SMOOTH APPROXIMATIONS IN OLIGOMORPHIC CLONES

OR:

THE THREE DILEMMAS OF THE INFINITE SHEEP

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(JOINT WORK WITH ANTOINE MOTTET)

AAA 100 / KRAKÓW
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I OLIGOMORPHIC CLONES &
THE FIRST DILEMMA

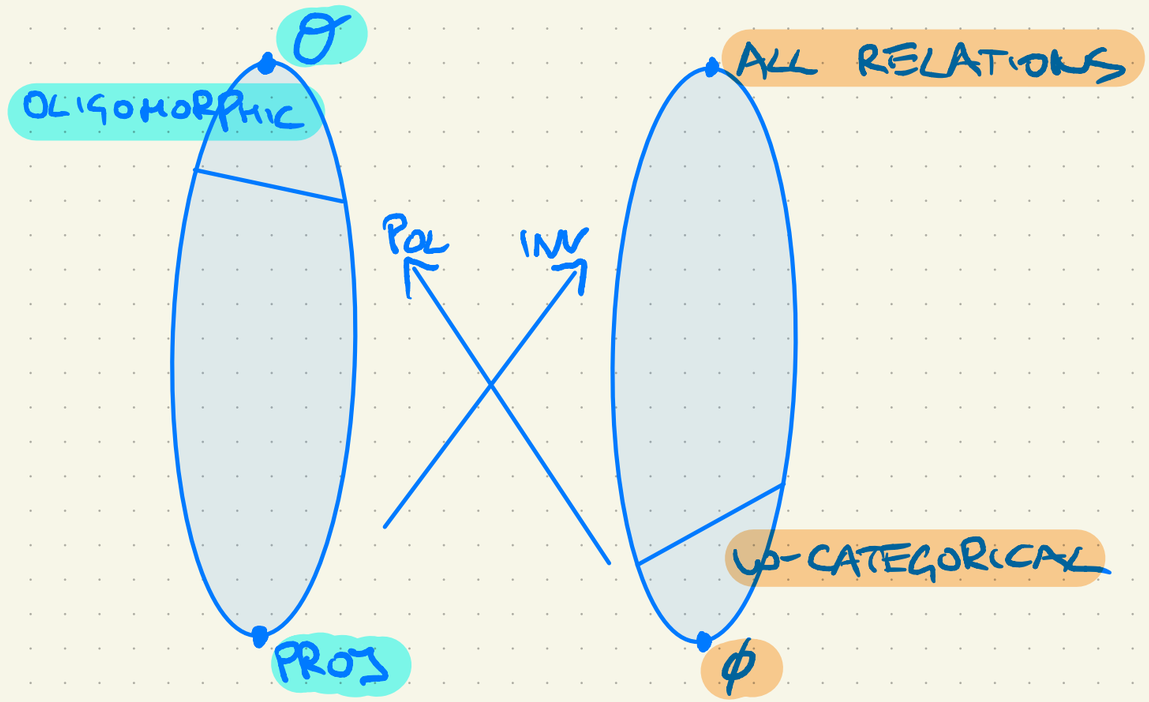
(BABY THEORY)

II CANONICAL FUNCTIONS &
THE THIRD AND SECOND DILEMMA

(ADOLESENT THEORY)

III SMOOTH APPROXIMATIONS

(ADULT THEORY)



CLONES ON IN

RELATIONAL STRUCTURES ON IN

$\mathcal{C}$ OLIGOMORPHIC :⇔

UNARY INVERTIBLES IN $\mathcal{C}$ ARE OLIGOMORPHIC PERMUTATION GROUP:

$\forall n \quad A^n/_{\text{INVERTIBLES}}$ FINITE

$\text{Pol}(\mathbb{A}) = \{f \mid f \text{ PRESERVES } \mathbb{A}\}$ POLYMORPHISMS

$\mathbb{A}$ ω-CATEGORICAL :⇔

$\text{Aut}(\mathbb{A})$ OLIGOMORPHIC:

$\forall n \quad A^n/_{\text{Aut}(\mathbb{A})}$ FINITE

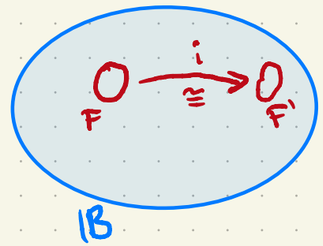
$\text{Inv}(\mathcal{C}) =$

$\{R \mid R \text{ INVARIANT UNDER } \mathcal{C}\}$

EXAMPLES

$\mathbb{A}$ FIRST-ORDER REDUCT OF:

- $(\mathbb{Q}, <)$
- RANDOM GRAPH
- FINITELY HOMOGENEOUS $\mathbb{B}$:



$\Rightarrow \exists \alpha \in \text{Aut}(\mathbb{B})$
$\alpha|_F = i$

... AND THEIR FIRST-ORDER REDUCTS

(CLOSED) CLONES CONTAINING

$\text{Aut}(\mathbb{Q}, <)$

$\text{Aut}(\mathbb{B})$ ↑ FINITELY HOMOGENEOUS

⋮

WHY OLIGOMORPHIC?

- BASIC THEORY OF UNIVERSAL ALGEBRA WORKS
- ω-CATEGORICAL STRUCTURES INTERESTING

- R pp-DEFINABLE IN $\mathbb{A}$ $\Leftrightarrow$ $\mathrm{Rel}_n(\mathrm{Pol}\,\mathbb{A})$ (BODIRSKY + NEŠETŘIL '03)
- IF FINITE:

$\mathrm{Pol}(\mathbb{A}) \longrightarrow \mathrm{Pol}(\mathbb{F})$ $\Leftrightarrow$ $\mathbb{F}$ PP-INTERPRETABLE IN $\mathbb{A}$ (BODIRSKY + P. '11)

$\mathrm{Pol}(\mathbb{A}) \dashrightarrow \mathrm{Pol}(\mathbb{F})$ $\Leftrightarrow$ $\mathbb{F}$ PP-CONSTRUCTIBLE IN $\mathbb{A}$ (BARTO + OPRŠAL + P. '15)

$\longrightarrow$ CLONE HOMOMORHISM: PRESERVES IDENTITIES/EQUATIONS

$\dashrightarrow$ H1 CLONE HOM: ONLY H1 IDENTITIES × NON-NESTED

UNIFORM CONTINUITY (POINTWISE CONVERGENCE)

WANTED:

ALGEBRAIC THEORY

GOALS

E.G.

NON-TRIVIALITY:

$Pol(\mathbb{A}) \not\to$ PROJ

NON-AFFINENESS:

$Pol(\mathbb{A}) \not\to$ + ... AFFINE MAPS / FINITE MODULE

WANT POSITIVE PROPERTIES: IDENTITIES?

TELL ME WHY!

$Pol(\mathbb{A})$ NOT IDEMPOTENT!

$(Pol(\mathbb{A}) \geq Aut(\mathbb{A}) \dots)$

FIRST DILEMMA OF THE INFINITE SHEEP

IDEMPOTENCY: CAN'T LIVE WITH IT
CAN'T LIVE WITHOUT IT

DEFINITION (IDEMPOTENCY "MODULO $Aut(\mathbb{A})$")

$\mathbb{A}$ MODEL-COMPLETE CORE: $\hookrightarrow$

$\overline{Aut}\mathbb{A} = Pol^{(n)}(\mathbb{A})$
$(= End(\mathbb{A}))$

THEOREM (BODIRSKY '08)

$\mathbb{A}$ ω-CATEGORICAL $\Rightarrow$

- $\exists \mathbb{A}'$ MC CORE, $\mathbb{A} \xrightarrow{hom} \mathbb{A}'$, $\mathbb{A}' \xrightarrow{hom} \mathbb{A}$
- $\mathbb{A}'$ ω-CATEGORICAL, UNIQUE

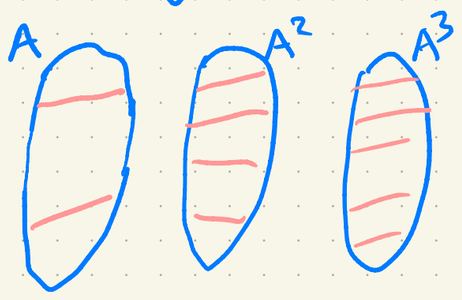
II CANONICAL FUNCTIONS &

THE THIRD AND SECOND DILEMMA

- LIVE INFINITE, ACT FINITE
- THEORY OF FINITE ALGEBRAS APPLIES
- ARE NOT EVERYTHING

GENERAL PARADIGM:

$\mathbb{A}$ „MODULO $Aut(\mathbb{A})$" $\rightsquigarrow$ $Pol(\mathbb{A})$ „MODULO $Aut(\mathbb{A})$"



ORBIT-EQUIVALENCE

BUT ... NOT A CONGRUENCE / INVARIANT UNDER $Pol(\mathbb{A})$!

DEFINITION

$$Pol(\mathbb{A})^{CAN}_{\mathbb{A}} := Pol(\mathbb{A}) \cap Pol(\text{ORBIT-EQUIVALENCES})$$

↑ ONE FOR EACH $n \geq 1$

CANONICAL POLYMORPHISMS WRT $Aut(\mathbb{A})$

- $\mathbb{A}$ MC CORE $\rightarrow$ $Pol(\mathbb{A})^{CAN}_{\mathbb{A}} \curvearrowright A^n / Aut\mathbb{A}$ (n-ORBITS) IDEMPOTENT

THIRD DILEMMA OF THE INFINITE SHEEP

ORBIT EQUIVALENCE: CAN'T LIVE WITH IT
CAN'T LIVE WITHOUT IT

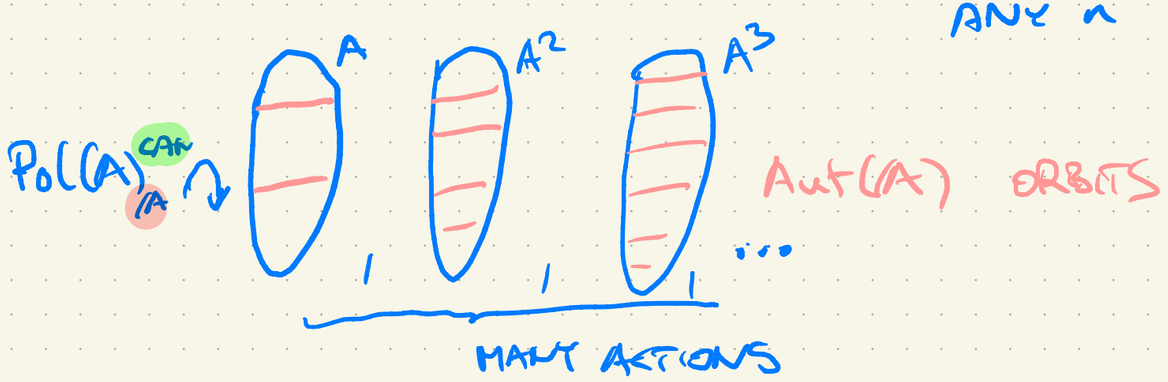
- THEORY OF SMOOTH APPROXIMATIONS

COMPARES

$$Pol(\mathbb{A})^{CAN}_{\mathbb{A}} \subseteq Pol(\mathbb{A})$$

PROBLEMS

- $\mathbb{A}$ NOT FINITELY HOMOGENEOUS
  $\Rightarrow$ n-ORBITS NOT SUFFICIENT FOR ANY n

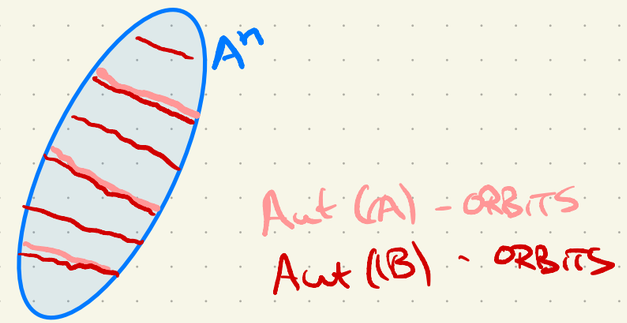


- $\mathbb{A}$ NO PARTICULAR COMBINATORIAL PROPERTIES THAT ALLOW TO **INTERPOLATE** SOME
  $g \in Pol(\mathbb{A})^{CAN}_{\mathbb{A}}$ FROM GIVEN $f \in Pol(\mathbb{A})$

IDEA

- TAKE FINITELY HOMOGENEOUS $\mathbb{B}$ WITH FINER ORBITS + RAMSEY PROPERTY
- CANONISE WITH $Aut(\mathbb{B})$ - ORBITS

$Pol(\mathbb{A})^{CAN}_{\mathbb{B}}$



$Pol(\mathbb{A}) \supseteq Aut(\mathbb{A}) \supseteq Aut(\mathbb{B})$
$\mathbb{A}$ FIRST-ORDER REDUCT OF $\mathbb{B}$.

LEMMA (BODIRSKY + P. + TSANKOV '13)
$\mathbb{B}$ HOMOGENEOUS RAMSEY STRUCTURE
$\Rightarrow \forall f \in Pol(\mathbb{A})\ \exists g \in Pol(\mathbb{A})^{CAN}_{\mathbb{B}}$ $g \in \overline{Aut\mathbb{B} \circ f \circ Aut\mathbb{B}}$

$\mathrm{Pol}(\mathbb{A})$

$\cup$ $\supseteq$

ACTS ON $\mathrm{Aut}(\mathbb{A})$-ORBITS → $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{A}}$ $\quad$ $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{B}}$ ← ACTS ON $\mathrm{Aut}(\mathbb{B})$-ORBITS

HOW GOOD IS IT?

$\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{B}} \not\to$ $\mathcal{C}$ FINITE

$\Rightarrow \mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{B}} \curvearrowright$ ORBITS $\not\to \mathcal{C}$

FINITE ALGEBRA $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{B}} \curvearrowright$ ORBITS $\models \Sigma$

FINITELY HOM. $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{B}} \models \Sigma$ MODULO $\overline{\mathrm{Aut}\,\mathbb{B}}$

(BODIRSKY + P. + PONGRACZ '16)

EXAMPLE

$\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{B}} \not\to$ PROJ

$\Rightarrow \mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{B}} \models u \circ f(x_1 \dots x_n) \approx v \circ f(x_2 \dots x_n x_1)$

(PSEUDO-CYCLIC)

WITHOUT CANONICITY:

FACES

2

"TRUE" (BARTO + P. '16)

$\models u_1 f(xx \dots y) \approx \dots \approx u_n f(yx \dots x)$

(PSEUDO-WNU)

$\models u f(xyxzyz) \approx v f(yxzxzy)$

(PSEUDO-SIGGERS)

- SIMILAR WITH $\mathbb{A}$ ... AFFINE MAPS / MODULE
- $\mathbb{B}$ FINITELY BOUNDED

$\Rightarrow$ SAME CONSEQUENCES FOR $\mathrm{CSP}(\mathbb{A})$ AS IN FINITE

(BODIRSKY + MOTTET '17)

→ DEFINED BY FINITE OBSTRUCTION SET

SOUNDS GREAT?

SECOND DILEMMA OF THE INFINITE SHEEP

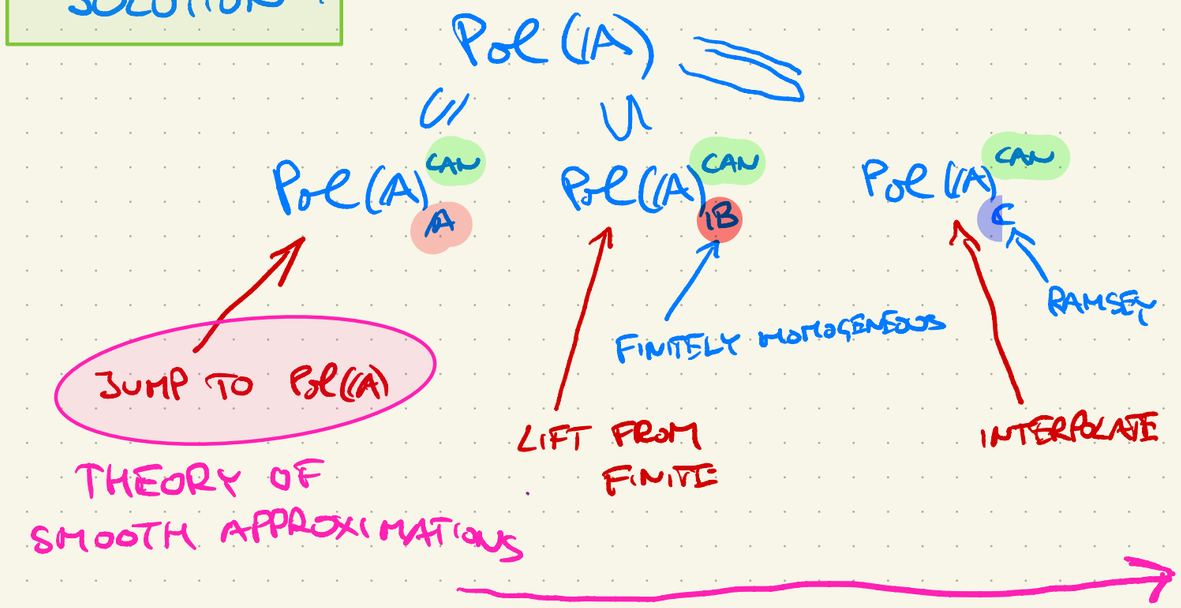
RAMSEY: CAN'T LIVE WITH IT
CAN'T LIVE WITHOUT IT

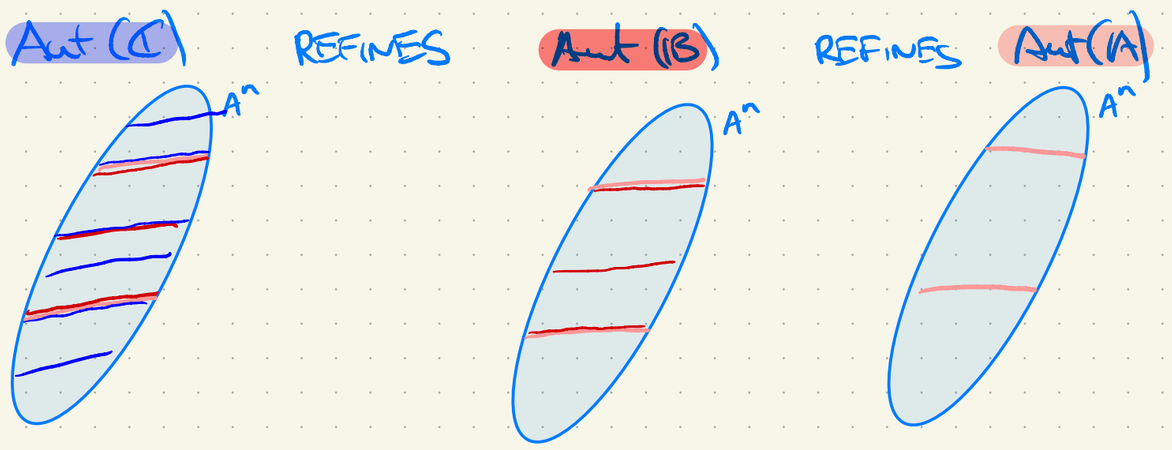
INTERESTING SITUATION: $Pol(\mathbb{A}) \not\to \mathcal{C}$

DOES NOT IMPLY $Pol(\mathbb{A})^{CAN\ \mathbb{B}} \not\to \mathcal{C}$

IN FACT THIS "NEVER" HAPPENS IF $\mathbb{B}$ RAMSEY

"SOLUTION":





III SMOOTH APPROXIMATIONS

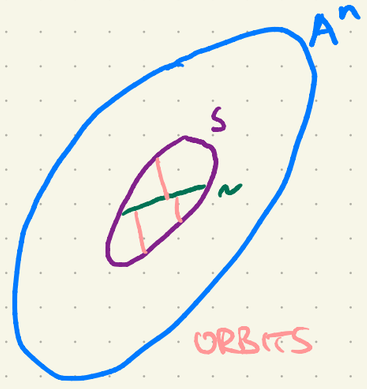
- CLOSE THE GAP CANONICAL / REAL
- GIVE US STRUCTURAL DICHOTOMIES

THEOREM (MOTTET + P. '20)

- $\mathbb{A}$ ω-CATEGORICAL MODEL-COMPLETE CORE
- $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{A}} \curvearrowright A^n / \mathrm{Aut}\mathbb{A}$ $\rightarrow$ $\mathcal{C}$:

$\mathrm{Aut}\mathbb{A}$-ORBITS

$\exists S$
$\exists \sim$



$\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{A}} \cap S/\sim \subseteq \mathcal{C}$

$\Rightarrow$ "LOOP LEMMA OF APPROXIMATIONS"
2 CASES

OR

$\exists \eta$ APPROXIMATION (= "REFINEMENT")
OF $\sim$
$\mathrm{Pol}(\mathbb{A})$-INVARIANT

QUALITY OF η :

SMOOTH

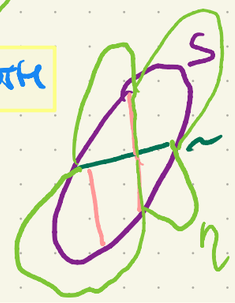


FUNDAMENTAL THEOREM

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A}) \dashrightarrow \mathcal{C}$

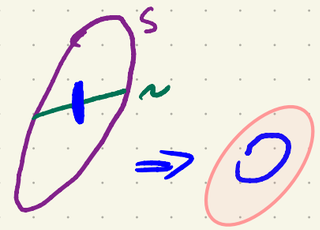
(IF $\mathrm{Pol}(\mathbb{A})$ INTERPOLATES $\mathrm{Pol}(\mathbb{A})^{\mathrm{FAP}}_{\mathbb{A}}$)

VERY SMOOTH



$\Rightarrow$ $\mathrm{Pol}(\mathbb{A}) \rightarrow \mathcal{C}$

CERTAIN RELATIONS
HAVE A PSEUDOLOOP
MODULO $\mathrm{Aut}(\mathbb{A})$



LEMMA

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ CONTAINS
"WEAKLY
SYMMETRIC"
FUNCTIONS

"PSEUDO-COMMUTATIVE"
"PSEUDO-CYCLIC" OF
ALL ARITIES

THEOREM (MOTTET + P. '20)

$\mathbb{T}$ ... UNIVERSAL HOMOGENEOUS TOURNAMENT

$\mathbb{A}$ ... FIRST-ORDER REDUCT OF $\mathbb{T}$
($\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \supseteq \mathrm{Aut}(\mathbb{T})$)

MODEL-COMPLETE CORE

TFAE:

- $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{PROJ}$
- $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{T}}$ HAS PSEUDO-CYCLIC FUNCTION
($\Rightarrow \mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$)

SAME PROOF FOR

- RANDOM GRAPH — OLD PROOF: BODIRSKY + P. '11
- UNIVERSAL HOMOGENEOUS K_n-FREE GRAPH — BODIRSKY + MARTIN + P. PONGRÁCZ '16
- RANDOM POSET — BODIRSKY + MOTTET '17
- UNARY STRUCTURES (WITH $\neq$) — KOMPATSCHER + PHAM '17

CAN ALSO REPROVE CSP DICHOTOMY FOR $(\mathbb{Q}, <)$

"PROOF":

- $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{T}} \not\to \mathrm{PROJ} \Rightarrow$ ✓ (2) HOLDS
- $\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{T}} \to \mathrm{PROJ}$

$\Rightarrow \mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathrm{Aut}} \to \mathrm{PROJ}$ (SIGGERS, POST, COMPACTNESS ...)

SMOOTH APPROX.

$\mathrm{Pol}(\mathbb{A}) \to \mathrm{PROJ}$

$\mathrm{Pol}(\mathbb{A})$ HAS WEAKLY SYMMETRIC FUNCTIONS

$\Downarrow$

$\mathrm{Pol}(\mathbb{A})^{\mathrm{CAN}}_{\mathbb{T}} \not\to \mathrm{PROJ}$ ↯

THEOREM (MOTTET + P. '20)

$\mathbb{T}$ ... UNIVERSAL HOMOGENEOUS TOURNAMENT

$\mathbb{A}$ ... FIRST-ORDER REDUCT OF $\mathbb{T}$
($\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \supseteq \mathrm{Aut}(\mathbb{T})$)

MODEL-COMPLETE CORE

TFAE:

- $\mathrm{Pol}(\mathbb{A}) \not\to A$ ... AFFINE/MODULE
- $\mathrm{Pol}(\mathbb{A})$ CAN $\to \mathbb{T}$ HAS PSEUDO-WNUs OF ALMOST ALL ARITIES

($\Rightarrow$ $\mathbb{A}$ HAS RELATIONAL WIDTH $(4,6)$ (MOTTET + NAGY + P. + WRONA '21))

PROOF:

ALMOST SAME AS FOR PROJ

SAME PROOF:

- RANDOM GRAPH
- UNIVERSAL HOMOGENEOUS K_n-FREE GRAPH
- RANDOM POSET
- UNARY STRUCTURES (MOTTET + NAGY + P. + WRONA '21) ($\not\to$)
- MMSNP (MOTTET + NAGY + P. + WRONA '21)

THANK YOU!

