Canonical functions and the Ramsey property, revisited

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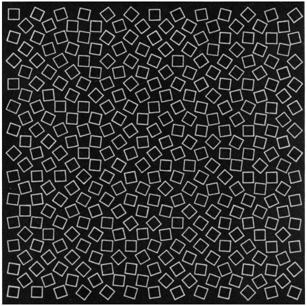
Canonical functions

I: Definition

II: Use

III: How to find

IV: How to avoid



I: What are canonical functions?

Regular behaviour of functions

Example 1

Let $f\colon A \to B$ be a function.
There exists an infinite $S \subseteq A$ on which f is injective or constant.

- f behaves “regularly" on S.

Example 2

Let $f\colon (\mathbb{N}, <) \to (\mathbb{Q}, <)$. There exists an infinite $S \subseteq \mathbb{N}$ on which f is strictly increasing, decreasing, or constant.

- f behaves on S "regularly" wrt the orders $(\mathbb{N}, <)$ and $(\mathbb{Q}, <)$.
- Note: f does not *preserve* the structure $(\mathbb{N}, <)$ on S.

Example 3

Let $f\colon (\mathbb{Q}, <, 0) \to (\mathbb{Q}, <)$. Every finite substructure of $(\mathbb{Q}, <, 0)$ has an isomorphic copy on which f behaves “regularly".

Canonical functions

Let $\mathbb{A}, \mathbb{B}$ be first-order structures, and $f \colon \mathbb{A} \to \mathbb{B}$.

Definition (model-theoretic)

f is canonical $:\leftrightarrow$
f sends tuples of the same type in $\mathbb{A}$ to tuples of the same type in $\mathbb{B}$.

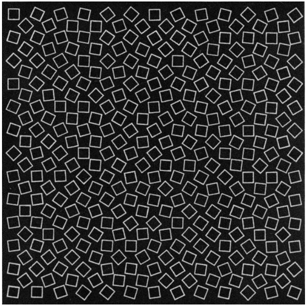
Definition (algebraic)

f is canonical $:\leftrightarrow$
$\forall$ tuples $\bar{t}$ in $\mathbb{A}$ $\forall \alpha \in \mathrm{Aut}(\mathbb{A})$ $\exists \beta \in \mathrm{Aut}(\mathbb{B})$ such that

$$f(\bar{t}) = \beta f \alpha^{-1}(\bar{t}).$$

Definitions equivalent when $\mathbb{A}, \mathbb{B}$ are countable ω-categorical, i.e., $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ act on n-tuples with finitely many orbits:

Tuples have same type $\leftrightarrow$ they are in the same orbit.



II: The use of canonical functions

Reducts

Theorem (Thomas '91)

Let (V, E) be the random graph.

There exist precisely 3 reducts of (V, E), i.e.,
closed permutation groups $\mathcal{G}$ with $\mathrm{Aut}(V, E) \subsetneq \mathcal{G} \subsetneq \mathrm{Sym}(V)$:

- $\mathrm{Aut}(V, E) + \gamma \in \mathrm{Sym}(V)$ which flips edges and non-edges.
- $\mathrm{Aut}(V, E) + \delta \in \mathrm{Sym}(V)$ which flips edges and non-edges around a fixed vertex $v \in V$.
- Their join.

Remarks.

- $\gamma \colon (V, E) \to (V, E)$ is canonical!
- $\delta \colon (V, E, v) \to (V, E)$ is canonical!

Coincidence?

Analyzing reducts

Proof of Thomas' theorem.

- Let $\mathrm{Aut}(V, E) \subsetneq \mathcal{G} \subsetneq \mathrm{Sym}(V)$ be closed.
- Pick $f \in \mathcal{G} \setminus \mathrm{Aut}(V, E)$.
- There is an edge $(a, b) \in V^2$ which f sends to a non-edge.
- Elsewhere f is *ugly*.
- There seems no way out.
- But suppose somebody steps in and tells us that $f\colon (V, E, a, b) \to (V, E)$ is *canonical*.
- Examine all possible *behaviours* for f. (Deletes edges around a? Other edges? Non-edges?...)
- Conclude that in each case, $\mathcal{G}$ contains one of the two groups.
- Continue if $\mathcal{G}$ is even bigger ...

Q.E.D.

More motivation

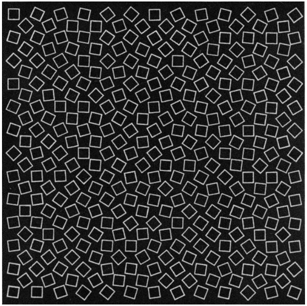
- Canonicity convenient when one has finite information about a function $f\colon \mathbb{A} \to \mathbb{B}$;
  want f to be nice outside that information.

- Polymorphisms $f\colon \mathbb{A}^n \to \mathbb{A}$ for Constraint Satisfaction Problems:

  Every structure $\mathbb{A}$ defines computational problem:

  Given a *primitive positive* sentence ϕ, does $\mathbb{A} \models \phi$ hold?

  The more polymorphisms $f\colon \mathbb{A}^n \to \mathbb{A}$ has, the easier the problem.
  Use polymorphism f to prove polynomial-time algorithm.
  Good if f is canonical...



III: Canonical functions in an ideal world

Obtaining canoncial functions

Let $f\colon \mathbb{A} \to \mathbb{B}$.

- Where?
- How?

Definition

The orbit of f (under action of $\mathrm{Aut}(\mathbb{B}) \times \mathrm{Aut}(\mathbb{A})$ on $\mathbb{B}^{\mathbb{A}}$):

$$O(f) := \{\beta f \alpha^{-1} \mid \beta \in \mathrm{Aut}(\mathbb{B}),\ \alpha \in \mathrm{Aut}(\mathbb{A})\}$$

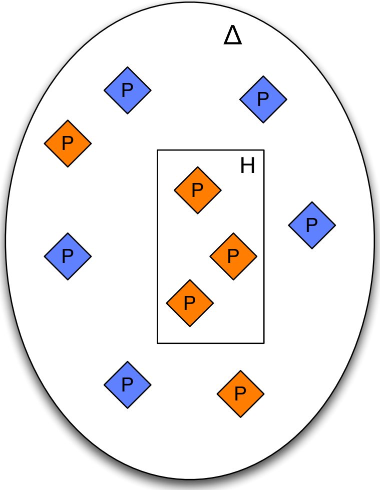
Note: All functions or no function in an orbit are canonical.
But we may be lucky in the (pointwise) closure $\overline{O(f)}$ in $\mathbb{B}^{\mathbb{A}}$.

Definition

$\mathbb{A}$ is Ramsey $:\leftrightarrow$
$\forall$ finite $\mathbb{P}, \mathbb{H} \subseteq \mathbb{A}$ $\quad \forall \chi\colon \binom{\mathbb{A}}{\mathbb{P}} \to 2 \quad \exists\, \mathbb{H}' \cong \mathbb{H}$ on which χ is constant.

Examples in beginning: Ramsey's theorem.

The Ramsey property



The Canonical Functions Proposition

Proposition (Bodirsky + MP + Tsankov '11)

Let $\mathbb{A}$ be homogeneous ordered Ramsey, and $\mathbb{B}$ be ω-categorical.

Let $f\colon \mathbb{A} \to \mathbb{B}$.

Then $\overline{O(f)}$ contains a canonical function $f'\colon \mathbb{A} \to \mathbb{B}$.

New proof.

For $g, g' \in \overline{O(f)}$ set $g \sim g'$ if $g \in \overline{\{\beta g' \mid \beta \in \mathrm{Aut}(\mathbb{B})\}}$.

Factoring by $\sim$ yields compact space (ω-categoricity).

Consider action of $\mathrm{Aut}(\mathbb{A})$ on that space: $([g]_\sim, \alpha) \mapsto [g\alpha^{-1}]_\sim$.

By Kechris-Pestov-Todorčević '05, $\mathrm{Aut}(\mathbb{A})$ is extremely amenable:

Cont. actions of $\mathrm{Aut}(\mathbb{A})$ on compact Hausdorff spaces have fixed point.

Let $[f']_\sim$ be a fixed point of our action.

f' is canonical.

Enjoying canoncial functions

In proof of Thomas' theorem:

$f \in \mathcal{G} \setminus \mathrm{Aut}(V, E)$

$f \colon (V, E, a, b) \to (V, E)$, where $(a, b) \in E$ is sent to a non-edge.

$\overline{O(f)}$ (with respect to action of $\mathrm{Aut}(V, E) \times \mathrm{Aut}(V, E, a, b)$) contains canonical function f'.

- f' is still "contained" in $\mathcal{G}$
- f' still sends (a, b) to non-edge.

Note:

(V, E) is not Ramsey $\Longrightarrow$
work with Ramsey expansion $(V, E, <)$.

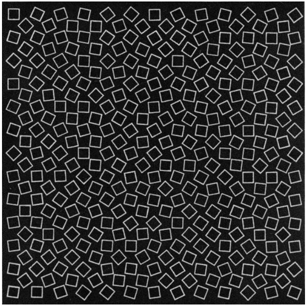
More applications

Reducts:

- Rational numbers $(\mathbb{Q}, <)$ (3) (Cameron '76)
- Random partial order $(P, \leq)$ (3)
  (Pach + MP + Pluhár + Pongrácz + Szabó '14)
- Random ordered graph $(V, E, <)$ (42)
  (Bodirsky + MP + Pongrácz '15)
- Random digraph (9) (Agarwal '16)
- Homogeneous binary branching C-relation (1)
  (Bodirsky + Jonsson + Van Pham '16)
- Homogeneous binary branching semilinear order (1)
  (Bodirsky + Bradley-Williams + MP + Pongrácz '17)

Constraint Satisfaction Problems:

- Homogeneous binary branching C-relation
  (Bodirsky + Jonsson + Van Pham '16)
- All homogeneous graphs (Bodirsky + Martin + MP + Pongrácz '16)



IV: Canonical functions in the real world

Necessity of the Ramsey property

What if $f\colon \mathbb{A} \to \mathbb{B}$, where $\mathbb{A}$ is not Ramsey?

Ramsey property: all continuous actions of Aut($\mathbb{A}$) on compact Hausdorff spaces have fixed point.

Canonical functions: fixed points of a particular action.

Problem

Suppose $\mathbb{A}$ is ordered homogeneous, and
$\overline{O(f)}$ contains a canonical function
for all $f\colon \mathbb{A} \to \mathbb{B}$, for all ω-categorical $\mathbb{B}$.

Is $\mathbb{A}$ Ramsey?

Theorem (Van Pham '17)

Yes, if $\mathbb{A}$ has an ω-categorical Ramsey expansion $\mathbb{A}'$.

Proof idea. Consider the identity function $id\colon \mathbb{A} \to \mathbb{A}'$.

Ramsey expansions

Problem (Hubička + Nešetřil, Bodirsky + MP '11)

Does every ω-categorical structure
have an ω-categorical Ramsey expansion?

Theorem (Evans '15)

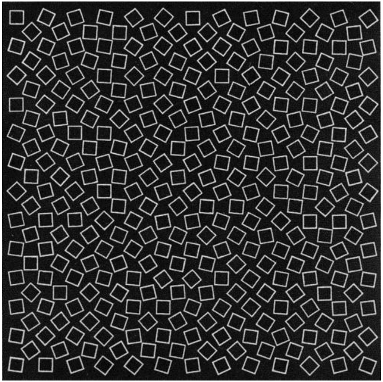
No.

Open for homogeneous structures in finite relational language.

Conjecture (Thomas '91)

Let $\mathbb{A}$ be homogeneous in a finite language.
Then $\mathbb{A}$ has only finitely many reducts.

- Ramsey expansion $\Longrightarrow$ canonical functions.
- No Ramsey expansion $\Longrightarrow$ perhaps no canonical functions...



Gerhard von Graevenitz *Regularity - Irregularity V*