Topology: relevant or irrelevant?

Michael Pinsker

Technische Universität Wien /
Univerzita Karlova

Funded by Austrian Science Fund (FWF) grant P32337
and GAČR grant 18-20123S

57th Summer School on General Algebra and Ordered Sets (SSAOS)

Karolinka 2019

Topology: relevant or irrelevant?

Depends on:

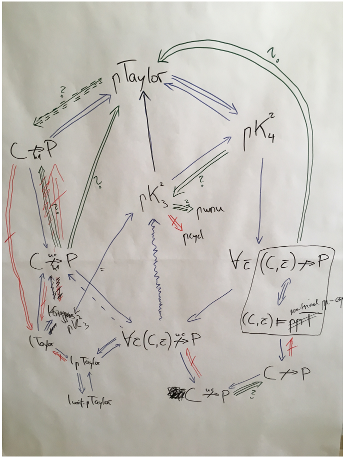
- Topology
- What for?

We only consider one particular topology
on the functions of an algebra / clone:

The *pointwise convergence topology*.

"What for?" varies.

- **Part I:** Global identities
- **Part II:** Local identities
- **Part III:** Topology is irrelevant
- **Part IV:** Topology is relevant



I: Global identities

Identities

Let $\mathbf{A} = (A; (f_i)_{i \in I})$ be an algebra.

$\mathrm{Clo}(\mathbf{A})$ ... clone of all *term functions* of $\mathbf{A}$

(= composites of fundamental operations of $\mathbf{A}$ and projections).

Universal algebra:

Identities of $\mathbf{A}$ / $\mathrm{Clo}(\mathbf{A})$ $\Leftrightarrow$ structure / properties of $\mathbf{A}$ / $\mathrm{Clo}(\mathbf{A})$.

Identities $\Sigma_{\mathbf{A}}$ of $\mathbf{A}$: All true statements of the form

$$\forall x_1, \ldots, x_n \; t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$$

where s, t are abstract terms over the language for $\mathrm{Clo}(\mathbf{A})$.

Depends only on $\mathrm{Clo}(\mathbf{A})$.

We write

$$t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_n) \; .$$

Example, HSP

Example: Let $\mathbf{G} = (G; +, -)$ be a group.

Let $m \in \mathrm{Clo}(\mathbf{G})$ be the function given by $(x, y, z) \mapsto x + ((-y) + z)$.

Then $\Sigma_{\mathbf{G}}$ contains (for example):

- $m(x, y, z) \approx x + ((-y) + z)$
- $m(x, x, y) \approx y$
- $m(y, x, x) \approx m(x, x, y)$

$\Sigma_{\mathbf{A}} \Leftrightarrow$ structure / properties of $\mathbf{A}$. **Which properties?**

- Independent of the fundamental operations of $\mathbf{A}$ / depend only on $\mathrm{Clo}(\mathbf{A})$.
- Closed under H, S, P / properties of the *variety* generated by $\mathbf{A}$.

HSP($\mathbf{A}$)... variety of $\mathbf{A}$... all algebras obtained by taking

(H) *factors* by congruence relations (aka *homomorphic images*);

(S) *subalgebras*;

(P) *powers*.

Birkhoff's theorem

H, S, P can be applied to the *term clone* Clo($\mathbf{A}$) of $\mathbf{A}$:
can be viewed as algebra $(A; \mathrm{Clo}(\mathbf{A}))$.

Function clone $\mathcal{A}$: set of finitary functions on a fixed domain A which

- contains all projections $\pi_i^n(x_1, \ldots, x_n) \approx x_i$;
- is closed under composition.

Every function clone is the term clone of an algebra.

$$\text{Identities } \Sigma_{\mathcal{A}} \quad \Leftrightarrow \quad \text{structure / properties of } \mathcal{A}.$$

Theorem (Birkhoff 1935)

Let $\mathcal{A}, \mathcal{B}$ be function clones. TFAE:

- $\mathcal{B} \in \mathrm{EHSP}(\mathcal{A})$, i.e., $\mathrm{HSP}(\mathcal{A})$ contains a function clone $\subseteq \mathcal{B}$;
- $\Sigma_{\mathcal{A}}$ " $\subseteq$ " $\Sigma_{\mathcal{B}}$, i.e., $\exists \phi \colon \mathcal{A} \to \mathcal{B}$ preserving identities.

We write $\mathcal{A} \to \mathcal{B}$, or $\Sigma_{\mathcal{A}} \leq \Sigma_{\mathcal{B}}$. ϕ is called a *clone homomorphism*.

Mal'cev conditions

- Properties invariant under EHSP = properties invariant under existence of clone homomorphisms.
- More identities $\Rightarrow$ more such properties.
- Characterized by *existence* of functions satisfying identities.

Strong Mal'cev condition:
Set Σ of identities over some abstract functional signature σ.

A function clone $\mathcal{A}$ *satisfies* Σ (we write $\Sigma \leq \Sigma_{\mathcal{A}}$) $:\leftrightarrow$
$\exists \phi\colon \sigma \to \mathcal{A}$ making all Σ true.

Example: Groups satisfy $q(x,x,y) \approx q(y,x,x) \approx y$.

Mal'cev condition: $\bigvee_{n \geq 1} \Sigma_n$, where each Σ_n is strong.

Example: Having a near unanimity (nu) term of some arity:

$$n(x,\ldots,x,y) \approx n(x,\ldots,x,y,x) \cdots \approx n(y,x,\ldots,x) \approx x$$

Examples

Classically often related to congruences
(= invariant equivalence relations of clones).

- $\mathcal{A}$ satisfies $q(x,x,y) \approx q(y,x,x) \approx y \quad \Longrightarrow$
  EHSP($\mathcal{A}$) has permuting congruences (CP). *(Mal'cev '54)*

  Example: groups $(A, +, -)$.

- $\mathcal{A}$ satisfies *near unanimity (nu)* equations $\Longrightarrow$
  EHSP($\mathcal{A}$) is congruence distributive (CD).

  Example: lattices. Equivalent to CD: *Jónsson '68 equations*.

- $\mathcal{A}$ finitely related, EHSP($\mathcal{A}$) *congruence modular* $\Rightarrow$
  The number of subalgebras of $\mathcal{A}^n$ grows only exponentially with n.
  (Barto '12)

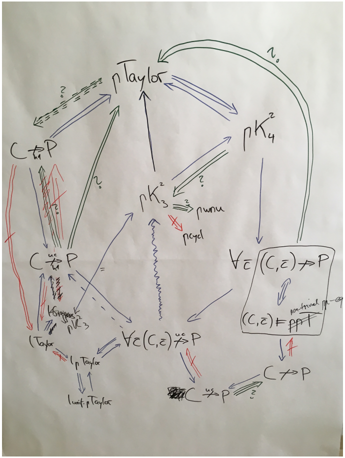
- $\mathcal{A}$ finitely related, EHSP($\mathcal{A}$) is CD $\Longrightarrow$
  $\mathcal{A}$ has a near unanimity function. *(Barto '10)*

Summary

Part I: Global identities

- Identities of algebras $\Leftrightarrow$ structure of algebras
- Properties of clones, rather than algebras
- Clone homomorphisms characterize E, H, S, P
- Mal'cev conditions:

  - stipulate the existence of functions satisfying certain identities
  - characterize properties invariant under E, H, S, P



II: Local identities

Finite powers

Consider properties invariant under *finite powers*.

$\mathsf{HSP}^{\mathsf{fin}}(\mathcal{A})$ ... *pseudovariety* generated by a function clone $\mathcal{A}$.

Examples:

- Complexity of the Constraint Satisfaction Problems (CSPs):
  Certain computational problems encoded by function clones.
  Function clones in the pseudovariety encode easier problems.
- Things definable from a structure within classical logic.
  (relational structure $\mathbb{A} \leftrightarrow$ function clone $\mathcal{A}$)
- Local properties.

Finite powers and local identities

Let $\mathcal{A}$ be a function clone on domain A.

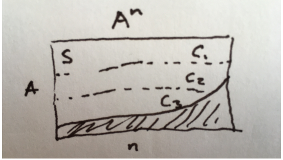
For any $F \subseteq^{\text{fin}} A$, let $\Sigma_{\mathcal{A}}^F$ be the identities which hold on F:

$$\forall x_1, \ldots, x_n \in F \;\; t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$$

Then $\Sigma_{\mathcal{A}}^F \supseteq \Sigma_{\mathcal{A}}$.

Now let $\mathcal{B}$ be any function clone on a finite domain.
Suppose that for all $F \subseteq^{\text{fin}} A$ we have $\Sigma_{\mathcal{A}}^F \not\subseteq \Sigma_{\mathcal{B}}$.

Then $\mathcal{B} \notin \mathrm{EHSP}^{\text{fin}}(\mathcal{A})$!



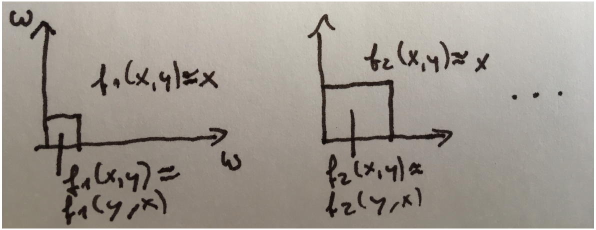
The converse also holds.

Local vs. global identities

$\mathcal{A} \to \mathcal{B}$ meant: $\exists \phi \colon \mathcal{A} \to \mathcal{B}$ preserving $\Sigma_{\mathcal{A}}$ (clone homomorphism).

Now we want: $\exists \phi \colon \mathcal{A} \to \mathcal{B}$ preserving *some* $\Sigma_{\mathcal{A}}^{F}$.

Example:



Let $\mathcal{A} := \mathrm{Clo}(\omega; (f_i)_{i \geq 1})$.

$\Sigma_{\mathcal{A}}$ is *trivial*, i.e., satisfiable in any clone (by its projections).

But every $\Sigma_{\mathcal{A}}^{F}$ "contains" $g(x,y) \approx g(y,x) \implies$ non-trivial.

The pointwise convergence topology

$A \ldots$ discrete.

$A^{A^n} = \{f\colon A^n \to A\} \ldots$ product topology

$\bigcup_{n\geq 1} A^{A^n} \ldots$ sum space.

Function clone $\mathcal{A} \subseteq \bigcup_{n\geq 1} A^{A^n} \ldots$ induced topology.

n-ary functions $(f_i)_{i\geq 1}$ converge to n-ary $f \leftrightarrow$
$f_i \restriction_{F^n} = f \restriction_{F^n}$ eventually, for all $F \subseteq^{\text{fin}} A$.

Topology induced by metric / uniformity.

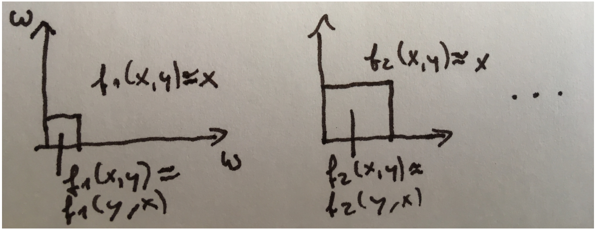
Theorem (Bodirsky + P. '11; Gehrke + P. '15)

Let $\mathcal{A}, \mathcal{B}$ be function clones, where $\mathcal{B}$ is finitely generated. TFAE:

- $\mathcal{B} \in \mathrm{EHSP}^{\text{fin}}(\mathcal{A})$;
- $\exists F \subseteq^{\text{fin}} A \;\; \exists \phi\colon \mathcal{A} \to \mathcal{B}$ (ϕ preserves $\Sigma_{\mathcal{A}}^{F}$);
- $\mathcal{A} \xrightarrow{uc} \mathcal{B}$:
  $\exists \phi\colon \mathcal{A} \to \mathcal{B}$ uniformly continuous clone homomorphism.

Topology is relevant

Previous example:



Let $\mathcal{A} := \mathrm{Clo}(\omega; (f_i)_{i\geq 1})$. $\Sigma_{\mathcal{A}}$ is *trivial*, but no $\Sigma^F_{\mathcal{A}}$ is trivial.

Let $\mathcal{P}$ be the clone consisting only of projections on $\{0, 1\}$.

$\Sigma_{\mathcal{P}}$ is trivial, i.e., satisfiable in every clone.

- $\mathcal{A} \to \mathcal{P}$ (hence $\mathcal{P} \in \mathrm{HSP}(\mathcal{A})$);
- $\mathcal{A} \overset{uc}{\not\to} \mathcal{P}$ (hence $\mathcal{P} \notin \mathrm{HSP}^{\mathrm{fin}}(\mathcal{A})$)

Topology is relevant!

Local closure

Topology: relevant or irrelevant?

- Local identities $\implies$ global identities?
- $\text{EHSP}^{\text{fin}} = \text{EHSP}$?
- Clone homomorphism $\implies$ u.c. clone homomorphism?

A clone $\mathcal{A}$ is *closed* / *topologically closed* / *locally closed* $:\leftrightarrow$ $\mathcal{A}$ contains all functions which it can interpolate on all finite sets.

$\mathcal{A}$ closed $\leftrightarrow$ $\mathcal{A} = \text{Pol}(\mathbb{A})$ for some relational structure $\mathbb{A} = (A; (R_j)_{j \in J})$.

$\text{Pol}(\mathbb{A})$ ... polymorphism clone of $\mathbb{A}$:
all homomorphisms from finite powers of $\mathbb{A}$ into $\mathbb{A}$.

For non-closed clones, topology is relevant.

There exists a *closed* clone $\mathcal{A}$ with $\mathcal{A} \to \mathcal{P}$ and $\mathcal{A} \stackrel{uc}{\not\to} \mathcal{P}$. *(Barto + P. '17)*

Compactness

- In closed clones, functions converge.
- When do local identities converge to global ones?

A permutation group $\mathcal{G}$ acting on a set G is *oligomorphic* $:\leftrightarrow$
$\mathcal{G}$ acts on G^n with finitely many orbits, for all $n \geq 1$.

A function clone $\mathcal{A}$ is *oligomorphic* $:\leftrightarrow$
$\mathcal{A}$ contains an oligomorphic permutation group.

<u>Lemma</u> (Mottet + P.)

Let $\mathcal{A}$ be a closed oligomorphic clone, and $\mathcal{E}$ be a reasonable finite set of identities.

$\forall F \subseteq^{\text{fin}} A \; (\mathcal{E} \subseteq \mathcal{E}_{\mathcal{A}}^{F}) \implies \mathcal{E} \subseteq \mathcal{E}_{\mathcal{A}}$

Reason:

For $f, g \in \mathcal{A}$, set $f \sim g :\leftrightarrow f \in \overline{\{\alpha \circ g \mid \alpha \in \mathcal{G}\}}$
(where $\mathcal{G}$ is the oligomorphic group in $\mathcal{A}$).

Then $(\mathcal{A} \cap A^{A^n})/\sim$ is compact for all $n \geq 1$.

Topology: relevant or irrelevant?

Open problem

Let $\mathcal{A}$ be a closed oligomorphic clone. TFAE?

- $\mathcal{A} \to \mathcal{P}$ (i.e., $\Sigma_{\mathcal{A}}$ is trivial);
- $\mathcal{A} \xrightarrow{uc} \mathcal{P}$ (i.e., $\Sigma_{\mathcal{A}}^F$ is trivial for some $F \subseteq^{\text{fin}} A$).

Search for *weakest* non-trivial strong Mal'cev condition Σ, locally:

$\mathcal{A} \stackrel{uc}{\not\to} \mathcal{P} \implies \Sigma \leq \Sigma_{\mathcal{A}}^F$ for all $F \subseteq^{\text{fin}} A$.

Related problem:

Is there a strong Mal'cev condition for non-triviality of finite clones?

Is there Σ such that any finite $\mathcal{A}$ with non-trivial $\Sigma_{\mathcal{A}}$ satisfies Σ?

Recently: weakest condition *which depends on* $|A|$ (for *cores*) *(Barto + Kozik)*.

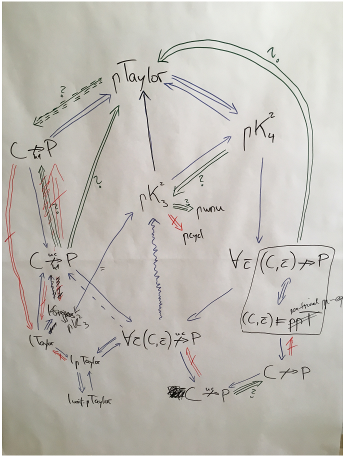
Irrelevance of topology $\Leftrightarrow$ existence of strong Mal'cev conditions

Summary

Part I: Global identities

Part II: Local identities

- Local identities determine which finitely generated clones can be constructed using E, H, S, P^{fin}
- Characterized by the existence of uniformly continuous clone homomorphisms
- In closed oligomorphic clones: local identities $\Longrightarrow$ global identities



Thank you!

Topology: relevant or irrelevant?

Michael Pinsker

Technische Universität Wien /
Univerzita Karlova

Funded by Austrian Science Fund (FWF) grant P32337
and GAČR grant 18-20123S

57th Summer School on General Algebra and Ordered Sets (SSAOS)

Karolinka 2019

What happened so far

Part I: Global identities

Part II: Local identities

- Local identities in a function clone determine which finite clones can be constructed using E, H, S, P^{fin}
- Characterized by the existence of uniformly continuous clone homomorphisms
- In closed oligomorphic clones:
  local identities $\Longrightarrow$ global identities

  Inspires search for strong Mal'cev conditions

ω-categoricity

A countable relational structure $\mathbb{A}$ is ω-categorical $:\leftrightarrow$
$\mathbb{A}$ first-order defines only finitely many n-ary relations, for all $n \geq 1$ (without parameters).

Theorem (Ryll-Nardzewski, Engeler, Svenonius 1959)

Let $\mathbb{A}$ be a countable structure. TFAE:

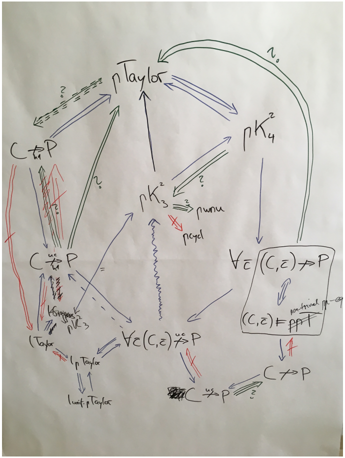
- $\mathbb{A}$ is ω-categorical;
- Aut($\mathbb{A}$) (equivalently, Pol($\mathbb{A}$)) is oligomorphic.

Examples:

- $(\mathbb{Q}; <)$ is ω-categorical.
- $(\mathbb{Z}; <)$ is not ω-categorical.

High degree of symmetry.

Close to finite structures:
Finitely many tuples of every arity modulo Aut($\mathbb{A}$).



III: Topology is irrelevant

pp-interpretations

HSP$^{\text{fin}}$ on function clones

$\Leftrightarrow$

interpretations on relational structures.

Let $\mathbb{A}, \mathbb{B}$ be relational structures.
$\mathbb{B}$ has a *first-order (fo) interpretation* in $\mathbb{A}$ $:\leftrightarrow$ $\mathbb{B}$ can be constructed by

- taking a power A^n for some finite $n \geq 1$;
- defining a subset $S \subseteq A^n$;
- defining an equivalence relation $\sim$ on S;
- defining relations on the equivalence classes of $\sim$.

Example: $(\mathbb{Q}; +, \cdot)$ has a fo-interpretation in $(\mathbb{Z}; +, \cdot)$.

An interpretation is *primitive positive (pp)* $\leftrightarrow$
all used formulas are primitive positive, i.e.,
of the form $\exists x_1, \ldots, x_n \; R_1(\cdots) \wedge \cdots \wedge R_m(\cdots)$.

pp interpretations and HSP$^{\text{fin}}$

Theorem (Bodirsky + P. '11)

Let $\mathbb{A}$ be ω-categorical, and let $\mathbb{B}$ be finite. TFAE:

- $\mathbb{B}$ has a pp interpretation in $\mathbb{A}$;
- $\mathrm{Pol}(\mathbb{B}) \in \mathrm{EHSP}^{\mathrm{fin}}(\mathrm{Pol}(\mathbb{A}))$;
- $\mathrm{Pol}(\mathbb{A}) \xrightarrow{uc} \mathrm{Pol}(\mathbb{B})$.

Remarks:

- $\mathrm{Pol}(\mathbb{A}) \xrightarrow{uc} \mathcal{P} \Leftrightarrow$ all finite structures have a pp interpretation in $\mathbb{A}$.
- Is topology relevant for this property?

  $\mathrm{Pol}(\mathbb{A}) \xrightarrow{uc} \mathcal{P} \quad \Leftrightarrow \quad \mathrm{Pol}(\mathbb{A}) \rightarrow \mathcal{P}$?

Constraint Satisfaction Problems (CSPs)

Let $\mathbb{A}$ be a relational structure.
CSP($\mathbb{A}$)…problem of deciding the primitive positive theory of $\mathbb{A}$:

Input: pp sentence $\varphi = \exists x_1, \ldots, x_n\; R_1(\cdots) \wedge \cdots \wedge R_m(\cdots)$.
Question: Does φ hold in $\mathbb{A}$?

Example:

$\mathbb{A} = (\mathbb{Q}; <)$.
$\varphi = \exists x_1, x_2, x_3, x_4 \;\; (x_1 < x_3) \;\wedge\; (x_3 < x_2) \;\wedge\; (x_2 < x_4) \;\wedge\; (x_4 < x_1)$

- $\mathbb{B}$ has a pp interpretation in $\mathbb{A} \implies$ CSP($\mathbb{B}$) reduces to CSP($\mathbb{A}$).
- Hence: Pol($\mathbb{A}$) $\xrightarrow{uc} \mathcal{P} \implies$ CSP($\mathbb{A}$) is NP-hard.
- A “finite reason" for NP-hardness.
- Not the only reason (but almost?)

pp interpretations with parameters

$\mathbb{A}, \mathbb{B} \ldots$ relational structures.

$\mathbb{A}$ pp-interprets $\mathbb{B}$ *with parameters* $:\leftrightarrow$
$\exists n \geq 1 \; \exists \bar{a} \in A^n \;\; (\mathbb{A}, \bar{a})$ pp interprets $\mathbb{B}$.

$\mathrm{Pol}(\mathbb{A}, \bar{a}) \ldots$ *stabilizer* of $\bar{a}$ in $\mathrm{Pol}(\mathbb{A})$.

$\mathbb{A}$ pp interprets all finite structures with parameters $\Leftrightarrow$
$\exists \bar{a} \;\; \mathrm{Pol}(\mathbb{A}, \bar{a}) \xrightarrow{uc} \mathcal{P}$.

Question:

$$\exists \bar{a} \;\; \mathrm{Pol}(\mathbb{A}, \bar{a}) \xrightarrow{uc} \mathcal{P} \quad \Leftrightarrow \quad \exists \bar{a} \;\; \mathrm{Pol}(\mathbb{A}, \bar{a}) \to \mathcal{P} \quad ?$$

pp interpretations with parameters $\not\Rightarrow$ complexity reductions:
Need an additional assumption on $\mathbb{A}$.

Mc cores

Let $\mathbb{A}$ be a relational structure.

$\mathbb{A}$ is a *model-complete (mc) core* $:\leftrightarrow$
Aut($\mathbb{A}$) is dense in End($\mathbb{A}$) $\leftrightarrow$
all endomorphisms agree with an automorphism on every finite set $\leftrightarrow$
the unary functions in Pol($\mathbb{A}$) are the closure of Aut($\mathbb{A}$).

Equivalent to **local idempotency of the clone**:
$\forall F \subseteq^{\text{fin}} A\ \forall f \in \text{Pol}(\mathbb{A})\ \exists \alpha \in \text{Aut}(\mathbb{A})$ such that $\alpha \circ f \restriction_{F^n}$ is idempotent.

Theorem (Bodirsky '03)

Every ω-categorical structure is homomorphically equivalent to a (unique, ω-categorical) mc core.

- Homomorphically equivalent structures have equal CSPs.
- pp interpretations with parameters *in ω-categorical mc cores* $\Longrightarrow$ CSP reductions.

Topology is irrelevant

Theorem (Barto + P. '16)

Let $\mathcal{A}$ be a closed oligomorphic, mc core. TFAE:

- $\forall \bar{a}\ (\mathcal{A}, \bar{a}) \overset{uc}{\not\to} \mathcal{P}$;
- $\forall \bar{a}\ (\mathcal{A}, \bar{a}) \not\to \mathcal{P}$;
- $\mathcal{A}$ satisfies $u \circ s(x, y, x, z, y, z) \approx v \circ s(y, x, z, x, z, y)$.

- For finite idempotent $\mathcal{A}$, non-triviality implies $\mathcal{A}$ satisfies $s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y)$ *(Siggers '11)*
- Conjectured to be tractability criterion for a certain class of ω-categorical CSPs. *(see talk of Antoine Mottet)*
- Tractability criterion for finite CSPs *(Bulatov, Zhuk '17)*

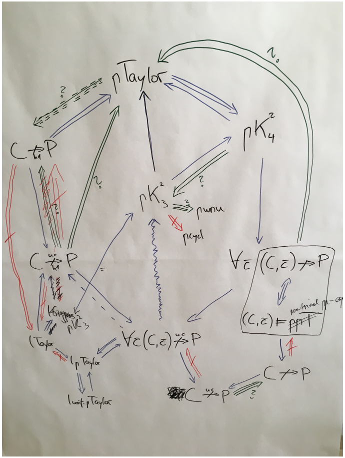
Summary

Part I: Global identities

Part II: Local identities

Part III: Topology is irrelevant

- E, H, S, $\mathrm{P}^{\mathrm{fin}}$ corresponds to pp interpretations
- pp interpretations $\implies$ complexity reductions between CSPs
- pp interpretations with parameters $\implies$ complexity reductions in mc cores
- ω-categorical structures are homomorphically equivalent to mc cores
- Topology is irrelevant for pp interpreting all finite structures with parameters (in an ω-categorical mc core)



IV: Topology is relevant

h1 identities

A *height 1 (h1)* identity is of the form

$$s(x_1, \ldots, x_n) \approx t(y_1, \ldots, y_n) \ ,$$

where s, t are functional symbols (not arbitrary terms!).

Many Mal'cev conditions consist of h1 identities:

- $s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y)$
- $s(x, y, z, x) \approx s(y, x, y, z)$
- $w(x, \ldots, x, y) \approx w(x, \ldots, x, y, x) \approx \cdots \approx w(y, x, \ldots, x)$

$\tilde{\Sigma}_{\mathcal{A}}$ ...h1 identities of a function clone $\mathcal{A}$.

$\tilde{\Sigma}_{\mathcal{A}} \subseteq \Sigma_{\mathcal{A}} \implies$ weaker algebraic structure on $\mathcal{A}$.

$\mathcal{A} \dashrightarrow \mathcal{B} \ :\leftrightarrow \ \exists \phi \colon \mathcal{A} \to \mathcal{B}$ preserving h1 identities.

ϕ ... *minion homomorphism*.

The double shrink (R)

Let $\mathcal{A}$ be a function clone, B be a set, $p\colon A \to B$, $q\colon A \to B$.

$$\{p \circ f(q(x_1), \dots, q(x_n)) \mid f \in \mathcal{A}\}$$

is called a *reflexion* of $\mathcal{A}$.

R($\mathcal{A}$) ... all reflexions of $\mathcal{A}$. Generalizes H, S.

Theorem (Barto + Opršal + P. '16)

Let $\mathbb{A}, \mathbb{B}$ be relational structures,
where $\mathbb{A}$ is ω-categorical and $\mathcal{B}$ is finite. TFAE:

- Pol($\mathbb{B}$) $\in$ ERP$^{\text{fin}}$(Pol($\mathbb{A}$));
- Pol($\mathbb{A}$) $\overset{\text{uc}}{\dashrightarrow}$ Pol($\mathbb{B}$).
- $\mathbb{B}$ can be obtained from $\mathbb{A}$ by homomorphic equivalence and pp interpretations.

Slow orbit growth

Let $\mathbb{A}$ be a mc core.

- $\exists \bar{a}$ $\mathrm{Pol}(\mathbb{A}, \bar{a}) \xrightarrow{uc} \mathcal{P} \implies$ CSP($\mathbb{A}$) is NP-hard.
- $\mathrm{Pol}(\mathbb{A}) \overset{\text{uc}}{\dashrightarrow} \mathcal{P} \implies$ CSP($\mathbb{A}$) is NP-hard.
- $\mathrm{Pol}(\mathbb{A}) \overset{\text{uc}}{\dashrightarrow} \mathcal{P} \quad \Leftrightarrow \quad \exists \bar{a}$ $\mathrm{Pol}(\mathbb{A}, \bar{a}) \xrightarrow{uc} \mathcal{P}$
  when $\mathbb{A}$ has less than double exponential orbit growth.
  (Barto + Kompatscher + Olšák + Van Pham + P. '17)

Topology: relevant?

$$\mathrm{Pol}(\mathbb{A}) \overset{\text{uc}}{\dashrightarrow} \mathcal{P} \quad \Leftrightarrow \quad \mathrm{Pol}(\mathbb{A}) \dashrightarrow \mathcal{P} \quad ?$$

Topology is relevant

Theorem (Bodirsky + Mottet + Olšák + Opršal + P. + Willard '19)

For all non-trivial height 1 conditions $\tilde{\Sigma}$
there exists a closed oligomorphic clone $\mathcal{A}$ such that

- $\mathcal{A} \not\dashrightarrow \mathcal{P}$;
- $\mathcal{A}$ does not satisfy Σ.

Theorem (Bodirsky + Mottet + Olšák + Opršal + P. + Willard '19)

There exists a closed oligomorphic clone $\mathcal{A}$ such that

- $\mathcal{A} \not\dashrightarrow^{uc} \mathcal{P}$ (i.e., $\mathcal{A}$ satisfies non-trivial h1 identities locally);
- $\mathcal{A} \dashrightarrow \mathcal{P}$ (i.e., the global h1 identities of $\mathcal{A}$ are trivial).

- The first example is in contrast with finite structures.
- It lies within the infinite CSP dichotomy conjecture.
- The second one does not.

Summary

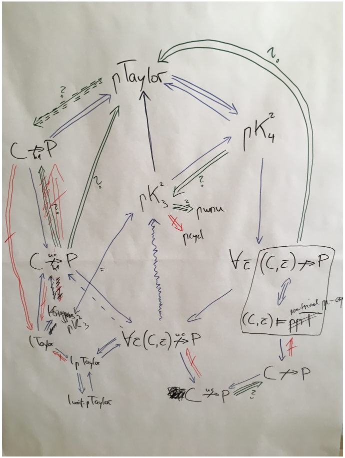
Part I: Global identities

Part II: Local identities

Part III: Topology is irrelevant

Part IV: Topology is relevant

- Height 1 identities characterize ERP / $\mathrm{ERP}^{\mathrm{fin}}$.
- $\mathrm{ERP}^{\mathrm{fin}}$ characterizes homomorphic equivalence + pp interpretations.
- There is no weakest h1 Mal'cev condition for closed oligomorphic clones.
- Topology is relevant for minion homomorphisms to projections.



V: Topology: relevant or irrelevant?

Open problems

- Is there a non-trivial Σ satisfied by every non-trivial finite $\mathcal{A}$?

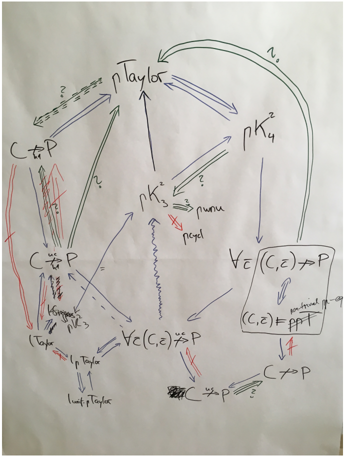
- $\mathcal{A} \to \mathcal{P} \quad \Leftrightarrow \quad \mathcal{A} \xrightarrow{uc} \mathcal{P}$

  for closed oligomorphic clones $\mathcal{A}$?

- $\mathcal{A} \dashrightarrow \mathcal{P} \quad \Leftrightarrow \quad \mathcal{A} \overset{uc}{\dashrightarrow} \mathcal{P}$

  for closed oligomorphic clones $\mathcal{A}$

  within the infinite CSP dichotomy conjecture?



Thank you!