SMOOTH APPROXIMATIONS

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JOINT WORK WITH ANTOINE MOTTET

CSP ONLINE SEMINAR

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I THE INFINITE DOMAIN CSP CONJECTURE

II FIRST GENERATION PROOFS

III SMOOTH APPROXIMATIONS

SCOPE

$\underline{CSP(\mathbb{A})}$, $\mathbb{A}$... RELATIONAL STRUCTURE IN FINITE SIGNATURE
"TEMPLATE", "CONSTRAINT LANGUAGE"

INPUT: PP-SENTENCE $\varphi = \exists x_1 \dots \exists x_n \; R_1(\text{variables}) \wedge \dots \wedge R_m(\text{variables})$

RELATIONAL SYMBOLS OF SIGNATURE OF $\mathbb{A}$

QUESTION: $\mathbb{A} \models \varphi$, i.e., DOES φ HOLD IN $\mathbb{A}$?

EXAMPLES

n-SAT, n-COLORING, SOLVING EQUATIONS, SOLVING LINEAR EQUATIONS, ACYCLICITY, ... (BASICALLY ANYTHING)

FINITE $\mathbb{A}$:

- $CSP(\mathbb{A}) \in NP$
- P/NP-c DICHOTOMY (BULATOV, ZHUK '17 : FEDER-VARDI-CONJECTURE)
- LOCAL CONSISTENCY METHODS UNDERSTOOD:
  - SCOPE (BARTO + KOZIK '09)
  - COLLAPSE (BARTO '16)
- SUCCESS STORY OF THE "ALGEBRAIC APPROACH"!

ALGEBRAIC APPROACH

TEMPLATE $\mathbb{A}$

$$\mathrm{Pol}(\mathbb{A}) = \bigcup_n \{ f : \mathbb{A}^n \to \mathbb{A} \mid f \text{ PRESERVES } \mathbb{A} \}$$

"POLYMORPHISM CLONE OF $\mathbb{A}$"

ALSO WORKS FOR INFINITE $\mathbb{A}$

(REASONABLE ASSUMPTIONS + ADAPTATIONS)

EQUATIONS ("IDENTITIES") OF $\mathrm{Pol}(\mathbb{A})$

$\boxed{\text{E.G.}}$ $w(x,x,\dots,x,y) \approx \dots \approx w(y,x,\dots,x)$ WEAK NU

WNU

$\mathrm{Pol}(\mathbb{A}) \not\models$ EQUATIONS

$\mathrm{Pol}(\mathbb{A}) \models$ EQUATIONS

CSP($\mathbb{A}$) HARD

$\boxed{\text{E.G.}}$
- NP-c,
- UNBOUNDED WIDTH

CSP($\mathbb{A}$) TRACTABLE

$\boxed{\text{E.G.}}$ $\in$ P, SOLVABLE BY LOCAL CONSISTENCY METHODS (BOUNDED WIDTH)

- EQUATIONS STUDIED IN UNIVERSAL ALGEBRA FOR 70 YEARS
- NEW METHODS: BULATOV'S GRAPHS, ZHUK'S CENTERS, BARTO & KOZIK'S ABSORPTION

WHY FINITE $\mathbb{A}$?

① $\mathrm{CSP}(\mathbb{A}) \in$ NP GUARANTEED,
P / NP-c DICHOTOMY POSSIBLE
$\mathbb{A}$ INFINITE $\Rightarrow$ ANY COMPLEXITY POSSIBLE!

② ALGEBRAIC APPROACH WORKS (+ FINITE ALGEBRAS FOR $\mathbb{A}$) WELL-STUDIED BEFORE)

$\mathrm{CSP}(\mathbb{A})$ MORE INTERESTING / NATURAL / RELEVANT FOR FINITE $\mathbb{A}$?

PROBABLY NOT.

EXAMPLES

- ACYCLICITY = $\mathrm{CSP}(\mathbb{Q}, <)$
- BETWEENNESS = $\mathrm{CSP}(\mathbb{Q}, \mathrm{BETW}(x,y,z))$
- MORE GENERALLY: CSPs THAT STEM FROM $(\mathbb{Q}, <)$ "<u>TEMPORAL CSPs</u>"
- MMSNP (FEDER + VARDI '93)
- SOLVING EQUATIONS
- CSPs OVER INTEGERS

⋮

- WHICH CLASS SATISFIES ① + ② ?
- GENERAL REASONS FOR TRACTABILITY?

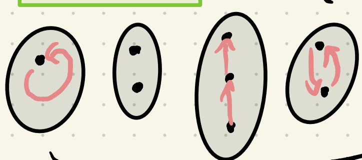
THE LARGER CLASS

INSTEAD OF A FINITE SET, THE SOLUTION SPACE IS GIVEN BY AN INFINITE ~~SET~~ STRUCTURE $\mathbb{B}$:
"SOLUTION SPACE"

$\mathbb{B}$ HAS FINITE SIGNATURE
+ IS **FINITELY BOUNDED**:

FINITE SUBSTRUCTURES GIVEN BY FINITE SET OF "FORBIDDEN SUBSTRUCTURES"

EXAMPLE $\mathbb{B} = (\mathbb{Q}, <)$:

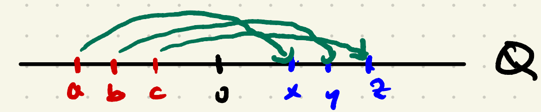


FORBIDDEN

$\mathbb{B}$ IS **HOMOGENEOUS**:

EVERY FINITE PARTIAL ISOMORPHISM OF $\mathbb{B}$ EXTENDS TO AN AUTOMORPHISM

EXAMPLE $\mathbb{B} = (\mathbb{Q}, <)$



$\Downarrow$

- $\mathbb{B}$ HAS FINITE DESCRIPTION
- $\mathrm{Aut}(\mathbb{B})$ ACTS ON $\mathbb{B}^n$ BY FINITELY MANY ORBITS

CSP TEMPLATE $\mathbb{A}$ OVER SOLUTION SPACE $\mathbb{B}$:

ANY STRUCTURE WITH FIRST-ORDER DEFINITION IN $\mathbb{B}$

EXAMPLE $(\mathbb{Q}, \mathrm{BETW}(x,y,z))$

- EVERY INSTANCE OF $\mathrm{CSP}(\mathbb{A})$ HAS ONLY FINITELY MANY POSSIBLE SOLUTIONS (UP TO $\mathrm{Aut}(\mathbb{B})$)
- RELATIONS OF $\mathbb{A}$ ARE UNIONS OF ORBITS

5

CONJECTURE (BODIRSKY + P. '11)

LET $\mathbb{A}$ BE FIRST-ORDER DEFINABLE IN $\mathbb{B}$ WHICH IS FINITELY BOUNDED + HOMOGENEOUS
THEN $CSP(\mathbb{A}) \in P$ OR NP-c

- BASED ON COUPLE OF EXAMPLES
- SEEMS MOST REASONABLE
  (E.G. ω-CATEGORICITY ALONE = NIGHTMARE)
  BODIRSKY + GROHE '08
  GILLIBERT + JONUŠAS + KOMPATSCHER + MOTTET + P. '22

- NOW CONFIRMED FOR MANY SOLUTION SPACES $\mathbb{B}$

- STILL OPEN

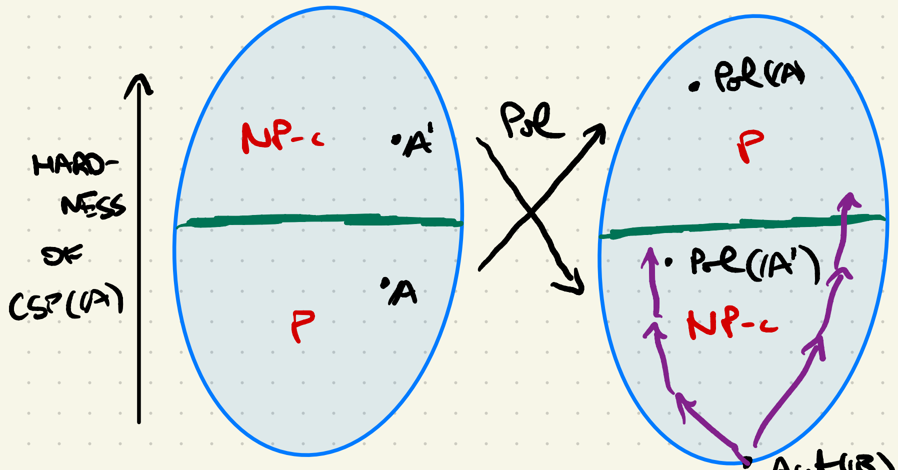
EXAMPLES $\mathbb{B} = \ldots$

- $(\mathbb{Q}, <)$ BODIRSKY + KÁRA '07
- RANDOM GRAPH BODIRSKY + P. '11
- EQUIVALENCE RELATION
- K_n-FREE GRAPH
- ANY UNDIRECTED GRAPH

  BODIRSKY + MARTIN + PONGRÁCZ + P. '16
- RANDOM POSET KOMPATSCHER + PHAM '17
- UNARY STRUCTURES BODIRSKY + MOTTET '17
- "MMSNP" BODIRSKY + MADELAINE + MOTTET '18
- ...

FIRST GENERATION

FIRST-GENERATION PROOFS



TEMPLATES $\mathbb{A}$ FO-DEFINABLE IN $\mathbb{B}$

POLYMORPHISM CLONES $\supseteq \mathrm{Aut}(\mathbb{B})$

- TRY TO GUESS THE $\boxed{\text{BORDER}}$, OR
- $\boxed{\text{„CLIMB UP"}}$ FROM $\mathrm{Aut}(\mathbb{B})$
- USE $\boxed{\text{CANONICAL FUNCTIONS}}$ TO DISTINGUISH POLYMORPHISM CLONES

$f \in \mathrm{Pol}(\mathbb{A})$ $\boxed{\text{CANONICAL}}$ (WRT $\mathbb{B}$) $:\Leftrightarrow$

$$\left.\begin{array}{l} \forall \bar{a}_1, \dots, \bar{a}_n \\ \forall \bar{b}_1, \dots, \bar{b}_n \\ \forall i \ \ \bar{a}_i, \bar{b}_i \text{ SAME } \mathrm{Aut}(\mathbb{B})\text{-ORBIT} \end{array}\right\} \Rightarrow \begin{array}{l} f(\bar{a}_1, \dots, \bar{a}_n), \\ f(\bar{b}_1, \dots, \bar{b}_n) \\ \text{SAME } \mathrm{Aut}(\mathbb{B})\text{-ORBIT} \end{array}$$

EXAMPLES

- $f: (\mathbb{Q}, <) \to (\mathbb{Q}, <)$
  MONOTONE OR ANTITONE
- $\mathbb{B}$... GRAPH

$f: \mathbb{B}^2 \to \mathbb{B}$

$f(E,E) = E$
$f(N,N) = N$
$f(E,N) = E$
$f(N,E) = N$

$\dots f(\cdot, E) = E$
$\vdots$

E ... EDGES
N ... NON-EDGES

(!) • CANONICAL FUNCTIONS ARE "FINITE" OBJECTS

• THEY ARE NOT RARE IF $\mathbb{B}$ IS $\boxed{\text{RAMSEY}}$

$$\forall f \in \mathrm{Pol}(\mathbb{A})\ \exists g \text{ canonical } g \in \overline{\{\alpha f \bar{\beta} \mid \alpha, \bar{\beta} \in \mathrm{Aut}(\mathbb{B})\}}$$

BODIRSKY & P. + TSANKOV '13

FIRST-GENERATION PROOFS:

ALL INFORMATION ABOUT CSP($\mathbb{A}$) → TEMPLATE $\mathbb{A}$

↓

SUFFICIENT INFORMATION → $\mathrm{Pol}(\mathbb{A})$

↓

SOMETIMES SUFFICIENT INFORMATION → $\mathrm{Pol}^{\mathrm{can}}(\mathbb{A}) = \mathrm{Pol}(\mathbb{A}) \cap \{f \mid f \text{ canonical}\}$

↓

DISTINGUISH CASES & HOPE FOR THE BEST

TROUBLE:

• ANNOYING
• NOT SCALEABLE

SECOND-GENERATION PROOFS

(RECENT MOTTET-CLASSIFICATIONS:
- UNARY STRUCTURES
- MMSNP)

TEMPLATE $\mathbb{A}$

$\mathrm{Pol}(\mathbb{A}) \not\models$ EQUATIONS → **NP-c** (BODIRSKY + P.)

$\mathrm{Pol}(\mathbb{A}) \models$ EQUATIONS → $\mathrm{Pol}(\mathbb{A}) \models e \circ s(x,y,x,z,y,z) \approx f \circ s(y,x,z,x,z,y)$ (BARTO + P. '16)

- $\mathrm{Pol}^{\mathrm{can}}(\mathbb{A}) \not\models$ EQUATIONS → **?** (")
- $\mathrm{Pol}^{\mathrm{can}}(\mathbb{A}) \models$ EQUATIONS → **P** (BODIRSKY + MOTTET '17) "LIFT FROM FINITE"

BAD SITUATION:
- $\mathrm{Pol}(\mathbb{A}) \models$ EQUATIONS (AND CSP SHOULD BE IN **P**)
- $\mathrm{Pol}^{\mathrm{can}}(\mathbb{A}) \not\models$ EQUATIONS

THEORY OF SMOOTH APPROXIMATIONS:

DESIGNED TO SHOW **BAD SITUATION** DOES NOT OCCUR.

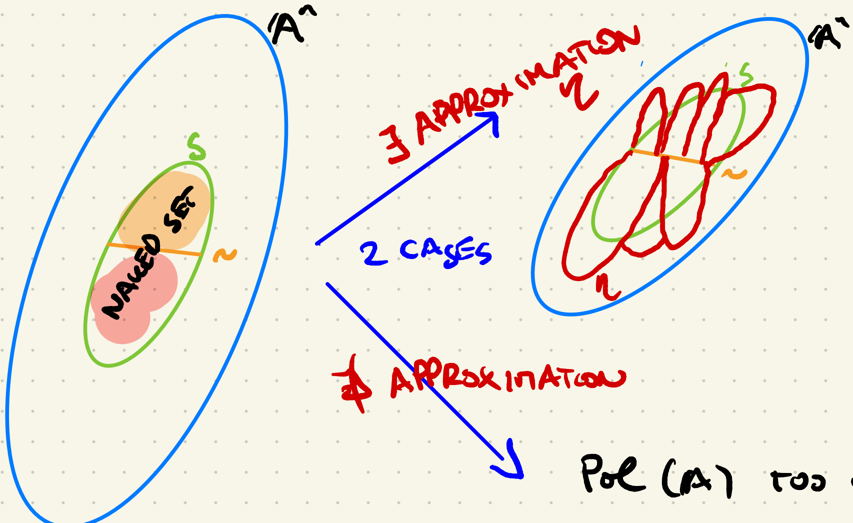
... IF IT DOESN'T...

HOW?

ASSUME: - $\mathrm{Pol}(\mathbb{A}) \models$ EQUATIONS

- $\mathrm{Pol}^{\mathrm{can}}(\mathbb{A}) \not\models$ EQUATIONS : "<u>NAKED</u>"

$\Rightarrow \mathrm{Pol}^{\mathrm{can}}(\mathbb{A})$ BEHAVES LIKE PROJECTIONS <u>SOMEWHERE</u>: (BIRKHOFF 1935)



η SMOOTH APPROXIMATION ON WHICH $\mathrm{Pol}(\mathbb{A})$ ACTS LIKE PROJECTIONS

↯

$\mathrm{Pol}(\mathbb{A})$ TOO RICH FOR THIS

$\Rightarrow$ CONTAINS CERTAIN "COMMUTATIVE" FUNCTIONS WHICH ARE "ALMOST CANONICAL"

$\Rightarrow \mathrm{Pol}^{\mathrm{can}}(\mathbb{A}) \models$ EQUATIONS ↯

THEOREM (MOTTET + P. '20)

LET $\mathbb{A}$ BE FO-DEFINABLE IN:

- RANDOM TOURNAMENT NEW!
- HOMOGENEOUS GRAPH:
  - RANDOM GRAPH
  - K_n-FREE GRAPH
  - EQUIVALENCE RELATION
- UNARY STRUCTURE
- "MMSNP"
- $(\mathbb{Q},<)$
- RANDOM POSET

TRACTABILITY WITNESSED BY CANONICAL POLYMORPHISMS

$e \circ s(x,y,x,z,y,z) \approx f \circ s(y,x,z,x,z,y)$

$\Rightarrow$ CSP$(\mathbb{A})$ — NP-c / P

NEW PROOF:

- NO CASE DISTINCTIONS OVER ALL POSSIBLE CANONICAL FUNCTIONS
- EQUATIONS INSTEAD OF CONCRETE POLYMORPHISMS (AS IN FINITE)
- THEORY OF SMOOTH APPROXIMATIONS INDEPENDENT OF GROUND STRUCTURE $\mathbb{B}$
- APPLICATION NEEDS KNOWLEDGE ABOUT $\mathbb{B}$ AT TWO POINTS
- APPLICABLE TO OTHER EQUATIONAL DICHOTOMIES THAN TRIVIAL/NON-TRIVIAL

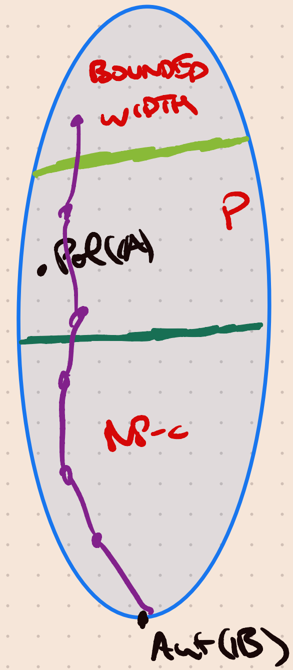
E.G. AFFINE/NON-AFFINE

THEOREM MOTTET + P. '20

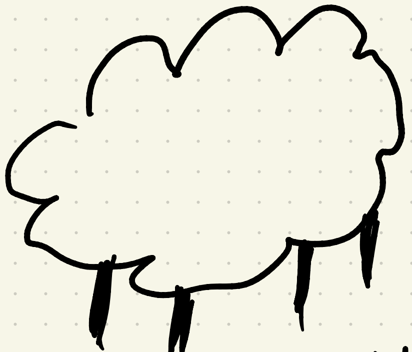
MOTTET + NAGY + P. + WRONA '21

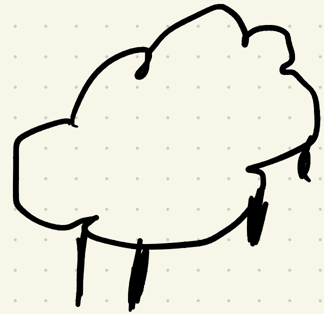
LET $\mathbb{A}$ BE FO-DEFINABLE IN:

- RANDOM TOURNAMENT
- HOMOGENEOUS GRAPH:
  - RANDOM GRAPH
  - K_n-FREE GRAPH
  - EQUIVALENCE RELATION
- UNARY STRUCTURE
- "MMSNP"



- THEN SOLVABILITY OF CSP($\mathbb{A}$) BY LOCAL CONSISTENCY METHODS IS WITNESSED BY (EQUATIONS OF) CANONICAL POLYMORPHISMS $e_n \circ w(x,\dots,x,y) \approx \dots \approx e_n \circ w(y,x,\dots,x)$
- THERE IS A COLLAPSE OF AMOUNT OF LOCALITY NEEDED TO SOLVE CSP($\mathbb{A}$): $(2k, \max(3k, \ell))$, $\mathbb{B}$ k-HOMOGENEOUS, ℓ-BOUNDED





THANK YOU!