THE INFINITE CSP DICHOTOMY CONJECTURE

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I MOTIVATION OF THE RANGE

II THE CONJECTURE

III RESULTS

DEFINITION (CSP($\mathbb{A}$))

$\mathbb{A} = (A; R_1, \ldots, R_m)$ RELATIONAL STRUCTURE

INPUT: PP-SENTENCE

$\varphi = \exists x_1 \ldots \exists x_n \, R_{i_1}(\text{variables}) \wedge \ldots$
$\ldots \wedge R_{i_\ell}(\text{variables})$

QUESTION: $\mathbb{A} \models \varphi$?

CSP($\mathbb{A}$) WELL-DEFINED COMPUTATIONAL PROBLEM FOR ANY $\mathbb{A}$!

? WHY IS CSP($\mathbb{A}$) FOR FINITE $\mathbb{A}$ PREVALENT?

(BY FAR) MOST RESULTS ...
OFTEN. EVEN SILENT ASSUMPTION!

RECALL: $\mathbb{A}$ FINITE $\Rightarrow$

- CSP($\mathbb{A}$) $\in$ P, IF Pol($\mathbb{A}$) HAS WNU FUNCTION
$w(x \ldots xy) = \ldots = w(yx \ldots x)$
- CSP($\mathbb{A}$) NP-c, OTHERWISE

(BULATOV, ZHUK, 2017)

TWO REASONS:

1. $\mathbb{A}$ FINITE $\Rightarrow$ CSP($\mathbb{A}$) $\in$ NP
NATURAL LARGE CLASS WHERE
P/NP-c DICHOTOMY MIGHT HOLD/HOLDS
(FEDER-VARDI-CONJECTURE) 1990s

$\mathbb{A}$ INFINITE $\Rightarrow$ CSP($\mathbb{A}$) CAN HAVE ARBITRARY COMPLEXITY!

2. ALGEBRAIC APPROACH WORKS:
$\mathbb{A} \rightsquigarrow$ Pol($\mathbb{A}$) $\rightsquigarrow$ identities/equations in Pol($\mathbb{A}$)
(BULATOV + JEAVONS + KROKHIN '05)

- Both reasons are of **practical** nature!

- Is $\mathrm{CSP}(\mathbb{A})$ more natural/interesting for finite $\mathbb{A}$?

Examples

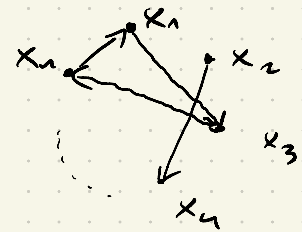
- Solving linear equations over field $\mathbb{F}$

  finite / infinite

- $\mathrm{CSP}(\mathbb{Q}, <)$:

  Input: $\varphi = \exists x_1 \ldots \exists x_n \; x_{i_1} < x_{j_1} \wedge \ldots \wedge x_{i_l} < x_{j_l}$

  Q: $(\mathbb{Q}, <) \models \varphi$?



φ represented as digraph G

Q: G acyclic?

Both problems in P!

? Can the two reasons:

1. containment in NP
2. algebraic approach possible

be met by a larger class?

Ad 2.:

Definition

$\mathbb{A}$ ω-categorical $:\Leftrightarrow$ $\mathbb{A}$ countable & $\forall n \geq 1$ $A^n/\mathrm{Aut}(\mathbb{A})$ finite

AD 2. (ALGEBRAIC APPROACH POSSIBLE)

DEFINITION

$\mathbb{A}$ ω-CATEGORICAL $:\Leftrightarrow$
$\mathbb{A}$ COUNTABLE &
$\forall n \geq 1$ $A^n/\mathrm{Aut}(\mathbb{A})$ FINITE

$A^n/_{\mathrm{Aut}(\mathbb{A})}$: $\bar{a}, \bar{b} \in A^n$
$\bar{a} \sim \bar{b} :\Leftrightarrow \exists \alpha \in \mathrm{Aut}(\mathbb{A})$
$\alpha(\bar{a}) = \bar{b}$

EXAMPLES

- $(\mathbb{Q}, <)$
- RANDOM GRAPH = UNIVERSAL HOMOGENEOUS GRAPH
- COUNTABLE ATOMLESS BOOLEAN ALGEBRA

THEOREM (BODIRSKY + NEŠETŘIL '03, BARTO + OPRŠAL + P. '15)

$\mathbb{A}$ ω-CATEGORICAL.
$\Rightarrow$ ALGEBRAIC APPROACH WORKS:

- $\mathrm{Pol}(\mathbb{A}) = \mathrm{Pol}(\mathbb{B})$
  $\Rightarrow$ $\mathrm{CSP}(\mathbb{A}) \sim_{\text{POLYTIME}} \mathrm{CSP}(\mathbb{B})$
- $\mathbb{B}$ FINITE, $\mathrm{Pol}(\mathbb{A}) \longrightarrow \mathrm{Pol}(\mathbb{B})$
  $\Rightarrow$ $\mathrm{CSP}(\mathbb{B}) \leq_{\text{POLYTIME}} \mathrm{CSP}(\mathbb{A})$

$\longrightarrow$: THERE EXISTS A <u>MORPHISM</u> φ

- PRESERVES HEIGHT 1 IDENTITIES
  (φ IS <u>MINION</u> HOMOMORPHISM)
- φ <u>UNIFORMLY CONTINUOUS</u>:
  DETERMINED ON FINITE SET
  $\forall n \geq 1$ $\exists F \subseteq A$ FINITE
  $\forall f, g \in \mathrm{Pol}(\mathbb{A})$ n-ary $f|_F = g|_F \Rightarrow \varphi(f) = \varphi(g)$

- $Pol(\mathbb{A}) \longrightarrow$ PROJ ... CLONE OF PROJECTIONS ON $\{0,1\}$

$\Rightarrow$ CSP($\mathbb{A}$) NP-HARD

- UNIFORM CONTINUITY VOID FOR FINITE $\mathbb{A}$.

2. ALGEBRAIC APPROACH ✓

1. CONTAINMENT IN NP ...

... DOES NOT WORK

"REASON":

COUNTABLE NUMBER OF RELATIONS (OF ARBITRARY ARITY) CAN BE ENCODED INTO SINGLE BINARY RELATION

MORE PRECISELY:

LET $\tau := (R_i)_{i \in \omega}$ R_i i-ary REL. SYMBOL

LET $S \subseteq \omega$ BE UNDECIDABLE (OR OTHERWISE NASTY)

LET $c \in A$, A COUNTABLE

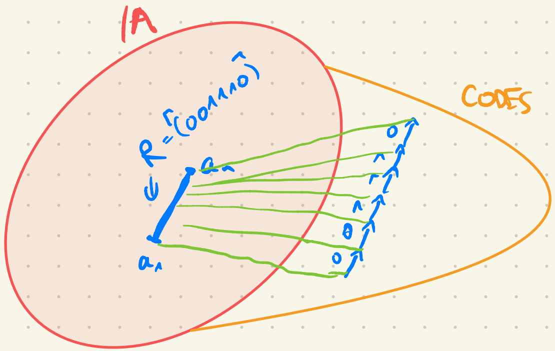
$$\mathbb{A} := (A; (R_i^{\mathbb{A}})_{i \in \omega})$$

$$R_i^{\mathbb{A}} := \begin{cases} \emptyset, & i \in S \\ \{(c,\dots,c)\}, & i \notin S \end{cases}$$

"CSP($\mathbb{A}$) UNDECIDABLE"!

OK SINCE $\mathbb{A}$ HAS INFINITE SIGNATURE

... BUT CAN BE ENCODED INTO FINITE



THEOREM (GILLIBERT + BODIRSKY...

(GILLIBERT + JONUŠAS + KOMPATSCHER + MOTTET + P. '20)

① $\exists \mathbb{A}$ ω-CATEGORICAL, FINITE SIGNATURE:

CSP($\mathbb{A}$) IS

- Π_n-COMPLETE $(n \geq 1)$
- PSPACE-C.
- EXPTIME-C.
- coNP-INTERMEDIATE
- UNDECIDABLE

AND $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{PROJ}$

② $\exists \mathbb{A}$ ω-CATEGORICAL, FINITE SIGNATURE:

$\mathrm{Pol}(\mathbb{A}) \longrightarrow \mathrm{PROJ}$

$\mathrm{Pol}(\mathbb{A}) \not\longrightarrow \mathrm{PROJ}$

③ $\forall \Sigma$ ... SET OF IDENTITIES, COUNTABLE

$\forall \mathcal{C}$ ... COMPLEXITY CLASS, "NON-TRIVIAL"

$\exists \mathbb{A}$ ω-CATEGORICAL:

$\mathrm{Pol}(\mathbb{A}) \models \Sigma$

$\mathrm{CSP}(\mathbb{A}) \notin \mathcal{C}$

SUMMARY: ω-CATEGORICAL TOO WIDE CLASS

BETTER CLASS?

BETTER CLASS?

CSP($\mathbb{A}$) ONLY DEPENDS ON

$$\mathrm{AGE}(\mathbb{A}) := \{ \mathbb{F} \mid \mathbb{F} \text{ FINITE}, \; \mathbb{F} \leq \mathbb{A} \}:$$

$$\mathrm{AGE}(\mathbb{A}) = \mathrm{AGE}(\mathbb{A}') \implies \mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathbb{A}')$$

WHICH CLASSES $\mathcal{K}$ OF FINITE STRUCTURES ARE THE AGE OF AN ω-CATEGORICAL STRUCTURE?

HP (HEREDITARY PROPERTY)

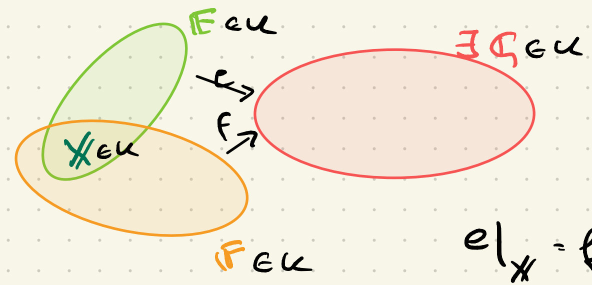
$$\mathbb{E} \leq \mathbb{F} \in \mathcal{K} \implies \mathbb{E} \in \mathcal{K}$$

JEP (JOINT EMBEDDING PROPERTY)

$$\mathbb{E}, \mathbb{F} \in \mathcal{K} \implies \exists \, \mathbb{G} \in \mathcal{K} \qquad \mathbb{E} \xrightarrow{\text{EMB.}} \mathbb{G}, \quad \mathbb{F} \xrightarrow{\text{EMB.}} \mathbb{G}$$

GUARANTEE FOR ω-CATEGORICITY (IF SIGNATURE FINITE):

AP (AMALGAMATION PROPERTY)



$$e|_{\mathbb{X}} = f|_{\mathbb{X}}$$

EXAMPLES

$\mathcal{K} =$

- (FINITE) LINEAR ORDERS
- GRAPHS
- BOOLEAN ALGEBRAS
- K_n-FREE GRAPHS
- POSETS
- TOURNAMENTS

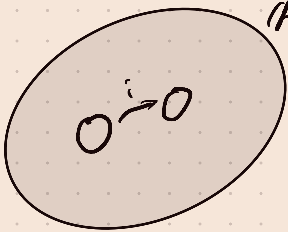
Fraïssé's Theorem

$\mathcal{K}$ with HP, JEP, AP

$\Updownarrow$

$\mathbb{A}$ homogeneous:

$\mathbb{A}$



$\mathcal{K}$ (and CSP($\mathbb{A}$)) can still be undecidable!

Solution to put $\mathcal{K}$ and CSP($\mathbb{A}$) into NP:

Finite Boundedness

Makes $\mathcal{K}$ (resp. $\mathbb{A}$) representable

Definition

$\mathcal{K}/\mathbb{A}$ finitely bounded $:\Longleftrightarrow$

$\exists \mathfrak{F}$ finite

$\forall \mathbb{E}$ finite:

$\mathbb{E} \in \mathcal{K} \,/\, \mathrm{Age}(\mathbb{A})$

$\Longleftrightarrow \forall \mathbb{F} \in \mathfrak{F}: \mathbb{F} \overset{\text{emb.}}{\not\hookrightarrow} \mathbb{E}$

$\mathfrak{F}$ ... forbidden substructures

Claim: CSP($\mathbb{A}$) $\in$ NP

Proof: given $\varphi = \exists x_1 \dots \exists x_n\, R_{i_1}(\ ) \wedge \dots \wedge R_{i_\ell}(\ )$

- guess $\mathbb{E}$ of size n

  check $\mathbb{E} \in \mathrm{Age}(\mathbb{A})$ (in P)

  yes $\rightarrow$ check $\mathbb{E} \models \varphi$ (in NP)

  no $\rightarrow$ reject

HAPPY?

- HOMOGENEOUS STRUCTURES QUITE RARE

  E.G. HOMOGENEOUS GRAPHS:

  - RANDOM GRAPH
  - K_n-FREE GRAPH
  - EQUIVALENCE RELATIONS

  } BASICALLY ALL (LACHLAN + WOODROW)

- NICE PROBLEMS EXCLUDED

  E.G. BETWEENNESS:

  B 3-ARY SYMBOL

  $\boxed{\text{INPUT}}$: FINITE B-STRUCTURE $\mathbb{S}$

  $\boxed{Q}$: IS THERE A LINEAR ORDER $<$ ON $\mathbb{S}$ SUCH THAT

  $$\forall x,y,z \quad B^{\mathbb{S}}(x,y,z) \to x<y<z \vee z<y<x$$

  ?

COMPARE WITH $CSP(\mathbb{Q},<)$:

INPUT: $\prec$-STRUCTURE $\mathbb{S}$

Q: IS THERE A LINEAR ORDER $<$ ON $\mathbb{S}$ SUCH THAT

$$\forall x,y \quad x \prec y \to x<y$$

👁 BETWEENNESS = $CSP(\mathbb{Q}, B)$

WHERE $B^{\mathbb{Q}}(x,y,z) :\Leftrightarrow x<y<z \vee z<y<x$

!

GENERAL

$\mathbb{A}$ HOMOGENEOUS τ-STRUCTURE

$\phi_1, \dots, \phi_m$ QUANTIFIER-FREE FIRST-ORDER FORMULAS

$R_1^{\mathbb{A}} \dots R_m^{\mathbb{A}}$ CORRESPONDING RELATIONS DEFINED IN $\mathbb{A}$

GIVEN $(R_1,\dots,R_m)$-STRUCTURE $\mathbb{S}$ IS THERE A τ-STRUCTURE ON $\mathbb{A}$ SUCH THAT $R_i \to \phi_i$ FOR ALL i?

EXAMPLES

- GRAPH-SAT

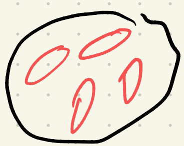
$\mathbb{A} = (V,E)$ RANDOM GRAPH

$$\phi_1 = (E(x,y) \wedge \neg E(y,z) \wedge \neg E(x,z)) \vee (\neg E(x,y) \wedge E(y,z) \wedge \neg E(x,z)) \vee (\neg E(x,y) \wedge \neg E(y,z) \wedge E(x,z))$$

$\rightsquigarrow$ R_1 DEFINED IN $\mathbb{A}$

$CSP(V, R_1)$:

INPUT S



Q: IS THERE A GRAPH ON S SUCH THAT $R_1^S \to \phi_1$?

- POSET-SAT
- TOURNAMENT-SAT

COMPLEXITY DEPENDS ON $\phi_1, \dots, \phi_m$

$\mathbb{A}$ σ-STRUCTURE

$\underbrace{\phi_1}_{R_1^{\mathbb{A}}}, \dots, \underbrace{\phi_m}_{R_m^{\mathbb{A}}}$ FO-FORMULAS in $\mathbb{A}$

$(A; R_1^{\mathbb{A}}, \dots R_m^{\mathbb{A}})$ FIRST-ORDER REDUCT OF $\mathbb{A}$

AGE($\mathbb{A}$) $\in P$

$\to$ $CSP(A; R_1 \dots R_m) \in NP$

CONJECTURE (2011)

$\mathbb{A}$ FINITELY BOUNDED HOMOGENEOUS

$\mathbb{B}$ FIRST-ORDER REDUCT OF $\mathbb{A}$

① $\mathrm{Pol}(\mathbb{B}) \to \mathrm{PROJ} \Rightarrow \mathrm{CSP}(\mathbb{B})$ NP-c

② $\mathrm{Pol}(\mathbb{B}) \not\to \mathrm{PROJ} \Rightarrow \mathrm{CSP}(\mathbb{B}) \in P$

① TRUE (BARTO + OPRŠAL + P. '16)

② IMPLIES $\mathrm{Pol}(\mathbb{B})$ HAS e, f, s SUCH THAT

$\forall x,y,z \quad e \circ s(xyxzyz) = f \circ s(yxzxzy)$

(BARTO + P. '17)

CONJECTURE TRUE FOR

- $(\mathbb{Q}, <)$
- RANDOM POSET
- HOMOGENEOUS GRAPHS
- UNARY STRUCTURES
- MMSNP

⋮

NEWS (2020)

CAN ALL BE PROVEN USING SMOOTH APPROXIMATIONS:

(MOTTET + P. '20)

REDUCTION TO FINITE CASE


THANK YOU !