ALGEBRAIC, LOGICAL & COMBINATORIAL METHODS FOR CONSTRAINT SATISFACTION PROBLEMS (CSPs)

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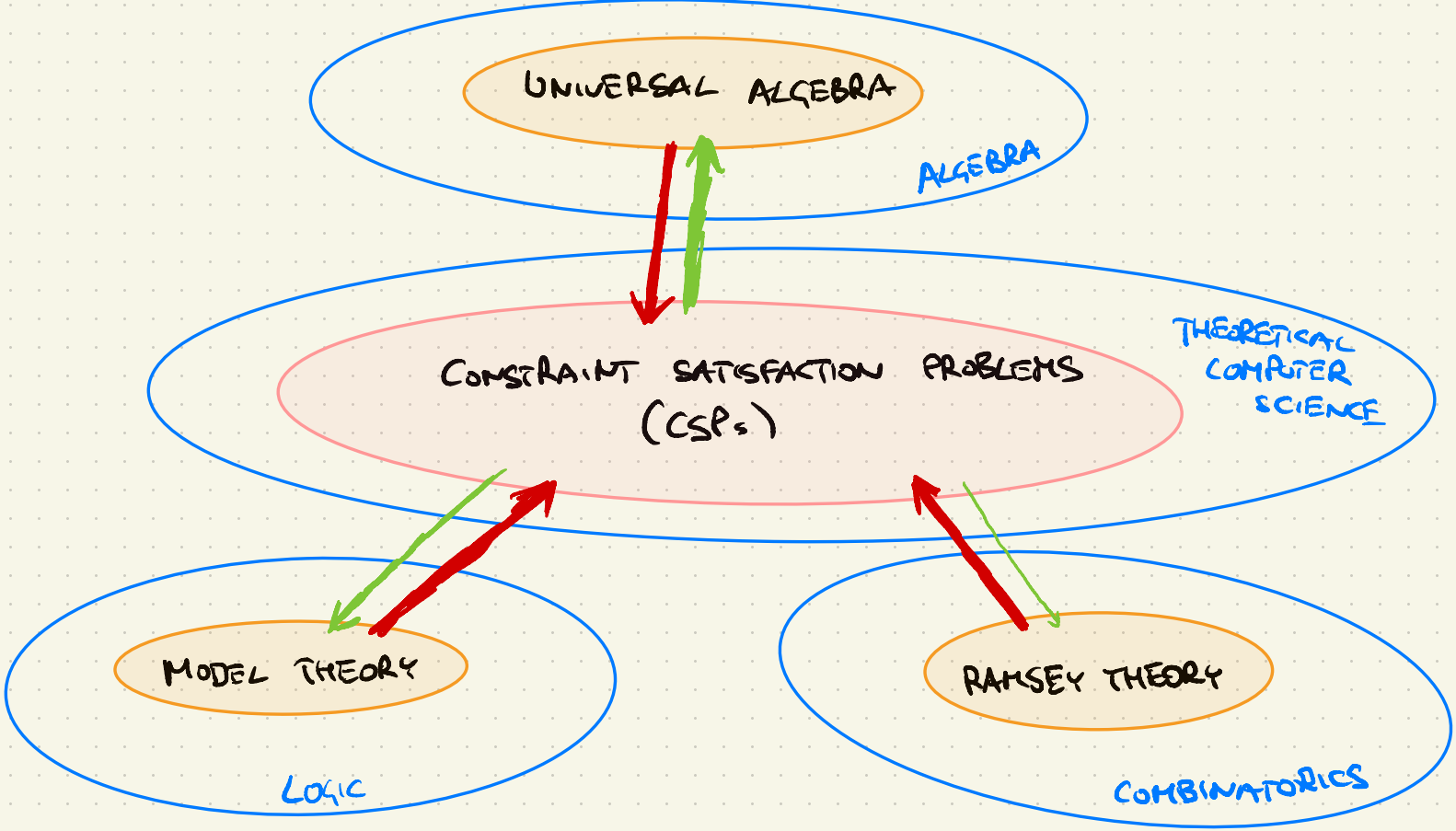

UNIVERSAL ALGEBRA
ALGEBRA
CONSTRAINT SATISFACTION PROBLEMS
(CSPs)
THEORETICAL COMPUTER SCIENCE
MODEL THEORY
LOGIC
RAMSEY THEORY
COMBINATORICS

CSP (CONSTRAINT SATISFACTION PROBLEM):

COMPUTATIONAL PROBLEM:

INPUT: VARIABLES + "CONSTRAINTS"

Q: $\exists$ SOLUTION?

COMPLEXITY?

MOST IMPORTANTLY, (FOR US) TIME COMPLEXITY (IN # OF VARIABLES)

EXAMPLES

- GIVEN EQUATIONS (OVER A RING, FIELD ...)

  $\exists$ SOLUTION?

  - DIOPHANTINE EQUATIONS (OVER $(\mathbb{Z}, +, \cdot, 0, 1)$)

    UNDECIDABLE

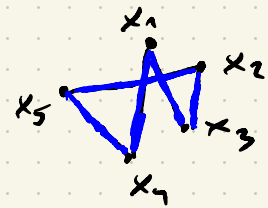
  - LINEAR EQUATIONS (E.G. OVER $(\mathbb{Q}, +, \cdot, 0, 1)$ OR $(\mathbb{Z}_p, +, \cdot, 0, 1)$)

    POLYNOMIAL TIME

MORE EXAMPLES

- GIVEN VARIABLES, SOME "CONNECTIONS" BETWEEN THEM

  (A.K.A. GRAPH)



Q: IS THERE A 3-COLORING?

(VARIABLES ASSIGNED VALUES IN $\{0,1,2\}$, NO ADJACENT VARIABLES GET SAME COLOR)

EXPONENTIAL TIME

NP-COMPLETE :- HARD TO FIND SOLUTION

- EASY TO CHECK SOLUTION

Q: IS THERE A 2-COLORING?

$\in$ P (POLYNOMIAL TIME)

MORE EXAMPLES

- GENERALIZED SUDOKU ($n^2 \times n^2$ GRID)

  NP-COMPLETE

- 3-SAT

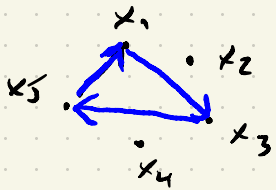
  GIVEN EXPRESSION

  $$\underbrace{(x_1 \vee \neg x_2 \vee x_3)}_{3} \wedge \underbrace{(\neg x_1 \vee x_2 \vee x_4)}_{3} \wedge \dots \wedge$$

  NP-COMPLETE

  Q: SATISFIABLE ?

- GIVEN DIGRAPH (VARIABLES + DIRECTED "CONNECTIONS")



  Q: CAN VARIABLES BE TOTALLY ORDERED SUCH THAT

  $$\forall i,j \quad x_i \to x_j \Rightarrow x_i \leq x_j$$

  $\in P$ "ACYCLICITY"

COMMON ABSTRACTION:

$\mathbb{A} = (A; R_1, \dots, R_m)$ RELATIONAL STRUCTURE

SET = "DOMAIN" RELATIONS ON DOMAIN

CSP($\mathbb{A}$)

INPUT: SENTENCE

$$\varphi = \underbrace{\exists x_1 \dots \exists x_n}_{\text{VARIABLES}} R_{i_1}(\text{SOME VARIABLES}) \wedge R_{i_2}(\text{SOME VARIABLES}) \wedge \dots R_{i_\ell}(\text{SOME VARIABLES})$$

Q: $\mathbb{A} \models \varphi$?

EXAMPLES

- $(\mathbb{Z}, +, \cdot, 0, 1)$
- $(\mathbb{Z}_p, +, \cdot, 0, 1)$ ($+, \cdot$ AS TERNARY RELATIONS)
- $(\mathbb{Q}, +, 0, 1)$
- $(\mathbb{Z}_p, +, 0, 1)$
- $(\{0,1\}, \neq)$
- $(\{0,1\}, \text{ALL 3-ARY REL.})$
- $(\mathbb{Q}, <)$
- $(\{0,1,2\}, \neq)$

- $\mathbb{A}$ INFINITE: $CSP(\mathbb{A})$ ⋮ UNDECIDABLE, NP-c, P, ⋮

EVERY COMPUTATIONAL PROBLEM POLYNOMIAL-TIME EQUIVALENT TO SOME CSP

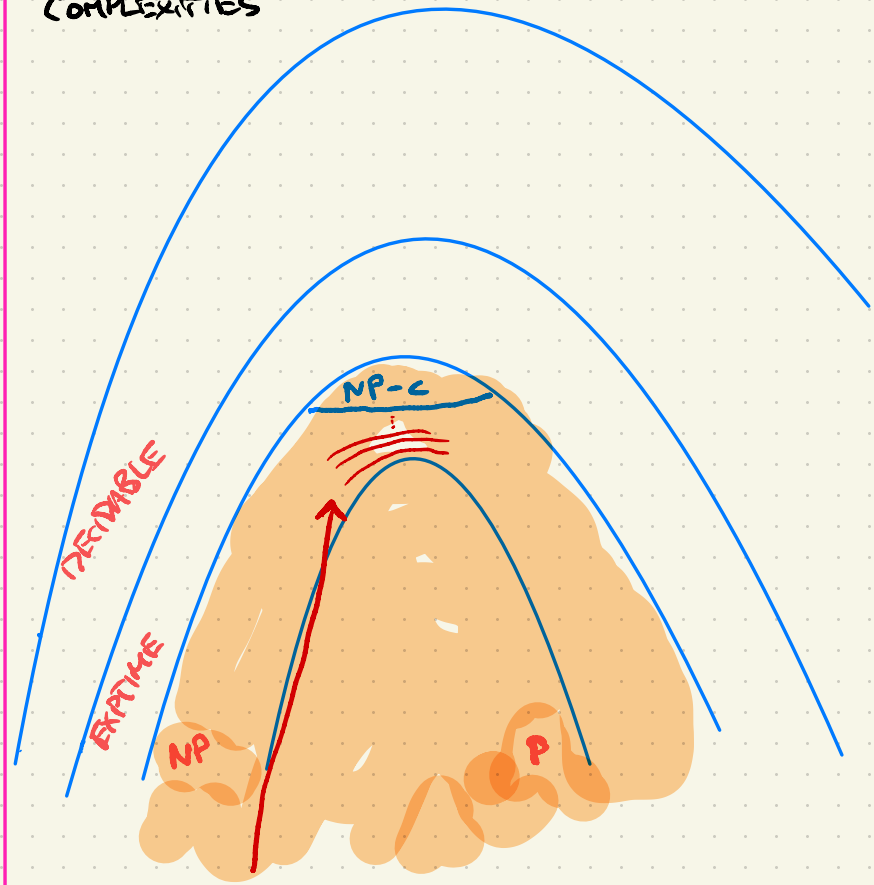
(BODIRSKY + GROHE '07)

- $\mathbb{A}$ FINITE $\Rightarrow$ $CSP(\mathbb{A}) \in$ NP
  - 3-SAT — NP-COMPLETE
  - $(\mathbb{Z}_p, +, 0, 1)$ — P
  - 2-COLORING — P
  - 3-COLORING — NP-c
  - $(\mathbb{Z}_2, +, \cdot, 0, 1)$ — NP-c

1-IN-3-SAT: $x \cdot y = 0$, $x \cdot z = 0$, $y \cdot z = 0$ $\quad (1-x)\cdot(1-y)(1-z) = 0$

$\{(1,0,0), (0,1,0), (0,0,1)\}$

COMPLEXITIES



LADNER'S THEOREM: MANY CLASSES INBETWEEN

CONJECTURE (FEDER + VARDI '93)

$\mathbb{A}$ FINITE $\Rightarrow$ CSP($\mathbb{A}$) — NP-c / P

THEOREM (BULATOV, ZHUK '17)

CONJECTURE TRUE.

GREAT?

- WHY IS THERE A DICHOTOMY?
- WHEN IS A COMPUTATIONAL PROBLEM EASY (P) / HARD (NP-c)?
- WHAT IS SYMMETRY?

$\Longrightarrow$ $\boxed{\text{UNIVERSAL ALGEBRA}}$

EXAMPLES

- CSP($\mathbb{Z}_p, +, 0, 1$)

  EASY BECAUSE OF GAUSS.

  SOLUTIONS: AFFINE SPACE:

  $\bar{u}, \bar{v}, \bar{w}$ SOLUTIONS $\Rightarrow$ $\bar{u} - \bar{v} + \bar{w}$ SOLUTION

  THEREFORE:

  $$m: \mathbb{Z}_p^3 \to \mathbb{Z}_p$$
  $$(x,y,z) \mapsto x - y + z$$

  PRESERVES SOLUTIONS/($\mathbb{Z}_p, +, 0, 1$)

  SYMMETRY: $m(x,x,y) = y = m(y,x,x)$

- CSP($\{0,1\}; <$)

  EASY: LOCAL CHECKING $x_i \to x_j \to x_k$

  SOLUTIONS: $\bar{s}, \bar{t}$ SOLUTIONS $\Rightarrow$ $\max(\bar{s}, \bar{t})$ SOL.

  SYMMETRY: $\max(x,y) = \max(y,x)$, ...

$\mathbb{A} = (A; R_1, \dots, R_m)$ RELATIONAL STRUCTURE

$k \geq 1$

$f: A^k \to A$ <u>POLYMORPHISM</u> $:\Leftrightarrow$

WHENEVER $\bar{r}_1, \dots, \bar{r}_k \in R_i$ FOR SOME i

$\Rightarrow f(\bar{r}_1, \dots, \bar{r}_k) \in R_i$

$\boxed{\mathrm{Pol}(\mathbb{A})}$ ... ALL POLYMORPHISMS OF $\mathbb{A}$

"POLYMORPHISM ALGEBRA"

(DOMAIN: A; FUNCTIONS: POLYMORPHISMS)

SIMILAR TO AUTOMORPHISM GROUP $\mathrm{Aut}(\mathbb{A})$

ENDOMORPHISM MONOID $\mathrm{End}(\mathbb{A})$

DIFFERENT KINDS OF <u>SYMMETRIES</u>

THEOREM (BULATOV + JEAVONS + KROKHIN '05)
(BARTO + OPRŠAL + P. '17)

- $\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

  $\mathrm{Pol}(\mathbb{A}) = \mathrm{Pol}(\mathbb{B}) \Rightarrow \mathrm{CSP}(\mathbb{A}) \sim \mathrm{CSP}(\mathbb{B})$

  PTIME

- $\mathbb{A}, \mathbb{B}$ FINITE,

  ALL NON-NESTED EQUATIONS SATISFIED IN $\mathrm{Pol}(\mathbb{A})$ ALSO SATISFIED IN $\mathrm{Pol}(\mathbb{B})$

  $\Rightarrow \mathrm{CSP}(\mathbb{B}) \leq \mathrm{CSP}(\mathbb{A})$

  PTIME

"THE <u>RICHER</u> $\mathrm{Pol}(\mathbb{A})$ (IN EQUATIONS), THE <u>EASIER</u> $\mathrm{CSP}(\mathbb{A})$"

SATISFACTION OF EQUATIONS, E.G.

$$m(x,x,y) = m(x,y,x) = m(y,x,x)$$

THERE <u>EXISTS</u> A POLYMORPHISM m FOR WHICH THE EQUATION IS TRUE

POOREST POSSIBLE Pol(𝔸):

ALL PROJECTIONS ON $\{0,1\}$

$\pi(x_1 \dots x_n) = x_i$

$\mathrm{PROJ} = \mathrm{Pol}(\underbrace{\text{1-IN-3SAT}}_{(\{0,1\};\{(0,0,1),(0,1,0),(1,0,0)\})})$

$\mathrm{Pol}(\mathbb{A})$ TRIVIAL $:\Leftrightarrow$ ALL NON-NESTED EQUATIONS OF $\mathrm{Pol}(\mathbb{A})$ ARE TRUE IN PROJ

THEOREM (BULATOV, ZHUK '17)

$\mathbb{A}$ FINITE

$\mathrm{Pol}(\mathbb{A})$

- TRIVIAL → $\mathrm{CSP}(\mathbb{A})$ NP-c
- NON-TRIVIAL → $\mathrm{CSP}(\mathbb{A}) \in P$

THEOREM (MAROTI + MCKENZIE; BARTO + KOZIK; SIGGERS; BARTO + OPRŠAL + P. 2005)

$\mathbb{A}$ FINITE

$\mathrm{Pol}(\mathbb{A})$

- TRIVIAL → NO NON-NESTED EQUATIONS
- NON-TRIVIAL → $\mathrm{Pol}\,\mathbb{A}$ SATISFIES:
  - $w(x,\dots,x,y) = \dots = w(y,x,\dots,x)$

  AND
  - $c(x_1,\dots,x_n) = c(x_2,\dots,x_n,x_1)$

  AND
  - $s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$

  AND
  - $s(a,r,e,a) = s(r,a,r,e)$

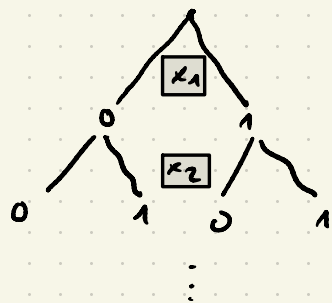
EQUATIONS = SYMMETRIES

EXAMPLE $\mathbb{A} = (\{0,1\}; \dots)$ $\quad \max(x,y) \in \mathrm{Pol}(\mathbb{A})$

CSP($\mathbb{A}$):

INSTANCE: VARIABLES $x_1, \dots, x_n$

+ CONSTRAINTS



EXPONENTIAL!

BUT BECAUSE <u>max</u> IS POLYMORPHISM ....

- CHECK <u>LOCALLY</u> IF $x_1 = 1, \dots x_n = 1$ POSSIBLE
  ↑ ON CONSTRAINTS
- IF NOT ⇒ REDUCED TO SMALLER INSTANCE WITH $x_i := 0$
- IF YES ⇒ $(1, \dots, 1)$ SOLUTION

... IN GENERAL: TOTALLY SYMMETRIC POLYMORPHISMS

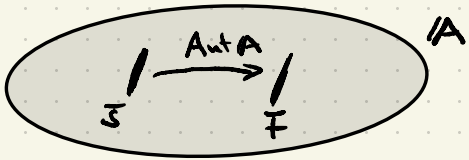
THIS DOES NOT WORK FOR INFINITE $\mathbb{A}$

MODEL THEORY

WANT: CONNECTION

$$\mathrm{CSP}(\mathbb{A}) \Leftrightarrow \mathrm{Pol}(\mathbb{A})$$

$\mathbb{A}$ ω-CATEGORICAL $:\Leftrightarrow$ $\mathbb{A}$ COUNTABLE & $\forall n \geq 1$ $\mathrm{Aut}\,\mathbb{A} \curvearrowright A^n$ HAS FINITELY MANY ORBITS



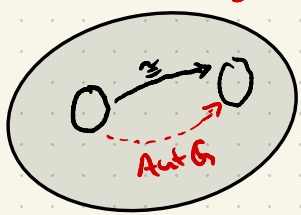
"$A^n / \mathrm{Aut}\,\mathbb{A}$ IS FINITE $\forall n \geq 1$"

CSP: SOLUTION SPACE IS INFINITE

<u>BUT</u> FOR EVERY GIVEN INSTANCE $(x_1 \dots x_n)$ IT IS "<u>FINITE</u>"!

EXAMPLES

- $(\mathbb{Q}, <)$
- RANDOM GRAPH = UNIVERSAL HOMOGENEOUS GRAPH G



- RANDOM POSET
- RANDOM BOOLEAN ALGEBRA = COUNTABLE ATOMLESS BOOLEAN ALGEBRA

⋮

+ ALL STRUCTURES DEFINABLE IN THEM:

E.G. $(\mathbb{Q}, B(x,y,z))$

$B(x,y,z) :\Leftrightarrow x<y<z \vee z<y<x$

$CSP(\mathbb{Q}, B(x,y,z))$:

GIVEN VARIABLES $x_1 \dots x_n$
+ CONSTRAINTS = SPECIFIED TRIPLES

Q: ARE THERE VALUES FOR $x_1 \dots x_n$ IN $\mathbb{Q}$ SUCH THAT ALL CONSTRAINTS SATISFY BETWEENNESS?

NP-COMPLETE

$\mathbb{A}$ DEFINABLE IN HOMOGENEOUS STRUCTURE WITH „REASONABLE DESCRIPTION" (FINITE BOUNDEDNESS)

$\Rightarrow CSP(\mathbb{A}) \in NP$

CONJECTURE MOM

IN THAT CASE:

$Pol(\mathbb{A})$

TRIVIAL $\rightarrow$ $CSP(\mathbb{A})$ NP-c

NON-TRIVIAL $\rightarrow$ $CSP(\mathbb{A}) \in P$

WHAT DOES "TRIVIAL" MEAN HERE?

$\mathrm{Pol}(\mathbb{A})$ TRIVIAL IF $\exists$ FINITE SET S A WHERE ALL NON-NESTED IDENTITIES SATISFIED ARE ALSO SATISFIED IN PROJ.

IN THAT CASE: $\mathrm{CSP}(\mathbb{A})$ is NP-c.

BARTO + OPRŠAL + P. '16

OTHERWISE: $\mathrm{Pol}(\mathbb{A}) \models e \circ s(x,y,x,z,y,z) = f \circ s(y,x,z,x,z,y)$

BARTO + P. '17

BARTO + KOMPATSCHER + OLŠÁK + PHAM + P. '18

METHODS:
- COMPACTNESS
- ORBIT GROWTH

WHY NO SOLUTION TO CONJECTURE?
- EQUATION NOT NICE
- THEORY BEHIND THE EQUATION NOT NICE
- FUNCTION s NOT NICE

RAMSEY THEORY

LET $f: \mathbb{Q} \to \mathbb{Q}$

$\Rightarrow \forall F \subseteq \mathbb{Q}$ FINITE $\exists F' \subseteq \mathbb{Q}$ FINITE $|F| = |F'|$, $f|_{F'}$ is INCREASING OR DECREASING

(RAMSEY'S THEOREM)

$\Rightarrow \overline{\mathrm{Aut}(\mathbb{Q},<) \circ f \circ \mathrm{Aut}(\mathbb{Q},<)}$ ← POINTWISE CLOSURE

CONTAINS DECREASING OR INCREASING FUNCTION

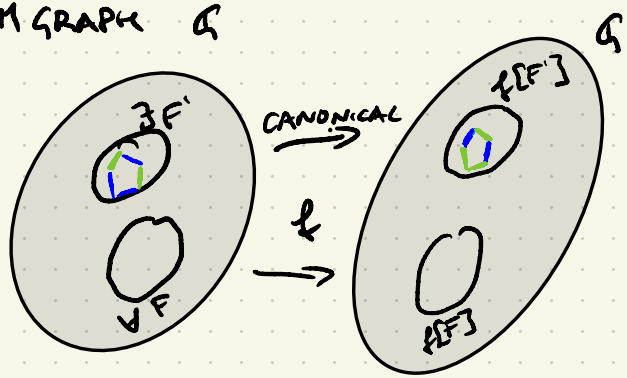
"SAME" FOR $f: \mathbb{Q}^n \to \mathbb{Q}$!

$\overline{\mathrm{Aut}(\mathbb{Q},<) \circ f \circ (\mathrm{Aut}(\mathbb{Q},<))^n}$

IF f is POLYMORPHISM OF $(\mathbb{Q},<)$ $\Rightarrow$ "GENERATES" CANONICAL POLYMORPHISM

ANOTHER EXAMPLE

RANDOM GRAPH G



$\mathrm{Pol}(\mathbb{A})$

TRIVIAL → $\mathrm{CSP}(\mathbb{A})$ NP-c

NON-TRIVIAL → $\mathrm{cons}(x,y,x,z,y,z) =$
$\mathrm{cons}(y,x,z,x,z,y)$

CANONICAL POLYMORPHISMS

TRIVIAL → ?

NON-TRIVIAL → $\mathrm{CSP}(\mathbb{A}) \in$ P

(BODIRSKY + MOTTET '17)

GOOD NEWS:

- CANONICAL POLYMORPHISMS ARE NOT RARE (RAMSEY THEORY) BODIRSKY + P. + TSANKOV '13
- ? OFTEN DOES NOT OCCUR "SMOOTH APPROXIMATIONS METHOD"

THEOREM (MOTTET + P. '20)

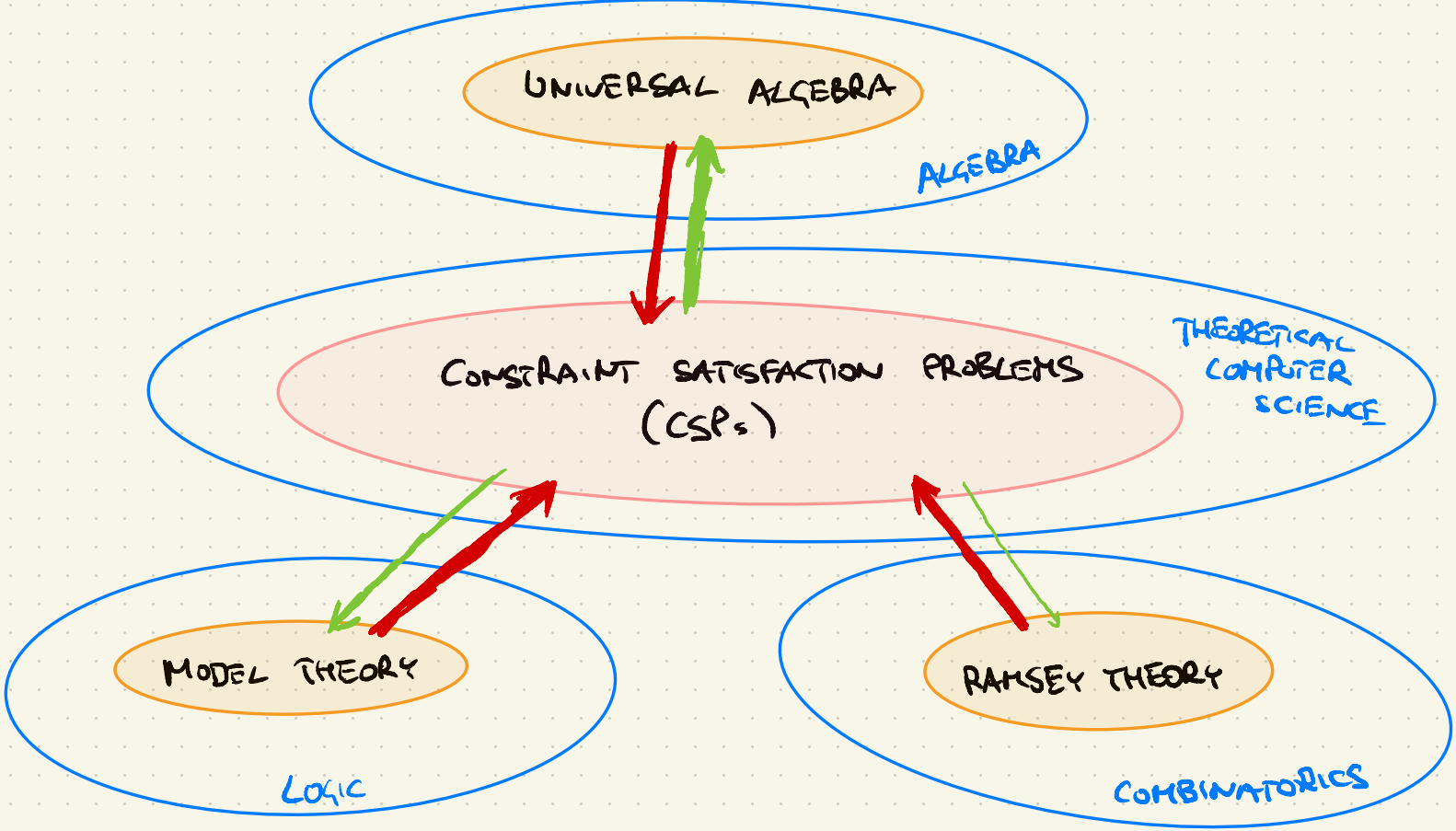
LET $\mathbb{A}$ BE DEFINABLE IN:

- $(\mathbb{Q},<)$
- RANDOM TOURNAMENT
- HOMOGENEOUS GRAPH
- RANDOM POSET
- ...

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in$ P OR NP-c

SUMMARY

- CSPs ARE COMMON COMPUTATIONAL PROBLEMS PARAMETRIZED BY RELATIONAL STRUCTURES $\mathbb{A}$
- SYMMETRY OF $\mathbb{A}$ AS MEASURED BY EQUATIONS OF POLYMORPHISM ALGEBRA DETERMINES COMPLEXITY IF $\mathbb{A}$ FINITE. STRUCTURAL DICHOTOMY.
- ALSO WORKS FOR ω-CATEGORICAL $\mathbb{A}$.
- USING RAMSEY THEORY, ONE ASSOCIATES A FINITE ALGEBRA (CANONICAL FUNCTIONS) WITH $\mathrm{Pol}(\mathbb{A})$.
- OFTEN, THIS YIELDS STRUCTURAL DICHOTOMY.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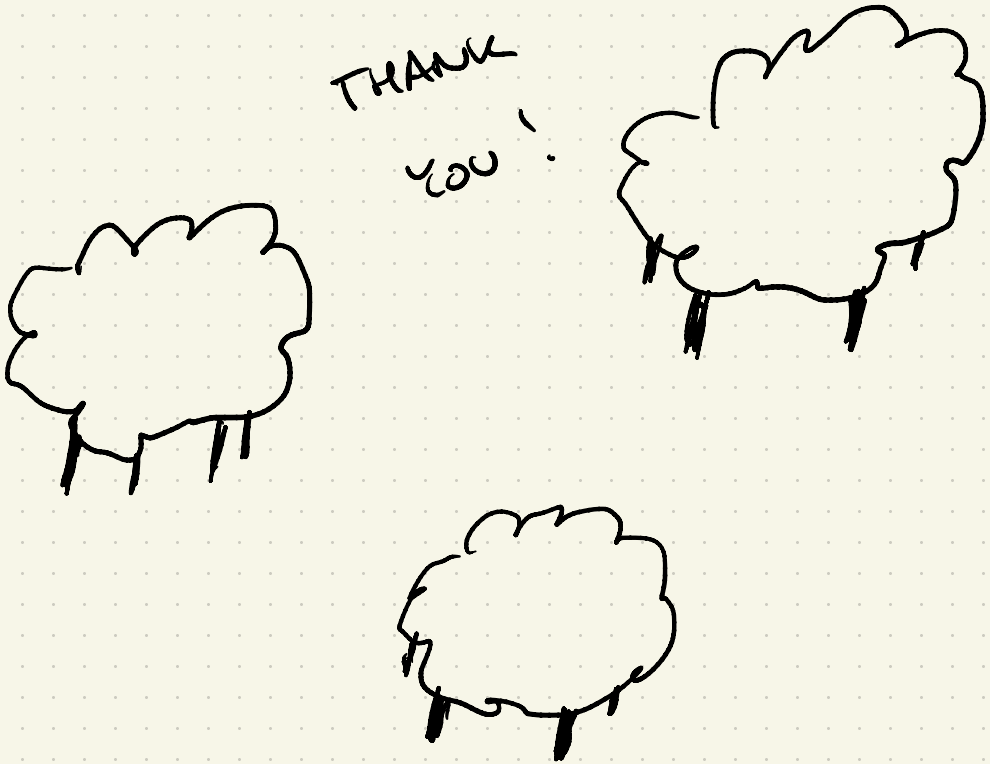
THANK YOU!