OLIGOMORPHIC CLONES

(PART I)

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MFO 1/2021

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WHAT OF THE THEORY OF FINITE ALGEBRAS HOLDS FOR INFINITE ONES?

$\rightsquigarrow$ TOPOLOGY

$\rightsquigarrow$ OLIGOMORPHIC CLONES

ANSWER:
- BASIC THEORY
- SOME ADVANCED THEORY ?

BARTO:

- "UNIVERSAL ALGEBRA STUDIES ALGEBRAS"

- BIRKHOFF'S HSP THEOREM:

  $\underline{A}, \underline{B}$ ALGEBRAS:

  $\underline{B} \in HSP(\underline{A}) \Leftrightarrow$ IDENTITIES $(\underline{B})$ SATISFIED IN $\underline{A}$

- (BASIC) SHAPE OF INVARIANT RELATIONS
  $\updownarrow$
  EQUATIONAL CONDITIONS

  E.G. MAL'TSEV, MAJORITY

  $m(x,x,y) = y = m(y,x,x)$

- "IF $\underline{A}, \underline{B}$ HAVE SAME TERM OPERATIONS $\Rightarrow$ CAN BE CONSIDERED EQUAL"

  $\rightsquigarrow$ CLONES

MOTIVATION: INVARIANT RELATIONS SAME

HOWEVER: $\mathcal{C} = \mathrm{Pol}(\mathrm{Inv}(\mathcal{C}))$

ONLY IF $\mathcal{C}$ closed:

$$\mathcal{C} \subseteq \underbrace{\bigcup_{n \geq 1}}_{\text{SUM}} C^{\overbrace{C^n}^{\text{PRODUCT}}}$$

DISCRETE

POINTWISE CONVERGENCE TOPOLOGY

(NOT NEEDED WITH INFINITARY RELATIONS)

SIMILAR: AUTOMORPHISM GROUPS = CLOSED PERMUTATION GROUPS

SO: $\mathbb{A}$ RELATIONAL STRUCTURE

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ CLOSED CLONE

TOPOLOGICAL CLONE WITH COMPOSITION

THEOREM $\mathbb{A}, \mathbb{B}$ ~~FINITE~~ ω-CATEGORICAL RELATIONAL STRUCTURES

1) $\mathbb{A}$ pp-DEFINES $\mathbb{B}$ $\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$

2) pp-INTERPRETS $\mathrm{Pol}(\mathbb{A}) \rightarrow \mathrm{Pol}(\mathbb{B})$ + LARGE RANGE

3) pp-CONSTRUCTS $\mathrm{Pol}(\mathbb{A}) \xrightarrow{h1} \mathrm{Pol}(\mathbb{B})$ + FINITE

UNIFORMLY CONTINUOUS

1) BODIRSKY + NEŠETŘIL '05 2) BODIRSKY + P. '10 3) BARTO + OPRŠAL + P. '15

$B \in HSP(\mathbb{A})$ ③

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REMARK: CONTINUITY $\Leftrightarrow$ FINITE POWERS

HSP^{fin} vs. HSP

CONTINUITY: IS IT NEEDED?

$$Pol(\mathbb{A}) \to Pol(\mathbb{B})$$

BUT ?

$$Pol(\mathbb{A}) \not\to Pol(\mathbb{B})$$

- HAPPENS (BODIRSKY + EVANS + KOMPATSCHER + P. '15)
- NOT ALWAYS E.G. RANDOM GRAPH (BODIRSKY + PONGRÁCZ + P. '14)

$\to$ PAOLINI ("RECONSTRUCTION")

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DEFINITION: $\mathcal{C} = Pol(\mathbb{A})$ CLOSED CLONE

OLIGOMORPHIC :$\Leftrightarrow$ CONTAINS OLIGOMORPHIC PERMUTATION GROUP

$\Leftrightarrow$ $\mathbb{A}$ ω-CATEGORICAL

GOOD NEWS: BASIC THEORY WORKS (INCLUDING $CSP(\mathbb{A})$ VIA $Pol(\mathbb{A})$)

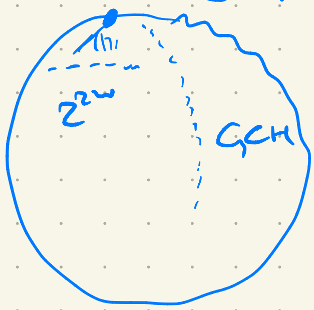
BAD NEWS:

BASIC THEORY = ABSTRACT NONSENSE

ADVANCED THEORY: "IT'S COMPLICATED"

GOAL: CLASSIFY:

CLONES ON ω



(GOLD* + SHELAH '02)

- ALL CLONES?
- CLOSED?
- OLIGOMORPHIC?

ω-CATEGORICAL STRUCTURES UP TO PP-DEFINABILITY...

NOT REALLY

- SOME ~~CLONES~~?

PROPERTIES E.G. NON-TRIVIALITY

ADDITIONAL MODEL-THEORETIC ASSUMPTIONS: OK.

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ADVANCED THEORY?

TRIVIALITY: $\mathcal{C} \to$ PROJ   CONTINUOUS

CLONE OF PROJECTIONS ON $\{0,1\}$

$\mathcal{C} = \mathrm{Pol}(\mathbb{A})$

$\Longleftrightarrow$ $\mathbb{A}$ pp-INTERPRETS ALL FINITE STRUCTURES

NO RESULT!

OPEN: $\Longleftrightarrow$ $\mathcal{C} \to$ PROJ ?

+ "IDEMPOTENCY":

BARTO: $\mathcal{C}$ TAYLOR $:\Longleftrightarrow$
- $\mathcal{C} \not\to$ PROJ
- $\mathcal{C}$ IDEMPOTENT: UNARY FUNCTIONS TRIVIAL

SO HE CAN PROVE STUFF

$\mathcal{C}$ OLIGOMORPHIC $\Rightarrow$ $\mathcal{C}$ NOT IDEMPOTENT

FIRST DILEMMA OF THE INFINITE SHEEP

IDEMPOTENCY: CAN'T LIVE WITH IT, CAN'T LIVE WITHOUT IT.

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IDEMPOTENCY MODULO $\mathrm{Aut}(\mathbb{A})$:

DEFINITION

$\mathbb{A}$ MODEL-COMPLETE CORE $:\Longleftrightarrow$

$$\overline{\mathrm{Aut}(\mathbb{A})} = \mathrm{End}(\mathbb{A}) \; (= \mathrm{Pol}^{(1)}(\mathbb{A}))$$

UNARY FCTS.

THEOREM (BODIRSKY '04)

$\mathbb{A}$ ω-CATEGORICAL

$\Rightarrow \exists \mathbb{A}'$ MC CORE SUCH THAT $\mathbb{A} \xrightarrow{\text{HOM}} \mathbb{A}'$, $\mathbb{A}' \xrightarrow{\text{HOM}} \mathbb{A}$

PLUS:
- $\mathbb{A}'$ UNIQUE
- $\mathbb{A}'$ ω-CATEGORICAL

$\mathbb{A}, \mathbb{A}'$ pp-INTER-CONSTRUCTIBLE

$$\frac{\;}{\mathrm{Th}_{pp}(\mathbb{A}) = \mathrm{Th}_{pp}(\mathbb{A}')}$$

THEOREM (BARTO + P. '16) TOPOLOGY IS IRRELEVANT

$\mathbb{A}$ ω-CATEGORICAL MC CORE. TFAE:

- $\mathbb{A}$ pp-INTERPRETS ALL FINITE STRUCTURES WITH PARAMETERS
- $\exists \bar{a}$ $\quad Pol(\mathbb{A}, \bar{a}) \rightarrow PROJ$
- $\exists \bar{a}$ $\quad Pol(\mathbb{A}, \bar{a}) \rightarrow PROJ$
- $Pol(\mathbb{A}) \not\models \exists e, f, s.$ $\; e \circ s(x,y,x,z,y,z) \approx f \circ s(y,x,z,x,z,y)$

$Pol(\mathbb{A}, \bar{a})$: STABILIZER OF TUPLE $\bar{a}$

$$= Pol(\mathbb{A}) \cap \bigcap_{a \in \bar{a}} Pol(\{a\})$$

THEOREM (BARTO + KOMPATSCHER + OLŠÁK + PHAM + P. '18)

$\mathbb{A}$ "SLOW ORBIT GROWTH" : $< 2^{2^n}$ TFAE:

- $\mathbb{A}$ pp-CONSTRUCTS ALL FINITE STRUCTURES
- $Pol(\mathbb{A}) \overset{h1}{\dashrightarrow} PROJ$
- $Pol(\mathbb{A}) \not\models e \circ s(x,y,x,z,y,z) \approx f \circ s(y,x,z,x,z,y)$

CONTINUITY NEEDED TOPOLOGY IS RELEVANT (BODIRSKY + MOTTET + OLŠÁK + OPRŠAL + WILLARD + P. '19)

$\rightsquigarrow$ **BODIRSKY :** + FINITE HOMOGENEITY
RAMSEY

$\rightsquigarrow$ **MOTTET :**

$\boxed{\text{SECOND DILEMMA OF THE INFINITE SHEEP}}$

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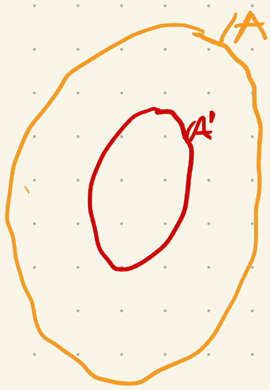
OPEN PROBLEM

$\mathbb{A}$ ω-CATEGORICAL

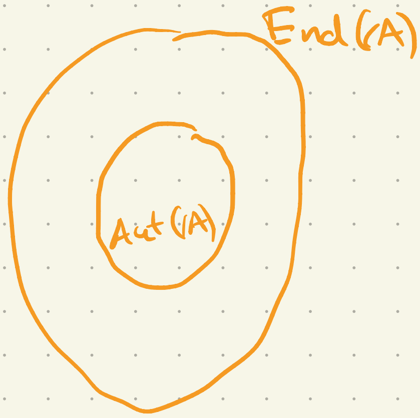
$\mathbb{A}'$ ITS MODEL-COMPLETE CORE:

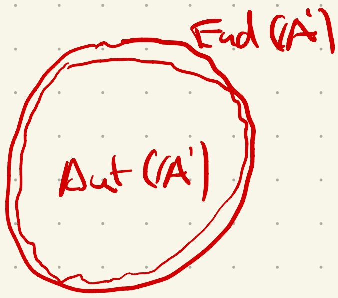


- $\mathbb{A} \underset{\longleftarrow}{\overset{\text{HOM}}{\longrightarrow}} \mathbb{A}'$
- $\overline{\mathrm{Aut}(\mathbb{A}')} = \mathrm{End}(\mathbb{A}')$

E.G.: $\mathbb{A} = (V, E)$ RANDOM GRAPH

$\mathbb{A}' = (V', E)$ INFINITE CLIQUE





$\mathbb{A}$		$\mathbb{A}'$
ω-CAT.	$\Rightarrow$	ω-CAT.
SLOW ORBIT GROWTH	$\Rightarrow$	SLOW ORBIT GROWTH
HOMOGENEOUS	$\Rightarrow$	HOMOGENEOUS
RAMSEY	$\Rightarrow$	RAMSEY

HAVE ALGEBRAIC THEOREMS FOR $\mathrm{Pol}(\mathbb{A}')$ WHEN

$\mathbb{A}$ / $\mathbb{A}'$ ARE ω-CATEGORICAL

/ HAVE SLOW ORBIT GROWTH

WANT BETTER THEOREMS WHEN

$\mathbb{A}$ IS FO-REDUCT OF FINITELY HOMOGENEOUS.

IF $\mathbb{A}$ HAS FINITELY HOMOGENEOUS EXPANSION ?
$\Rightarrow$ $\mathbb{A}'$ ——— " ———

THEOREM (MOTTET + P. '20) .... + RAMSEY $\Rightarrow$ ... + RAMSEY

SAME ?