UNIQUENESS OF POLISH TOPOLOGIES ON ENDOMORPHISM MONOIDS OF ω-CATEGORICAL STRUCTURES

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PART I : MOTIVATION & RESULTS

- ALGEBRAIC - TOPOLOGICAL STRUCTURES
- AUTOMORPHISM GROUPS, ENDOMORPHISM MONOIDS, POLYMORPHISM CLONES
- A WEE FORAY INTO CSP
- THE GRAND TABLE OF RESULTS

PART II : METHODS

- COUNTEREXAMPLES
- METHODS FOR GROUPS
- OLD METHODS FOR MONOIDS/CLONES
- PROPERTY X

MANY MATHEMATICAL OBJECTS CARRY: • ALGEBRAIC STRUCTURE • TOPOLOGICAL STRUCTURE } COMPATIBLE: ALGEBRAIC OPERATIONS CONTINUOUS.

EXAMPLES

- GROUP $(\mathbb{R},+)$ • GROUP $(\mathbb{R}^2,+)$
- VECTOR SPACES OVER $\mathbb{R}/\mathbb{C}/\mathbb{K}$: SUBSETS OF $\mathbb{R}^I / \mathbb{C}^I / \mathbb{K}^I$

  PRODUCT TOPOLOGIES

  IN PARTICULAR: $\mathbb{K}$ DISCRETE

  $\Rightarrow$ TOPOLOGY OF POINTWISE CONVERGENCE

  $(f_n)_{n\in\omega} \to f \iff \forall a \; f_n(a) = f(a)$ EVENTUALLY
- IN PARTICULAR: VECTOR SPACE $\mathbb{R}^n$
- GROUP $\mathrm{Aut}(\mathbb{Q},<) \subseteq \mathbb{Q}^{\mathbb{Q}}$: PRODUCT TOPOLOGIES
- IN GENERAL: GROUP $\mathrm{Aut}(\mathbb{A})$, $\mathbb{A}$ FIRST-ORDER STRUCTURE

  POINTWISE CONVERGENCE
- SEMIGROUPS (MONOIDS): $\mathrm{End}(\mathbb{A})$ ENDOMORPHISM MONOID

  $\mathrm{Emb}(\mathbb{A})$ EMBEDDING MONOID

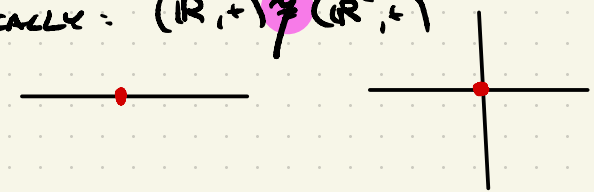
  $\mathrm{Aut}(\mathbb{A}) \subseteq \mathbb{A}^{\mathbb{A}}$
- CLONE $\mathrm{Pol}(\mathbb{A}) = \bigcup_{n\in\omega} \{f : \mathbb{A}^n \to \mathbb{A} \mid f \text{ PRESERVES } \mathbb{A}\}$

  ALGEBRAIC STRUCTURE: COMPOSITION

  $\mathrm{Pol}\,\mathbb{A} \cap \mathbb{A}^{\mathbb{A}^n}$ POINTWISE CONV., CLOPEN
- ALGEBRA $(A; (f_i)_{i\in I})$: COMPOSITION + POINTWISE CONVERGENCE.

HOW MUCH CHOICE FOR THE TOPOLOGIES?

- **UNIQUE**: VECTOR SPACE $\mathbb{R}^n$ HAS UNIQUE HAUSDORFF TOPOLOGY
- **MANY**: GROUPS $(\mathbb{R},+) \cong (\mathbb{R}^2,+)$ BUT NOT TOPOLOGICALLY: $(\mathbb{R},+) \not\cong (\mathbb{R}^2,+)$



QUESTION WHAT ARE THE COMPATIBLE TOPOLOGIES FOR A GIVEN ALGEBRAIC OBJECT?

QUESTION (MARKOV 44)

IS THERE AN INFINITE NON-TOPOLOGIZABLE GROUP?
(ONLY TOPOLOGIES = DISCRETE, TRIVIAL)

ANSWER YES (SHELAH (CM), HESSE, OLSHANSKII)

QUESTION CAN THERE BE A UNIQUE NON-TRIVIAL, NON-DISCRETE TOPOLOGY?

- VARIANTS: ASSUMPTIONS ON TOPOLOGY, E.G., HAUSDORFF, POLISH

POLISH = SEPARABLE + COMPLETELY METRIZABLE

EXAMPLE POINTWISE CONVERGENCE ON $\mathrm{Sym}(\mathbb{N}) \subseteq \mathbb{N}^{\mathbb{N}} \subseteq \bigcup_{n\in\omega} \mathbb{N}^{\mathbb{N}^n}$ PW

POLISH GROUPS

- **EXAMPLE** $(\mathbb{R},+) \cong (\mathbb{R}^2,+)$, $(\mathbb{R},+) \not\cong (\mathbb{R}^2,+)$

- <u>ON THE OTHER HAND</u>: CONSISTENT WITH ZF$\cancel{C}$ THAT ANY POLISH GROUP HAS UNIQUE POLISH TOPOLOGY (PETTIS, SOLOVAY, SHELAH)

- **EXAMPLE** FREE GROUPS HAVE NO NON-DISCRETE POLISH TOPOLOGY (DUDLEY '61)

QUESTION (ULAM, SCOTTISH BOOK #96)

DOES $\mathrm{Sym}(\mathbb{N})$ HAVE LOCALLY COMPACT POLISH TOPOLOGY?

ANSWER <u>No</u>:

- EVERY T_1 TOPOLOGY $\geq$ PW (GAUGHAN '70s)
- $\mathcal{T} \subseteq \mathcal{T}'$, BOTH POLISH $\Rightarrow$ $\mathcal{T} = \mathcal{T}'$

$\Rightarrow$ PW UNIQUE POLISH

(EVEN SEPARABLE)

(ROSENDAL + SOLECKI '07)

UNIQUE POLISH TOPOLOGY:

- ISOMETRY GROUP OF THE URYSOHN SPACE / SPHERE (SABOK '13)
- HOMEOMORPHISM GROUPS OF $[0,1]^{\mathbb{N}}$, $2^{\mathbb{N}}$ (KALLMAN '80s)
- $\mathrm{Aut}(\mathbb{Q},<)$ (HODGES, HODKINSON, LASCAR, SHELAH '93)
- $\mathrm{Aut}(G)$ G ... RANDOM GRAPH (KRUSKOVSKI '92)

POLISH SEMIGROUPS / CLONES

UNIQUE POLISH TOPOLOGY:

- $\mathbb{N}^{\mathbb{N}}$
- $\bigcup_{n \in \omega} \mathbb{N}^{\mathbb{N}^n}$

(ELLIOTT + JONUŠAS + MESYAN + MITCHELL + MORAYNE + PÉRESSE '18)

INFINITELY MANY POLISH TOPOLOGIES:

$\mathrm{Inj}(\mathbb{N}) := \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ INJECTIVE}\}$ $\left(= \overline{\mathrm{Aut}(\mathbb{N}, \neq)}^{PW}\right)$

SMALLEST: PW : GENERATED BY $U_{x,y} = \{f \in \mathbb{N}^{\mathbb{N}} \mid f(x) = y\}$ $(x, y \in \mathbb{N})$

LARGEST: PW + $N_x := \{f \in \mathbb{N}^{\mathbb{N}} \mid x \notin f(\mathbb{N})\}$ $(x \in \mathbb{N})$

$M_n := \{f \in \mathbb{N}^{\mathbb{N}} \mid |\mathbb{N} \setminus f(\mathbb{N})| = n\}$ $(n \leq \omega)$

IN BETWEEN: RESTRICT n

(NO CLASSIFICATION)

WHAT FOR?

INSIGHT INTO RELATIONSHIP BETWEEN ALGEBRAIC & TOPOLOGICAL STRUCTURE

PRACTICAL EVEN IF A TOPOLOGY IS "IRRELEVANT" (UNIQUELY DETERMINED), IT MIGHT BE SO (AND USUALLY IS) FOR NON-TRIVIAL REASONS. KNOWLEDGE OF THIS FACT IS USEFUL.

EXAMPLES LET $\mathbb{A}, \mathbb{B}$ BE COUNTABLE ω-CATEGORICAL.

- $\mathbb{A}, \mathbb{B}$ FIRST-ORDER BI-INTERPRETABLE $\Leftrightarrow \mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ (AHLBRANDT + ZIEGLER '80s)
  IF $\mathbb{A}$ HAS UNIQUE POLISH TOP. : $\Leftrightarrow \mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$
- $\mathbb{A}, \mathbb{B}$ EXISTENTIAL-POSITIVE BI-INT. $\Leftrightarrow \mathrm{End}(\mathbb{A}) \cong \mathrm{End}(\mathbb{B})$ (BODIRSKY + JUNKER '06)
- $\mathbb{A}, \mathbb{B}$ PRIMITIVE-POSITIVE BI-INT. $\Leftrightarrow \mathrm{Pol}(\mathbb{A}) \cong \mathrm{Pol}(\mathbb{B})$ (BODIRSKY + P. '11)

VARIANT AUTOMATIC CONTINUITY (AC)

$\mathrm{Aut}(\mathbb{A})$ HAS AC $:\Leftrightarrow$ $\forall \varphi : \mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$ (φ CONTINUOUS)

$\mathrm{End}(\mathbb{A})$ HAS AC $:\Leftrightarrow$ $\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$

$\mathrm{Pol}(\mathbb{A})$ HAS AC $:\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \to \bigcup_{n \in \mathbb{N}} \mathbb{N}^{\mathbb{N}^n}$

MEANING FOR ω-CATEGORICAL $\mathbb{A}$:

$\mathbb{B}$ HAS FO-INTERPRETATION IN $\mathbb{A}$ $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$ (MORE OR LESS)

EP- $\mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$

PP- $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

OK ... BUT WHO CARES ABOUT $\mathrm{Pol}(\mathbb{A})$?

MY PERSONAL STORY

$\mathbb{A}$ ... RELATIONAL STRUCTURE, FINITE SIGNATURE $R_1, \dots, R_m$

$CSP(\mathbb{A})$ = CONSTRAINT SATISFACTION PROBLEM OF $\mathbb{A}$:

<u>GIVEN</u>: $\varphi \equiv \exists x_1 \dots \exists x_n \; R_{i_1}(\text{variables}) \wedge \dots \wedge R_{i_\ell}(\text{variables})$ PP-SENTENCE

<u>Q</u>: $\mathbb{A} \models \varphi$?

- EVERY COMPUTATIONAL PROBLEM CAN BE MODELLED AS $CSP(\mathbb{A})$ FOR SOME $\mathbb{A}$.
- EXAMPLES
  - $CSP(\mathbb{Q}, <)$: ACYCLICITY (IN P)
  - $CSP(\mathbb{Q}, Betw(x,y,z))$: BETWEENNESS (NP-c)
- WANT ALGEBRAIC TOOLS TO DISTINGUISH COMPLEXITY.
- $\mathbb{B}$ HAS PP-INTERPRETATION IN $\mathbb{A}$

  $(\Leftrightarrow Pol(\mathbb{A}) \to Pol(\mathbb{B}))$

  $\Rightarrow CSP(\mathbb{B})$ REDUCES TO $CSP(\mathbb{A})$

- IMPORTANT: CAN $\mathbb{A}$ PP-INTERPRET

  $\mathbb{B} = SAT := (\{0,1\}, \{(0,0,1), (0,1,0), (1,0,0)\})$?

  $Pol(SAT) = \{\text{PROJECTIONS ON } \{0,1\}\}$

- NEED ALGEBRAIC CONSEQUENCES OF

  $Pol(\mathbb{A}) \not\to Pol(SAT)$

- THEY EXIST!

  $Pol(\mathbb{A}) \ni e \circ s(x,y,x,z,y,z) = f \circ s(y,x,z,x,z,y)$

(BARTO + P. '16, NOT TRUE AS STATED)

RECONSTRUCTING THE TOPOLOGY OF $\mathrm{Aut}(\mathbb{A})$, $\mathrm{End}(\mathbb{A})$, $\mathrm{Pol}(\mathbb{A})$

$\mathbb{A}$ ... COUNTABLE ω-CATEGORICAL

CONSIDER THREE NOTIONS:

UP: PW IS UNIQUE POLISH TOPOLOGY ON $\mathrm{Aut}(\mathbb{A})$ / $\mathrm{End}(\mathbb{A})$ / $\mathrm{Pol}(\mathbb{A})$

AC: EVERY φ: $\mathrm{Aut}(\mathbb{A}) \to \mathrm{Sym}(\mathbb{N})$
$\mathrm{End}(\mathbb{A}) \to \mathbb{N}^{\mathbb{N}}$
$\mathrm{Pol}(\mathbb{A}) \to \bigcup_{n \in \omega} \mathbb{N}^{\mathbb{N}^n}$
IS CONTINUOUS

(AUTOMATIC CONTINUITY)

AH: EVERY ISOMORPHISM φ: $\mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$
$\mathrm{End}(\mathbb{A}) \to \mathrm{End}(\mathbb{B})$
$\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$
IS A HOMEOMORPHISM

(AUTOMATIC HOMEOMORPHICITY)

AR: EVERY ISOMORPHISM $\uparrow$ IS INDUCED BY BIJECTION $f: \mathbb{A} \to \mathbb{B}$

(ACTION RECONSTRUCTION)

$$\begin{tikzcd} & & \text{AC} \arrow[dl] \\ \text{UP} \arrow[r] & \text{AH} & \\ & & \text{AR} \arrow[ul] \end{tikzcd}$$

FOR GROUPS (OPEN FOR MONOIDS)

AR USUALLY RESTRICTED TO $\mathbb{B}$ SATISFYING CONDITIONS: NO ALGEBRAIC

THE GRAND TABLE OF RESULTS

(INCOMPLETE + PROBABLY INCORRECT)

	UNIQUE POLISH (UP)	AUTOMATIC CONT. (AC)	ACTION REC. (AR)	AUTOMATIC HOMEO (AH)
Aut	$(\mathbb{N},=)$ Gaughan '70s $(\mathbb{Q},<)$ Rosendal + Solecki '07 $\mathcal{G}$ Kechris + Rosendal '07	$(\mathbb{N},=)$ Dixon, Neumann, Thomas '86, Simmons '81 $(\mathbb{Q},<)$, $\mathbb{B}$ Truss '89 $\mathcal{G}$, ω-stable Hodges + Hodkinson + Lascar + Shelah '80s $\mathbb{H}_n$ Henig '80s	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94 Hypergraphs, Henson digraphs Bastida Zuniga + Macpherson '07	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94
End	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20		$(\mathbb{N},+)$, $\mathcal{G}$, $\mathbb{T}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{Q},<)$, $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-Garcia '17
Pol	$\mathcal{G}$, $\mathbb{D}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ Elliott + Jonušas + Meysen + Mitchell + Morayne + Péresse '20	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-Garcia '19	$\geq \mathbb{N}^{\mathbb{N}}$, $\mathcal{G}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{P},\leq)$, $(\mathbb{P},<)$ Peech + Peech '18 $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-Garcia '17 $(\mathbb{Q},<)$ Behrisch + Vargas-Garcia '19

$\mathcal{G}$ ... RANDOM GRAPH
$\mathbb{P}$ ... RANDOM POSET
$\mathbb{T}$ ... RANDOM TOURNAMENT
$\mathbb{D}$ ... RANDOM DIGRAPH
$\mathbb{E}$ ... RANDOM EQUIVALENCE RELATION
$\mathbb{H}_n$ ... RANDOM K_n-FREE GRAPH
$\mathbb{B}$ ... COUNTABLE ATOMLESS BOOLEAN ALG

ω-CAT
NO ALGEBRAICITY

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COUNTEREXAMPLES

- $\neg$UP : $\mathrm{Inj}(\mathbb{N}) = \mathrm{End}(\mathbb{N}, \neq)$: INFINITELY MANY POLISH TOPOLOGIES (BUT AH)

- $\neg$AH : $\exists \mathbb{A}, \mathbb{B}$ ω-CATEGORICAL, $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$, $\mathrm{Aut}(\mathbb{A}) \not\cong \mathrm{Aut}(\mathbb{B})$ (EVANS + HEWITT '90)

  $\exists \mathbb{A}, \mathbb{B}$ — " — $\mathrm{End}(\mathbb{A}) \cong \mathrm{End}(\mathbb{B})$, $\mathrm{End}(\mathbb{A}) \not\cong \mathrm{End}(\mathbb{B})$

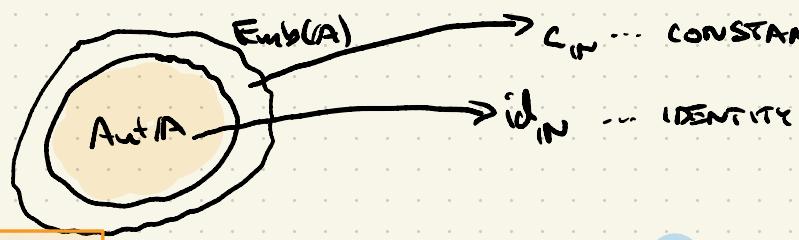
  $\mathrm{Pol}(\mathbb{A}) \cong \mathrm{Pol}(\mathbb{B})$, $\mathrm{Pol}(\mathbb{A}) \not\cong \mathrm{Pol}(\mathbb{B})$ (BODIRSKY + EVANS + KOMPATSCHER + P. '15)

NOT NATURAL!

- $\neg$AC : **OBSERVATION** (BODIRSKY + P. + PONGRACZ '13)

$\mathbb{A}$ ω-CATEGORICAL $\Rightarrow$ $\mathrm{Emb}(\mathbb{A})$ DOES NOT HAVE AUTOMATIC CONTINUITY.

PROOF:



COROLLARY $\mathrm{End}(\mathbb{Q}, <)$ DOES NOT HAVE AC. SIMILARLY: $\mathrm{End}(\mathbb{Q}, \leq)$

METHODS FOR GROUPS

- $\mathrm{Aut}(\mathbb{A})$ HAS SIP (SMALL INDEX PROPERTY)

  $:\Leftrightarrow$ EVERY SUBGROUP OF COUNTABLE INDEX IS OPEN

  $\Leftrightarrow$ EVERY SUBGROUP OF COUNTABLE INDEX CONTAINS A POINTWISE STABILIZER

  $\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A})$ HAS AC (DIXON, NEUMANN, THOMAS)

- $\mathrm{Aut}(\mathbb{A})$ HAS SSIP (STRONG SIP)

  $:\Leftrightarrow$ EVERY SUBGROUP OF COUNTABLE INDEX IS BETWEEN A POINTWISE & A SETWISE STABILIZER

  $\Rightarrow$ AR (ACTION RECONSTRUCTION) WITHIN THE CLASS OF ω-CAT., NO ALGEBRAICITY, SSIP STRUCTURES (PAOLINI + SHELAH)

- $\forall\exists$-INTERPRETATIONS:

  "INTERPRETATION" OF $\mathbb{A}$ IN $\mathrm{Aut}(\mathbb{A})$

  $\Rightarrow$ AH (AUTOMATIC HOMEOMORPHICITY)

  AR (ACTION RECONSTRUCTION)

  (RUBIN)

- $\mathrm{Aut}(\mathbb{A})$ HAS AMPLE GENERICS $:\Leftrightarrow$ $\mathrm{Aut}(\mathbb{A}) \curvearrowright \mathrm{Aut}(\mathbb{A})^n$ HAS COMEAGRE ORBIT.

  ↑ conjugation

  $\Rightarrow$ UP UNIQUE POLISH

  AC AUTOMATIC CONTINUITY

  (KECHRIS + ROSENDAL)

OLD METHODS FOR MONOIDS / CLONES

LAZY IDEA: LIFT AH FROM $\mathrm{Aut}(\mathbb{A}) \rightsquigarrow \overline{\mathrm{Aut}(\mathbb{A})} \rightsquigarrow \mathrm{End}(\mathbb{A}) \rightsquigarrow \mathrm{Pol}(\mathbb{A})$

METHOD: GATES (BODIRSKY + P. + PONGRACZ)

ROUGHLY: SAY WE KNOW $\mathrm{End}(\mathbb{A})$ HAS AH

FIND $f \in \mathrm{Pol}(\mathbb{A})$ (OR A FAMILY OF fs "GATE COVERING") SUCH THAT

$\forall\, (g_n)_{n\in\omega} \to g \in \mathrm{Pol}(\mathbb{A})$

$\exists\, (\alpha_n)_{n\in\omega}, (\overline{\beta_n})_{n\in\omega} \in \mathrm{End}(\mathbb{A})$

$\downarrow \alpha \qquad \downarrow \beta$

$\forall n: \quad g_n = \alpha_n f \overline{\beta_n}$

IF $\varphi: \mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

$\to \varphi(g_n) = \varphi(\alpha_n)\varphi(f)\varphi(\overline{\beta_n}) \to \varphi(\alpha)\varphi(f)\varphi(\beta) = \varphi(\alpha f \beta)$

$= \varphi(g)$

NEW METHOD PROPERTY X

DEFINITION

LET S TOPOLOGICAL SEMIGROUP, $A \subseteq S$ SUBSEMIGROUP.

S HAS PROPERTY X WITH RESPECT TO A $:\Leftrightarrow$

$\forall s \in S \ \exists f_s, g_s \in S \ \exists \alpha_s \in A$:

$$s = g_s \circ \alpha_s \circ f_s$$

AND

$\forall B$ NEIGHBOURHOOD OF α_s :

$g_s \circ (B \cap A) \circ f_s$ IS NEIGHBOURHOOD OF s.

$$s = g_s \circ \alpha_s \circ f_s$$

THEOREM (ELLIOTT + JONUSAS + MEKSAN + MITCHELL MORAYNE + PÉRESSE '19)

LET S SEMIGROUP, POLISH TOPOLOGY $\mathcal{J}$ ON S.

$A \leq S$ SUBSEMIGROUP, PROPERTY X.

- IF A IS POLISH SUBGROUP $\Rightarrow$ $\mathcal{J}$ IS MAXIMAL POLISH TOPOLOGY ON S.
- A HAS AC $\Rightarrow$ S HAS AC.

COROLLARY

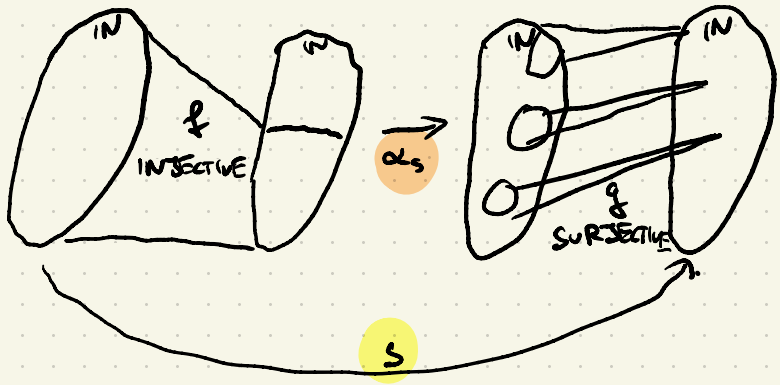
$\mathrm{End}(\mathbb{A})$ HAS PROPERTY X WRT $\mathrm{Aut}(\mathbb{A})$ $\Rightarrow$ PW MAXIMAL POLISH TOPOLOGY.

REMARK: OFTEN EASY TO SHOW THAT PW IS SMALLEST POLISH TOPOLOGY.

EXAMPLE $\mathbb{A} = (\mathbb{N}, =)$: $\mathrm{End}(\mathbb{A}) = \mathbb{N}^{\mathbb{N}}$, $\mathrm{Aut}(\mathbb{A}) = \mathrm{Sym}(\mathbb{N})$

PROPERTY X :

$s \in \mathbb{N}^{\mathbb{N}}$



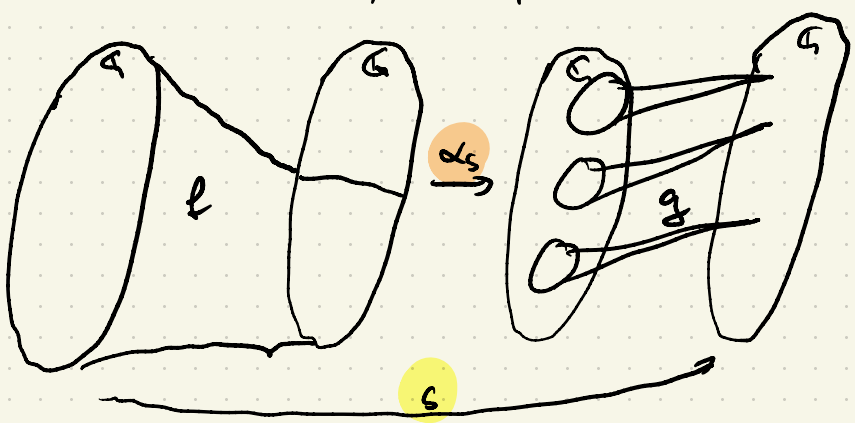
$s = g \circ \alpha_s \circ f$

SURJECTIVE INFINITE CLASSES

INJECTIVE CO-INFINITE RANGE

$\Rightarrow$ PW MAXIMAL POLISH TOP. ON $\mathbb{N}^{\mathbb{N}}$

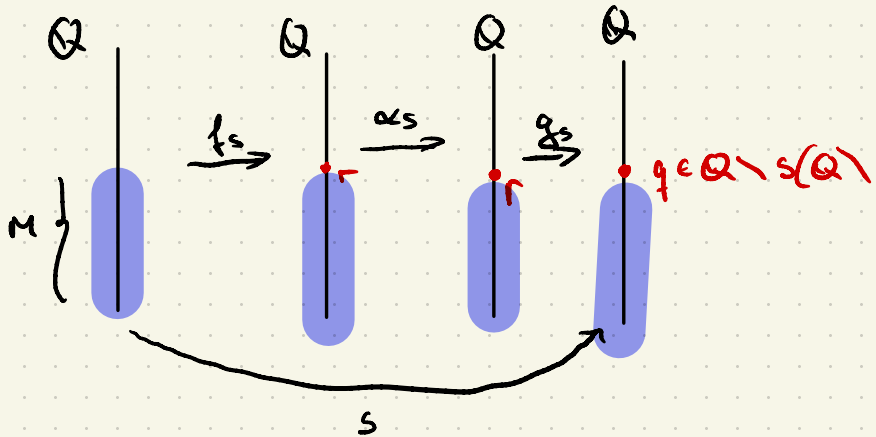
EXAMPLE $\mathbb{A} = G$ ... RANDOM GRAPH, $s \in \mathrm{End}\, G$



- g SURJECTIVE, "GENERIC"
- f EMBEDDING, GENERIC CO-IMAGE, PARAMETERS OF CO-IMAGE DO NOT DEFINE NEW TYPES IN THE IMAGE.
- α_s : BACK-AND-FORTH

NON-EXAMPLE

$\mathbb{A} = (Q, \leq)$



- $q \notin s(Q)$
- g_s HAS TO BE SURJECTIVE

$\Rightarrow \exists\, p \quad g_s(p) = q$

$\alpha_s(r) = p$

- ONCE $\alpha_s(r) = p$ IS FIXED, $g_s \circ \tilde{\alpha}_s \circ f_s$ MUST SEND M BELOW q

- NO NEIGHBOURHOOD OF s IS THAT RESTRICTIVE!

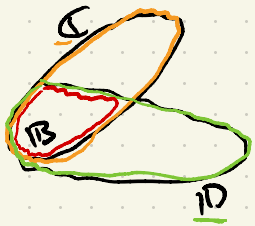
SUFFICIENT CONDITIONS FOR PROPERTY X

- $\mathbb{A}$ HOMOGENEOUS $:\Leftrightarrow$ EVERY PARTIAL FINITE ISOMORPHISM ON $\mathbb{A}$ EXTENDS TO AN AUTOMORPHISM.

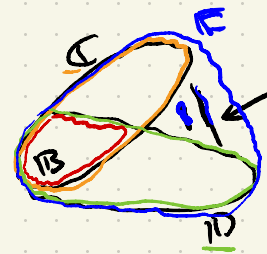
- $\mathbb{A}$ HOMOMORPHISM-HOMOGENEOUS $:\Leftrightarrow$ .... HOMOMORPHISM --- ENDOMORPHISM.

- $\mathbb{A}$ HAS STRONG AMALGAMATION PROPERTY $:\Leftrightarrow$

$\forall \mathbb{B}, \mathbb{C}, \mathbb{D}$ FINITE IN $\mathbb{A}$



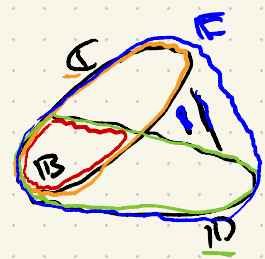
$\exists \mathbb{E}$ IN $\mathbb{A}$:



HERE CAN BE RELATIONS (BUT NO =)

SAP WITH HOMOMORPHISM GLUING:

SAP :



$\oplus$ WHENEVER $g : \mathbb{E} \to \mathbb{F}$

$g|_{\mathbb{C}}$ HOMO, $g|_{\mathbb{D}}$ HOMO

$\Rightarrow g$ HOMO

EXAMPLES

- FREE AMALGAMATION $\Rightarrow$ SAP + HG
- IN PARTICULAR: RANDOM GRAPH, DIGRAPH, ... HAVE SAP + HG
- RANDOM POSET $(\mathbb{P}, \leq)$, $(\mathbb{P}, <)$ HAVE SAP + HG
- EQUIVALENCE REL. $\mathbb{E}$ HAS SAP + HG
- $(\mathbb{Q}, \leq)$, $(\mathbb{Q}, <)$, $\mathbb{T}$ RANDOM TOURNAMENT: NO HG!

THEOREM (ELLIOTT + JONUŠAS + MITCHELL + PÉRESSE + P. '21)

LET $\mathbb{A}$ BE HOMOGENEOUS, HOMOMORPHISM-HOMOGENEOUS, SAP + HG.

$\Rightarrow$ $\mathrm{End}(\mathbb{A})$ HAS PROPERTY X WRT $\mathrm{Aut}(\mathbb{A})$,

PW IS THE MAXIMAL POLISH TOPOLOGY.

NON-UNIQUENES

PROPOSITION (ELLIOTT + JONUŠAS + MITCHELL + PÉRESSE + P. '21)

$\mathbb{A}$ ω-CATEGORICAL $\Rightarrow$ $\mathrm{Emb}(\mathbb{A})$ HAS AT LEAST TWO POLISH TOPOLOGIES.

PROOF

- $\mathrm{Inj}(\mathbb{N})$ HAS ≥ 2 POLISH TOP., PW AND ONE WHERE $\mathrm{Sym}(\mathbb{N})$ IS CLOSED
- INDUCE POLISH TOP. ON $\mathrm{Emb}(\mathbb{A})$ SINCE IT IS CLOSED.
- $\mathrm{Aut}(\mathbb{A})$ NOT CLOSED IN PW BUT IN THE OTHER ONE. $\square$

COROLLARY

$\mathrm{End}(\mathbb{Q},<)$ HAS ≥ 2 POLISH TOPOLOGIES.

WHAT ABOUT CLONES???

THEOREM (ELLIOTT + JONUŠAS + MITCHELL + PÉRÉBSE + P. '21)

LET $\mathbb{A}$ BE HOMOGENEOUS, HOMOMORPHISM-HOMOGENEOUS, SAP + HG.

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ HAS PROPERTY $\times$ WRT $\mathrm{Aut}(\mathbb{A})$,

PW IS THE MAXIMAL POLISH TOPOLOGY.

$\oplus$ CLOSED UNDER FINITE PRODUCTS

OPEN PROBLEMS

- DOES $\mathrm{End}(\mathbb{Q}, \leq)$ HAVE UNIQUE POLISH TOPOLOGY?
- DO $\mathrm{Aut}(\mathbb{P}, <)$, $\mathrm{End}(\mathbb{P}, <)$, $\mathrm{End}(\mathbb{P}, \leq)$ HAVE AUTOMATIC CONTINUITY?

Thank you!

THE GRAND TABLE OF RESULTS

(INCOMPLETE + PROBABLY INCORRECT)

	UNIQUE POLISH (UP)	AUTOMATIC CONT. (AC)	ACTION REC. (AR)	AUTOMATIC HOMEO (AH)
Aut	$(\mathbb{N},=)$ Gaughan '70s $(\mathbb{Q},<)$ Rosendal + Solecki '07 $\mathcal{G}$ Kechris + Rosendal '07	$(\mathbb{N},=)$ Dixon, Neumann, Thomas '86 $(\mathbb{Q},<)$, $\mathbb{B}$ Truss '89 $\mathcal{G}$, ω-stable Hodges + Hodkinson + Lascar + Shelah '80s $\mathbb{H}_n$ Henig '80s	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94 Hypergraphs, Henson digraphs Bastida Zuniga + Macpherson '07	$(\mathbb{Q},<)$, $\mathcal{G}$, $\mathbb{H}_n$, $\mathbb{P}$, $\mathbb{T}$ Rubin '94
End	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20	$\mathcal{G}$, $\mathbb{D}$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ EJMMMP '20		$(\mathbb{N},+)$, $\mathcal{G}$, $\mathbb{T}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{Q},<)$, $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-Garcia '17
Pol	$\mathcal{G}$, $\mathbb{D}$, $(\mathbb{P},<)$, $(\mathbb{P},\leq)$, $\mathbb{E}$ Elliott + Jonušas + Mitchell + Péresse + P. '21 $(\mathbb{N},=)$ Elliott + Jonušas + Meysen + Mitchell + Morayne + Péresse '20	$(\mathbb{N},=)$ EJMMMP '20	$(\mathbb{Q},<)$, $(\mathbb{P},<)$, $\mathcal{G}$, $\mathbb{H}_n$ Behrisch + Vargas-Garcia '19	$\geq \mathbb{N}^{\mathbb{N}}$, $\mathcal{G}$ Bodirsky + P. + Pongrácz '13 $(\mathbb{P},\leq)$, $(\mathbb{P},<)$ Peech + Peech '18 $(\mathbb{Q},\leq)$ Behrisch + Truss + Vargas-Garcia '17 $(\mathbb{Q},<)$ Behrisch + Vargas-Garcia '19

$\mathcal{G}$ ... RANDOM GRAPH
$\mathbb{P}$ ... RANDOM POSET
$\mathbb{T}$ ... RANDOM TOURNAMENT
$\mathbb{D}$ ... RANDOM DIGRAPH
$\mathbb{E}$ ... RANDOM EQUIVALENCE RELATION
$\mathbb{H}_n$ ... RANDOM K_n-FREE GRAPH
$\mathbb{B}$ ... COUNTABLE ATOMLESS BOOLEAN ALG

ω-CAT
NO ALGEBRAICITY