SURPRISES IN INFINITE-DOMAIN CONSTRAINT SATISFACTION

(AN ORDER OUT OF NOTHING)

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CONJECTURE (Bodirsky-P '11)

$\mathbb{U}$... FINITELY BOUNDED HOMOGENEOUS STRUCTURE

$\mathbb{A}$... FIRST-ORDER DEFINABLE IN $\mathbb{U}$

$\Rightarrow$ CSP($\mathbb{A}$) IN P OR NP-c

FINITE FORMULATION OF CSP($\mathbb{A}$):

- $\tau = \{R_1, \dots, R_s\}$ ... RELATIONAL LANGUAGE
- $\mathcal{F} = \{F_1, \dots, F_t\}$ ... FINITE "FORBIDDEN" τ-STRUCTURES
- $\mathcal{C} = \{C_1, \dots, C_u\}$ ... CONSTRAINT LANGUAGE: EACH C_i QUANTIFIER-FREE τ-FORMULA

Forb-CSP($\mathcal{F}, \mathcal{C}$)

INPUT

- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $C(x_{i_1}, \dots, x_{i_k})$, $C \in \mathcal{C}$

QUESTION

- POSSIBLE TO SATISFY CONSTRAINTS IN Forb($\mathcal{F}$)?
(STRUCTURE NOT CONTAINING FORBIDDEN SUBSTRUCTURE)

EXAMPLE

- $\tau = \{<\}$ BINARY SYMBOL
- Forb: $\mathcal{F} = $  : Forb = TOTAL ORDERS!
- CONSTRAINT LANGUAGE $\mathcal{C}$: $x<y$

Forb-CSP$(\mathcal{F}, \mathcal{C})$

GIVEN

- VARIABLES: $x_1 \dots x_n$
- CONSTRAINTS: $x_1 < x_2$, $x_2 < x_3$, $x_3 < x_1$

QUESTION

- SATISFIABLE IN Forb$(\mathcal{F})$ = A TOTAL ORDER ?

EXAMPLE

- $\tau = \{E\}$ BINARY
- Forb: $\mathcal{F} = $  UNDIRECTED GRAPHS WITHOUT LOOPS
- $\mathcal{C}$: $x \neq y$, $\varphi(x,y,z) := E(x,y) \wedge \neg E(y,z) \wedge \neg E(z,x)$

$$\vee\ \neg E(x,y) \wedge E(y,z) \wedge \neg E(z,x)$$

$$\vee\ \neg E(x,y) \wedge \neg E(y,z) \wedge E(z,x)$$

Forb-CSP$(\mathcal{F}, \mathcal{C})$

... ESSENTIALLY 1-IN-3SAT ON EDGES/NON-EDGES OF GRAPHS

CONNECTION WITH "NORMAL" CSP

EXAMPLE
- $\mathrm{Forb}(\mathcal{F})$ = TOTAL ORDERS
- $\mathcal{C} = \{x<y\}$

$\mathrm{Forb}\text{-}\mathrm{CSP}(\mathcal{F},\mathcal{C})$

GIVEN
- $x_1, \dots, x_n$
- CONSTRAINTS $x_1<x_2,\ x_2<x_3,\ x_3<x_1$

QUESTION
- SATISFIABLE IN TOTAL ORDER?

REDUCTION TO FINITE-DOMAIN CSP:

$\boxed{\mathrm{FinCSP}(\mathcal{F},\mathcal{C})}$

- FOR EACH PAIR $(x_i,x_j) \rightsquigarrow$ VARIABLE y_{ij}
- DOMAIN OF y_{ij}: $\{<,>,=\}$
- CONSTRAINTS $x_i<x_j \rightsquigarrow y_{ij}=\,<$ ETC.
- + CONSTRAINTS FROM Forb:

$$(y_{ij}=\,<)\wedge(y_{jk}=\,<)\to(y_{ik}=\,<)$$
$$(y_{ij}=\,=)\wedge(y_{jk}=\,=)\to(y_{ik}=\,=)$$
ETC.

$\mathrm{FinCSP}(\mathcal{F},\mathcal{C})$ CAN BE HARD EVEN IF $\mathrm{Forb}\text{-}\mathrm{CSP}(\mathcal{F},\mathcal{C})$ IS NOT

(REDUCTION PRODUCES VERY SPECIFIC INSTANCES)

TRUE REPRESENTATION AS $\mathrm{CSP}(\mathbb{A})$ (WITH ORIGINAL VARIABLES):

INSTANCE $x_1<\dots<x_n$ HAS SOLUTION $\forall n$

$\Rightarrow \mathbb{A}$ HAS TO BE INFINITE

$\mathrm{Forb}(\mathcal{F}) \rightsquigarrow \mathbb{U}$ UNIVERSAL FOR $\mathrm{Forb}(\mathcal{F})$

τ-STRUCTURE, COUNTABLE

"FINITELY BOUNDED"

CONSTRAINT LANGUAGE $\mathcal{C}$

$\rightsquigarrow$ RELATIONS DEFINED IN $\mathbb{U}$

$\rightsquigarrow$ STRUCTURE $\mathbb{A}$

$$\boxed{\mathrm{CSP}(\mathbb{A}) = \mathrm{Forb}\,\mathrm{CSP}(\mathcal{F},\mathcal{C})}$$

CONJECTURE (BODIRSKY-P '11)

$\mathbb{U}$... FINITELY BOUNDED HOMOGENEOUS STRUCTURE

$\mathbb{A}$... FIRST-ORDER DEFINABLE IN $\mathbb{U}$

$\Rightarrow$ CSP($\mathbb{A}$) IN P OR NP-C

- FOR CERTAIN Forb($\mathcal{F}$), THERE IS $\underbrace{\text{HOMOGENEOUS}}_{\text{" HIGHLY SYMMETRIC}}$ $\mathbb{U}$: FRAÏSSÉ: AMALGAMATION PROPERTY
- $\mathcal{F}$ CLOSED UNDER HOMOMORPHISMS $\Rightarrow$ GOOD $\mathbb{U}$ (CHERLIN-SHELAH-SHI)

PROBLEM OF MODELLING

HOW TO SOLVE Forb CSP($\mathcal{F}$, $\mathcal{E}$) = CSP($\mathbb{A}$)?

HEAVEN

BY REDUCTION TO FinCSP($\mathcal{F}$, $\mathcal{E}$) = FinCSP($\mathbb{A}$)

Forb($\mathcal{F}$) =
- UNDIRECTED GRAPHS
- " K_n-FREE GRAPHS
- EQUIVALENCE RELATIONS
- TOURNAMENTS
- UNARY STRUCTURES
- MMSNP
- RCC-5

HELL Forb($\mathcal{F}$) = TOTAL ORDERS

WHAT THE HELL IS WRONG WITH $CSP(\mathbb{Q},<)$ ($Forb(\mathcal{J})$ = TOTAL ORDERS)?

POLYMORPHISMS OF $FinCSP(\mathbb{Q},<) := FinCSP(\text{TOTAL ORDERS}, <)$:

= POLYMORPHISMS OF $(\mathbb{Q},<)$ WHICH SURVIVE REDUCTION

= " WHICH ACT ON DOMAIN $\{<,>,=\}$

= " f FOR WHICH $f(<,>,=)$ ETC. MAKES SENSE

= $\boxed{\text{CANONICAL POLYMORPHISMS } Pol^{CAN}(\mathbb{Q},<) \subseteq Pol(\mathbb{Q},<)}$

EXAMPLE $\max(x,y) : \mathbb{Q}^2 \to \mathbb{Q}$, $\in Pol(\mathbb{Q},<)$

- $\max(<,<)$:
  $$\begin{array}{c} x_1 < x_2 \\ y_1 < y_2 \\ \hline \end{array}$$
  max: $\cdot < \cdot$ ✓

  $\max(<,<) = <$

- $\max(<,>)$:
  $$\begin{array}{c} x_1 < x_2 \\ y_1 > y_2 \\ \hline \end{array}$$
  max: $\cdot\ ?\ \cdot$ ✗

  $\max(<,>)$ UNDEFINED

PROPOSITION $Pol^{CAN}(\mathbb{A})$ SATISFIES NO IDENTITIES WHEN $\mathbb{U}$ HAS TOTAL ORDER

PROOF IDEA: $Pol^{CAN}(\mathbb{A})$ SATISFIES CYCLIC IDENTITY $\Rightarrow$ LOOP IN ORDER

HEAVEN UNORDERED STRUCTURES

HELL ORDERED STRUCTURES

SMOOTH APPROXIMATIONS METHOD FOR STRUCTURES IN HEAVEN

$\mathbb{A}$ FO-DEFINABLE IN UNORDERED $\mathbb{U}$ (HEAVEN)

- ASSUME $\mathbb{A}$ DOES NOT PP-CONSTRUCT 3SAT (OTHERWISE CSP IS HARD)
- $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ CONTAINS $e \circ s(xyxzyz) = f \circ s(yxzxzy)$
- "$\overset{?}{\Rightarrow}$" $\mathrm{Pol}^{\mathrm{can}}(\mathbb{A})$ CONTAINS —"— : $\boxed{\text{SMOOTH APPROXIMATIONS}}$
- $\Rightarrow$ $\mathrm{FinCSP}(\mathbb{A}) \in P$
- $\Rightarrow$ $\mathrm{CSP}(\mathbb{A}) \in P$

FROM $\mathrm{Pol}(\mathbb{A})$ TO $\mathrm{Pol}^{can}(\mathbb{A})$ IN HEAVEN, OR:

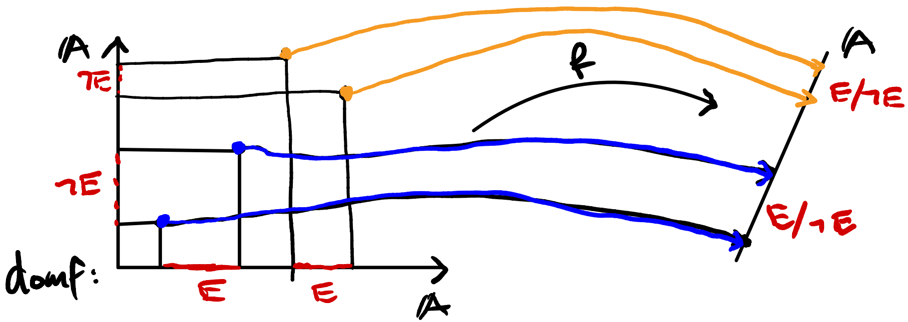
THE PACT WITH THE DEVIL

- WANT TO FIND CANONICAL BEHAVIOUR IN ARBITRARY $F \in \mathrm{Pol}(\mathbb{A})$

EXAMPLE: $\mathrm{Forb}(\mathcal{J})$ = GRAPHS $\Rightarrow$ $\mathbb{U}$: UNIVERSAL GRAPH ("RADO GRAPH / RANDOM GRAPH")

$\mathbb{A}$ ... FO-DEFINABLE IN $\mathbb{U}$

$f : \mathbb{A}^2 \to \mathbb{A}, \in \mathrm{Pol}(\mathbb{A})$



ALWAYS THE SAME!

- WANT F TO BEHAVE CANONICALLY ON BIG $\boxed{} \subseteq \mathbb{A}^2$ (RAMSEY?)
- IMPOSSIBLE: IMAGINE ORDER $<$ ON $\mathbb{A}$

  F COULD BEHAVE DIFFERENTLY ON $\cdot\ \cdot$ AND $\cdot\ \cdot$!
- ABSENCE OF ORDER = PROBLEM!

THE PACT WITH THE DEVIL II

CONVERSE: PRESENCE OF SUITABLE ORDER "$\Rightarrow$" CAN ALWAYS FIND CANONICAL BEHAVIOUR!

HEAVEN (NO ORDER) = HELL?
- CAN'T FIND CANONICAL BEHAVIOUR
$\Rightarrow$ CAN'T REDUCE TO FinCSP($\mathbb{A}$)

HELL (ORDER) = HEAVEN?
- CANONICAL POLYMORPHISMS CANNOT SATISFY IDENTITIES
$\Rightarrow$ CAN'T REDUCE TO FinCSP($\mathbb{A}$)

APPROACH (UNORDERED CASE)

- ADD ORDER
- FIND CANONICAL FUNCTIONS
- FORGET ORDER, DESTROY IT WITH POLYMORPHISMS MESSING IT UP
- OBTAIN CANONICAL FUNCTIONS (WITHOUT ORDER) SATISFYING IDENTITIES
- $\Rightarrow$ FinCSP($\mathbb{A}$) $\in$ P
- $\Rightarrow$ CSP($\mathbb{A}$) $\in$ P

BELIEVED TO BE ALWAYS POSSIBLE

"WHY SHOULD AN ORDER OUT OF NOTHING MATTER?"

"WHAT COULD GO WRONG WITH THE DEVIL?"

RECALL CSP($\mathbb{A}$) = Forb CSP($\mathcal{F}$, $\mathcal{C}$)

- INSTANCES $x_1 \ldots x_n$, CONSTRAINTS
- BY ADDING THE UNRELATED ORDER, WE DESCRIBE ARBITRARILY ORDERED SATISFIABLE INSTANCES
- ON META-LEVEL: WE INCREASE THE POWER OF OUR "LOGIC"
- "LOGIC" INCOMPATIBLE WITH IDENTITIES

HYPERGRAPHS, OR: THE PACT WITH THE DEVIL III

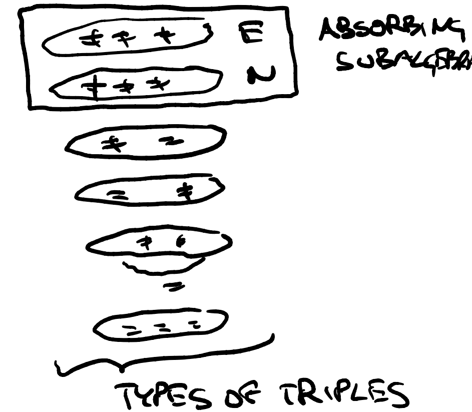
THEOREM $\mathbb{H}$ … k-UNIFORM HYPERGRAPH, $k \geq 3$
$\mathbb{A}$ … FO-DEFINABLE IN $\mathbb{H}$

MOTTET, NAGY '23, P.

$\Leftrightarrow$ $\mathrm{Pol}(\mathbb{A}) \models$ $e \circ s(xyxzyz) = f \circ s(yxzxzy)$ (s NOT CANONICAL)

- $\mathrm{Pol}(\mathbb{A})$ HAS INJECTIVE f ACTING AS $\vee$, $\wedge$, MAJORITY ON $\{E, N\}$, $\mathrm{CSP}(\mathbb{A})$ HAS BOUNDED WIDTH, OR
- … MINORITY ON $\{E, N\}$, $\mathrm{CSP}(\mathbb{A}) \in P$, OR
- $\mathrm{CSP}(\mathbb{A})$ NP-c

DOMAIN OF FinCSP($\mathbb{A}$): ($k=3$)



- NEEDS ALGORITHMS / REDUCTION TO FinCSP($\mathbb{A}$) NOT POSSIBLE: **OPENING BLACKBOX**
- POLYMORPHISMS DEPEND ON ORDER (SURPRISE!)
- USES SMOOTH APPROXIMATIONS
  NOT POSSIBLE WITH OLD METHODS (# CASES UNBOUNDED)

THANK YOU :

- TO THE AUDIENCE FOR YOUR ATTENTION
- ALL UNCITED AUTHORS FOR THEIR WORK

  BARTO, KOZIK, BODIRSKY, TSANKOV, BODOR, KÁRA, WRONA, MADELAINE, KOMPATSCHER, VAN PHAM, ZHUK, BULATOV, CHERLIN, SHELAH, SHI, FRAÏSSÉ, MARTIN, OLŠÁK, MOTTET, NAGY

- THE DEVIL FOR HIS ASSISTANCE