A} \Leftrightarrow$ equations in $\mathrm{Clo}(\mathfrak{A})$.

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Theorem (Birkhoff 1935)

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras.

Then $\mathrm{Clo}(\mathfrak{B}) = \mathrm{Clo}(\mathfrak{C})$ for some $\mathfrak{C} \in \mathrm{HSP}(\mathfrak{A}) \leftrightarrow$
$\exists$ clone homomorphism from $\mathrm{Clo}(\mathfrak{A})$ onto $\mathrm{Clo}(\mathfrak{B})$.

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Theorem (Bodirsky + MP '11)

Let $\mathfrak{A}$, $\mathfrak{B}$ be countable.

Then $\mathrm{Clo}(\mathfrak{B}) = \mathrm{Clo}(\mathfrak{C})$ for some $\mathfrak{C} \in \mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A}) \leftrightarrow$
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- When can HSP^{fin} be described algebraically?
- Can we reconstruct the topological structure of function clones from their algebraic structure?

Part III

Reconstruction notions & results

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Fact. For groups (3) $\implies$ (2).

Groups: the small index property

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Automorphism groups with automatic continuity:

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Automorphism groups with automatic continuity:

- $(\mathbb{N}; =)$ (Dixon+Neumann+Thomas'86)
- $(\mathbb{Q}; <)$ and the atomless Boolean algebra (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)
- the random K_n-free graphs (Herwig'98)

Groups: Rubin's forall-exists interpretations

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  all homogeneous countable graphs
  various ω-categorical semilinear orders
  the random partial order
  the random tournament

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- the random *k*-hypergraphs
  the Henson digraphs

  (Barbina+MacPherson '07).

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Theorem (Bodirsky + Evans + Kompatscher + MP '16)

$\mathrm{Pol}(\Delta)$, $\mathrm{End}(\Delta)$, $\overline{\mathrm{Aut}(\Delta)}$ do not have reconstruction.

Method I: Automatic continuity via Birkhoff's theorem

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Theorem (Bodirsky + MP + Pongrácz '13)

Any closed subclone of **O** containing ω^ω has
automatic continuity and automatic homeomorphicity.

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- $\mathrm{Pol}(G)$;
- Various other famous clones containing $\mathrm{Aut}(G)$.

Method III: Rubin's interpretations

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Interpret structure Δ in the algebraic structure of its clone Pol(Δ).

Theorem (Maissel + Rubin '15)

Let Pol(Δ), Pol(Δ') contain all transpositions on their domain ω.

Then any clone isomorphism Pol(Δ) $\to$ Pol(Δ')
is induced by a permutation of ω.

Part IV

The open problem

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Let Δ be ω-categorical, with less than double exponential type growth.
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Let Δ be ω-categorical, with less than double exponential type growth. TFAE:

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- There is no linear uniformly continuous homomorphism $\mathrm{Pol}(\Delta) \to \mathbf{1}$;
- $\mathrm{Pol}(\Delta)$ contains functions u, v (unary) and s (6-ary) such that

$$\forall x, y, z \ \ (u \circ s(x, y, x, z, y, z) = v \circ s(y, x, z, x, z, y)) \ .$$

Thank you!