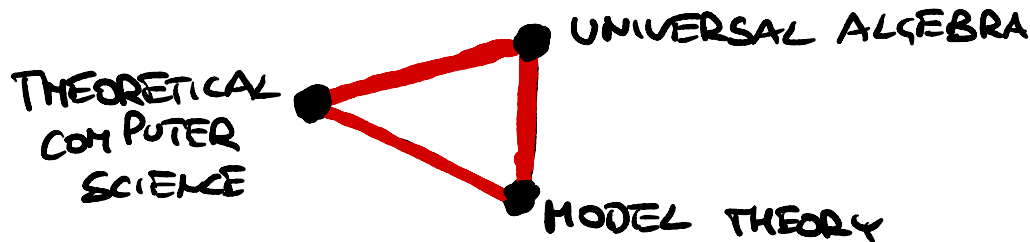


THE POWER OF POLYMORPHISMS



MICHAEL PINSKER

TU WIEN

FWF Austrian
Science Fund

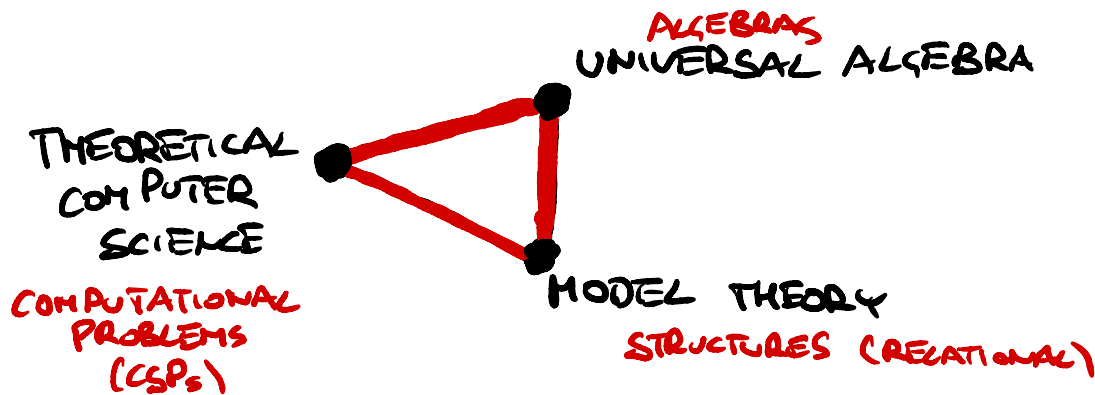


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PART I

THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II

INFINITE-DOMAIN CSPs MODELLING PROBLEMS
MATHEMATICS

PART I : THE MATHEMATICS OF FINITE-DOMAIN CSPs

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$
SATISFYING ALL CONSTRAINTS
?

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EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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- SUDOKU
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- SOLVING EQUATIONS

MODEL :

- SET V OF POSSIBLE VALUES (FIXED)
E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...
- ALLOWED CONSTRAINTS (FIXED)
 $C_1, \dots, C_m$ RELATIONS ON V
 $C_i \in V^{d_i}$

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 $C_i \in V^{d_i}$

SO $\mathcal{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE

CSP($\mathcal{A}$)

= "TEMPLATE"

GIVEN

- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $C_{i_1}(\text{VARIABLES})$
 $C_{i_2}(\text{VARIABLES})$
 $\vdots$

QUESTION

- $\exists$ SOLUTION $x_i \mapsto a_i \in A$?

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$\exists$ SOLUTION $x_i \mapsto a_i \in A$?

ALTERNATIVE: GIVEN PP-SENTENCE

$\varphi \equiv \exists x_1 \dots \exists x_n C_{i_1}(\text{VAR}) \wedge \dots \wedge C_{i_r}(\text{VAR})$

QUESTION: $\mathcal{A} \models \varphi$?

PP... PRIMITIVE
POSITIVE

CONSTRAINT SATISFACTION PROBLEM CSP

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- SCHEDULING
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MODEL:

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E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...
- ALLOWED CONSTRAINTS (FIXED)
 $C_1, \dots, C_m$ RELATIONS ON V
 $C_i \subseteq V^{\delta_i}$

SO $\langle A := (V, C_1, \dots, C_m) \rangle$ RELATIONAL STRUCTURE

CSP(A)

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PP... PRIMITIVE
POSITIVE

META-QUESTION

FOR WHAT $\langle A \rangle$ IS CSP(A) EASY/HARD?

P:

$\exists$ ALGORITHM PROVIDING ANSWER
IN $p(\# \text{ VARIABLES})$ STEPS, p POLYNOMIAL

NP:

NOT NECESSARILY IN P BUT VERIFYING
"SOLUTION" IN P

EXAMPLE

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$

TERNARY RELATIONS



CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$

$$x_2 + x_1 = x_3$$

$$x_3 = 1$$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

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UNDECIDABLE

(MATYASEWIC '77)

HILBERT'S 10TH PROBLEM

EXAMPLE

• $A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$

TERNARY RELATIONS
↙ ↘

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• SAME OVER $\mathbb{Z}_p$

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NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

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IN P (GAUSS)

$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

• SOLUTION IN $\mathbb{Q}$?

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$
 $x_2 + x_1 = x_3$
 $x_3 = 1$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIASSEMIC '77)

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NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

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IN P (GAUSS)

$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

• SOLUTION IN $\mathbb{Q}$?



EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$
 $x_2 + x_1 = x_3$
 $x_3 = 1$

QUESTION

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UNDECIDABLE (MATIASSEMIC '77)

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CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

• SOLUTION IN $\mathbb{Q}$?



IN P

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↓
↓

CSP(A)

GIVEN

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- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

- SOLUTION IN $\mathbb{Q}$?



IN P

$$\bullet A = \mathbb{R}_3 = \begin{matrix} & 0 \\ & \bullet \\ \bullet & & \bullet \\ 2 & & 1 \end{matrix} = (\{0, 1, 2\}, E)$$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $E(x_1, x_2)$
 $E(x_2, x_3)$
 $\vdots$

QUESTION

- SOLUTION IN $\{0, 1, 2\}$?

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↓
↓

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 + x_1 = x_2$
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QUESTION

- SOLUTION IN $\mathbb{Z}$?

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- SAME OVER $\mathbb{Z}_p$

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↳ IF IN P $\Rightarrow$ P = NP

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$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

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- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
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 $\vdots$

QUESTION

- SOLUTION IN $\mathbb{Q}$?



IN P

$$\bullet A = \mathbb{R}_3 = \begin{matrix} & 0 \\ & \bullet \\ \bullet & \text{---} & \bullet \\ 2 & & 1 \end{matrix} = (\{0, 1, 2\}, \mathcal{E})$$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $\mathcal{E}(x_1, x_2)$
 $\mathcal{E}(x_2, x_3)$
 $\vdots$

QUESTION

- SOLUTION IN $\{0, 1, 2\}$?

3-COLORING PROBLEM = NP-COMPLETE

ALTERNATIVE FORMULATION OF CSP(A):

GIVEN: FINITE B

QUESTION: $B \rightarrow A$?, i.e.

$\exists h: B \rightarrow A$ HOMOMORPHISM?

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"SAME" PROBLEM:

• PP-SENTENCE $\varphi = \exists x_1 \dots x_n C_1(x_1 \dots x_n) \wedge \dots \wedge C_k(x_1 \dots x_n)$

$\leadsto$ STRUCTURE: DOMAIN $\{x_1 \dots x_n\}$

B_e RELATIONS C_i GIVEN
BY CONSTRAINTS
"CANONICAL DATABASE"

THEN $B_e \rightarrow A \iff A \models \varphi$

ALTERNATIVE FORMULATION OF CSP(A):

GIVEN: FINITE IB

QUESTION: $IB \rightarrow A?$, i.e.

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"SAME PROBLEM":

- PP-SENTENCE $\varphi \equiv \exists x_1 \dots x_n C_1 \wedge \dots \wedge C_k()$

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IB_e RELATIONS C_i GIVEN
BY CONSTRAINTS
"CANONICAL DATABASE"

THEN $IB_e \rightarrow A \iff A \models \varphi$

- STRUCTURE $IB = (B, C_1, \dots, C_n)$

$\leadsto$ $\{b_1 \dots b_k\}$

$\varphi_{IB} \equiv \exists b_1 \dots b_k \wedge C_i()$

"CANONICAL QUERY OF IB"

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"CANONICAL QUERY OF B "

EXAMPLE

$A = K_2 = \text{!}$

GIVEN

DIGRAPH B

QUESTION

$B \rightarrow K_2$?

2-COLORING PROBLEM

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EXERCISE

$CSP(Q, <) = ?$

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"CANONICAL QUERY OF IB"

EXAMPLE

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DIGRAPH IB

QUESTION

$IB \rightarrow K_2?$

2-COLORING PROBLEM

EXERCISE

$CSP(Q, <) = ?$

COMPUTATIONAL COMPLEXITY OF
 $CSP(A)$ CAN BE:

- UNDECIDABLE
- IN P , NP-COMPLETE
- ANYTHING:

EVERY COMPUTATIONAL PROBLEM
IS POLYNOMIAL-TIME
TURING-EQUIVALENT TO SOME
 $CSP(A)$ (RUBINFELD + CROWE '06)



IA FINITE $\Rightarrow CSP(A)$ IN NP

THEOREM

(BOGATOV, ZHUK '17)

(CONJECTURE FEDER+ VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow \text{CSP}(\mathbb{A}) \in \text{P}$ OR

NP-COMPLETE

THEOREM

(BOGATOV, ZHUK '17)

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A FINITE

$\Rightarrow \text{CSP}(A) \in P$ OR

NP-COMPLETE



THEOREM

(BOGATOV, ZHUK '17)

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$\mathcal{A}$ FINITE

$\Rightarrow \text{CSP}(\mathcal{A}) \in \text{P}$ OR

NP-COMPLETE



$(\text{P} \neq \text{NP})$

$\text{CSP}(\mathcal{A}) \in \text{P} \Leftrightarrow \mathcal{A}$ HAS ALGEBRAIC
INVARIANT

$$S(x, y, x, z, y, z) =$$

$$S(y, x, z, x, z, y)$$

THEOREM

(BOGATOV, ZHUK '17)

(CONJECTURE FEDER + VARDI '93)

$\mathbb{A}$ FINITE

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NP-COMPLETE



$(\text{P} \neq \text{NP})$

$\text{CSP}(A) \in \text{P} \Leftrightarrow A$ HAS ALGEBRAIC
INVARIANT
 $S(x, y, x, z, y, z) =$
 $S(y, x, z, x, z, y)$



ALGEBRAIC INVARIANTS = POLYMORPHISMS

$A = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^r \rightarrow A$ POLYMORPHISM: $\Leftrightarrow$

f HOMOMORPHISM $A^r \rightarrow A \Leftrightarrow$

$\forall: \forall \bar{r}_1, \dots, \bar{r}_r \in R_i$

$$f(\overset{r_1}{\underset{|}{\text{O}}} \dots \overset{r_r}{\underset{|}{\text{O}}}) \in R_i$$

THEOREM

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A FINITE

$\Rightarrow \text{CSP}(A) \in P \text{ OR}$
NP-COMPLETE



$(P \neq NP)$

$\text{CSP}(A) \in P \Leftrightarrow A$ HAS ALGEBRAIC
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 $S(x, y, x, z, y, z) =$
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ALGEBRAIC INVARIANTS = POLYMORPHISMS

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f HOMOMORPHISM $A^r \rightarrow A \Leftrightarrow$

$\forall: \forall \bar{r}_1, \dots, \bar{r}_e \in R_i$

$$f\left(\begin{array}{c} \vdots \\ \bar{r}_1 \end{array} \dots \begin{array}{c} \vdots \\ \bar{r}_e \end{array}\right) \in R_i$$

$\text{POL}(A) := \{f \mid f \text{ POLYMORPHISM OF } A\}$

- CONTAINS PROJECTIONS $(x_1, \dots, x_e) \mapsto x_i$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS
OF AN ALGEBRA ON A !

EXAMPLE

- $\min(x, y) : \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

EXAMPLE

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$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{1, 1\}, \{1\})$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$$

 $\in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{1, 1\}, +)$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{F} \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{F} \quad = \quad \begin{pmatrix} \end{pmatrix} \in \mathbb{F}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑
NOT GREAT 😞

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

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 $\in \bar{+} \quad \in \bar{+} \quad \in \bar{+} \quad \in \bar{+}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOSTOV + ROMOV '01)
 GEISER '06

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \xrightarrow{①} A \text{ PP-DEFINES } B$
 $\xrightarrow{②} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

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- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, +, -, 0, 1, \bar{1})$

- $0 - 0 + 0 = 0$

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- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{F} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{F} = \begin{pmatrix} \end{pmatrix} \in \mathbb{F}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOTIGUT ROMOV '91 GEISER '96)

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③ PROOF

EXAMPLE

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SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

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(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOTIGUT ROMOV '91 GEISER '96)

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 $\xrightarrow{\text{Q}} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

③ PROOF:

LET $\varphi \equiv \exists x_1 \dots x_n C_{i_1}(\gamma) \wedge \dots \wedge C_{i_r}(\gamma)$

BE AN INSTANCE OF $\text{CSP}(B)$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION IN A
 $\leadsto \tilde{\varphi}$!

$$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$$

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

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SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM (BODNARČUK + GALUŽIN + KOSTOV + ROMOV '01)
GEISER '06

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IN A
 $\leadsto \tilde{\varphi}$!

$$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$$

④ PROOF

EXAMPLE

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 $\in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM (BODNARČUK + GALUŽNIN + KRIVONTSEV '04)
 ROMOV '04
 GEISER '06

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③ PROOF:

LET $\varphi \equiv \exists x_1, \dots, x_n C_{i_1}(\gamma) \wedge \dots \wedge C_{i_r}(\gamma)$

BE AN INSTANCE OF $\text{CSP}(B)$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION
 IN A
 $\leadsto \tilde{\varphi}$!

$$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$$

④ PROOF:

LET R BE RELATION

SHOW: R INVARIANT UNDER $\text{Pol}(A)$

$\Leftrightarrow$

R PP-DEFINABLE IN A

R INVARIANT UNDER $\text{POL}(A)$



R PP-DEFINABLE IN A

R INVARIANT UNDER $\text{POL}(A)$



R PP-DEFINABLE IN $1A$



EASY: $S(x_1, \dots, x_k), T(y_1, \dots, y_n)$ INVARIANT

$\Rightarrow S \wedge T$ INVARIANT,

$\exists x, S(x_1, \dots, x_k)$ INVARIANT

R INVARIANT UNDER $\text{POL}(A)$



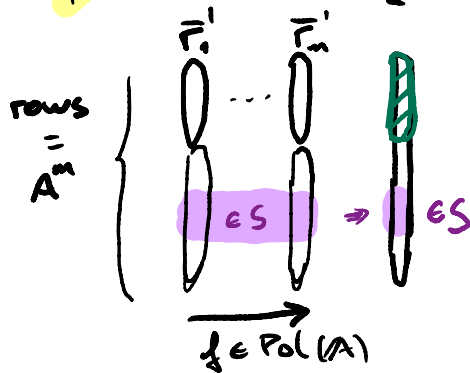
R PP-DEFINABLE IN A

← " EASY: $S(x_1, \dots, x_k), T(y_1, \dots, y_n)$ INVARIANT

$\Rightarrow S \wedge T$ INVARIANT,

$\exists x_i, S(x_1, \dots, x_k)$ INVARIANT

→ " LET $R = \{F_1, \dots, F_m\}$ (A FINITE!)



$R' := \{f(F_1, \dots, F_m) \mid f \in \text{POL}(A)\}$

R' IS $\{ \wedge, = \}$ -DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

□

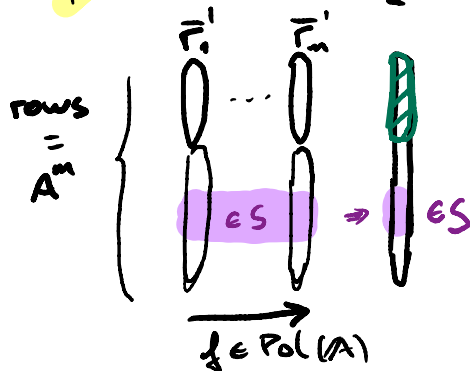
R INVARIANT UNDER $\text{Pol}(A)$



R PP-DEFINABLE IN A

EASY: $S(x_1, \dots, x_k), T(y_1, \dots, y_n)$ INVARIANT
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(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOSTOV + ROMOV '69)
GEISER '68

A, B FINITE, SAME DOMAIN

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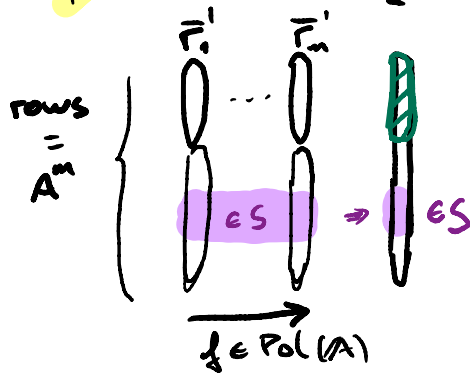
R INVARIANT UNDER $\text{POL}(A)$



R PP-DEFINABLE IN A

$\Leftarrow$ EASY: $S(x_1, \dots, x_k), T(y_1, \dots, y_n)$ INVARIANT
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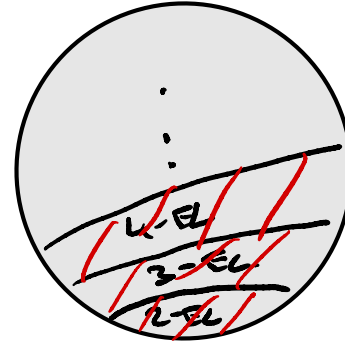
(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOSTOV + ROMOV '69) GEISER '68

A, B FINITE, SAME DOMAIN

$\text{POL}(A) \subseteq \text{POL}(B) \xrightarrow{①} A$ PP-DEFINES B

$\xrightarrow{②} \text{CSP}(B)$ REDUCES TO $\text{CSP}(A)$

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEFINABILITY



□

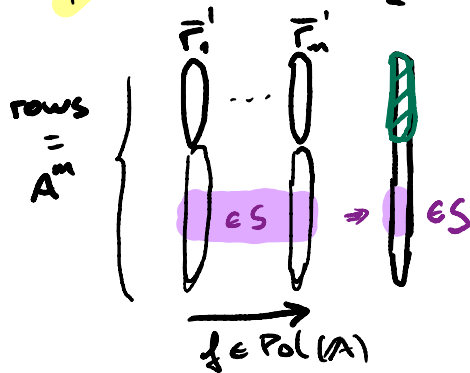
R INVARIANT UNDER $\text{Pol}(A)$



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 $\Rightarrow S \wedge T$ INVARIANT,
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$\rightarrow$ " LET $R = \{F_1, \dots, F_m\}$ (A FINITE!)



$R' := \{f(F_1, \dots, F_m) \mid f \in \text{Pol}(A)\}$

R' IS $\{1, =\}$ -DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

□

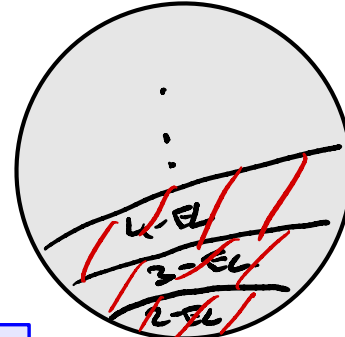
(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOSTOV + ROMOV '69)
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POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEFINABILITY



THEOREM (POST '41)

A BOOLEAN $\Rightarrow \text{Pol}(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJ
- MIN
- MINORITY

OR $\text{Pol}(A) = \{ \text{"UNARY" INJECTIONS} \}$

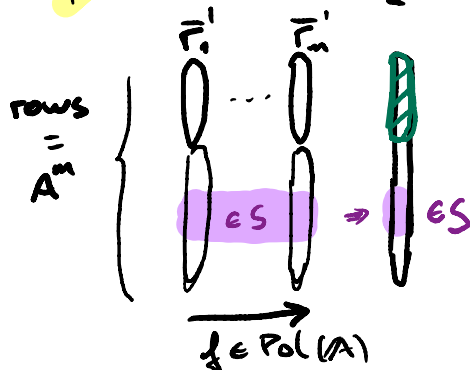
R INVARIANT UNDER $\text{Pol}(A)$



R PP-DEFINABLE IN A

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□

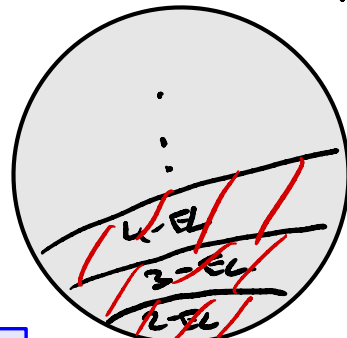
(BABY) THEOREM (BOUVARÉUR + KALWÉMIN + HOSIUVT ROMOV '69)
GEISER '68

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THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

THEOREM (POST '41)

$\{A\}$ BOOLEAN $\Rightarrow$ $\text{POL}(A)$ CONTAINS:

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THEOREM (SCHAEFER '78) COMPLEXITY DICOTOMY

HARDNESS $\text{NAE} = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

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ALGORITHMS

THEOREM (POST '41)

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- MINORITY

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THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $NAE = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

THEOREM (POST '41)

IF A BOOLEAN $\Rightarrow$ $POL(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $POL(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $NAE = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

- MINORITY: $m(x, x, y) = m(x, y, x) = m(y, x, x) = y$

$x - y + z$!

INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ POL(A) CONTAINS:

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THEOREM (SCHAEFER '78) COMPLEXITY DICOTOMY

HARDNESS $\text{NAE} = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

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INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

- MIN (AND DUALY MAX)

INVARIANT REL. HAVE SMALLEST TUPLE

SET EACH VARIABLE TO 0 UNLESS
SOME CONSTRAINT REQUIRES 1

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ POL(A) CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \frac{1}{2}$ "UNARY" INJECTIONS

THEOREM (SCHAEFER '78) COMPLEXITY DICOTOMY

HARDNESS $\text{NAE} = \frac{1}{2}(x_1, x_2, x_3)$ (NOT ALL EQUAL)

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

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INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

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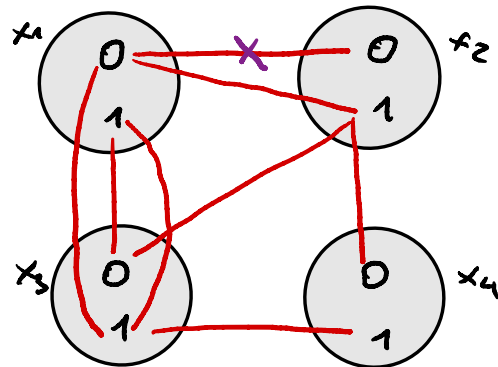
- MAJORITY $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$

LOCAL CONSISTENCY ALGORITHM

(2,3)-MINIMALITY

• KEEP LISTS OF POSSIBLE VALUES FOR
PAIRS OF VARIABLES (EDGES)

• REMOVE EDGES LOOKING AT CONSISTENCY
OF THE LISTS FOR 3 VARIABLES AT A
TIME



THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ POL(A) CONTAINS:

- CONSTANT
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THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $\text{NAE} = \frac{1}{2}(x_1, x_2, x_3) \text{ (NOT ALL EQUAL)}$

ALGORITHMS

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SET ALL VARIABLES OF INSTANCE TO 0

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INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

- MIN (AND DUALY MAX)

INVARIANT REL. HAVE SMALLEST TUPLE

SET EACH VARIABLE TO 0 UNLESS
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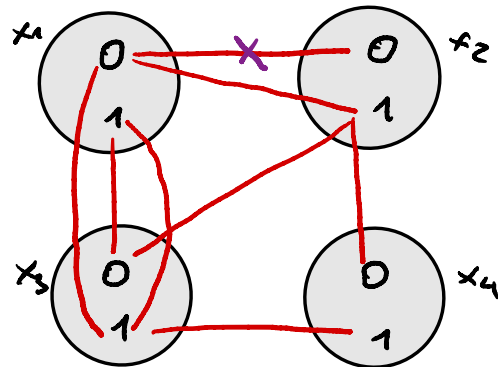
- MAJORITY $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$

LOCAL CONSISTENCY ALGORITHM

(2,3)-MINIMALITY

• KEEP LISTS OF POSSIBLE VALUES FOR
PAIRS OF VARIABLES (EDGES)

• REMOVE EDGES LOOKING AT CONSISTENCY
OF THE LISTS FOR 3 VARIABLES AT A
TIME



EXERCISE

SHOW LOCAL CONSISTENCY SOLVES CSP
CORRECTLY IF SOLUTIONS INVARIANT
UNDER MAJORITY

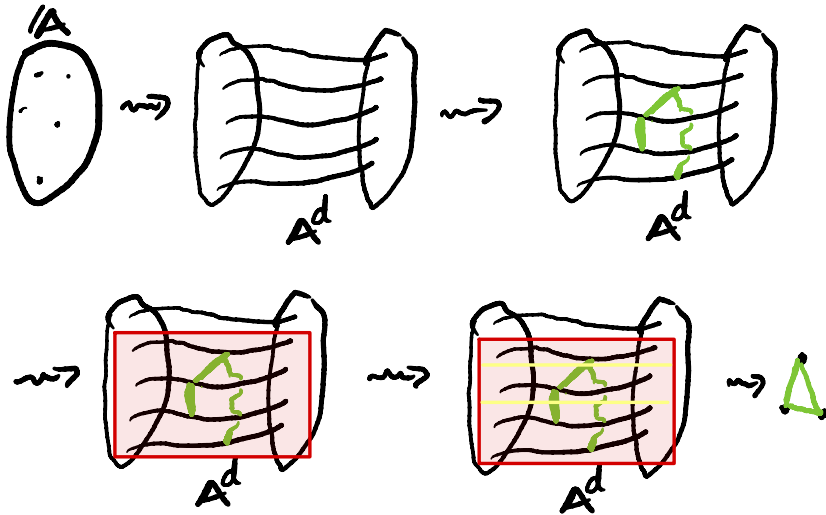
 PP-DEFINABILITY REQUIRES EQUAL
DOMAINS

 PP-DEFINABILITY REQUIRES EQUAL
DOMAINS

DEFINITION

1A PP-INTERPRETS 1B $\Leftrightarrow$

1B IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF A

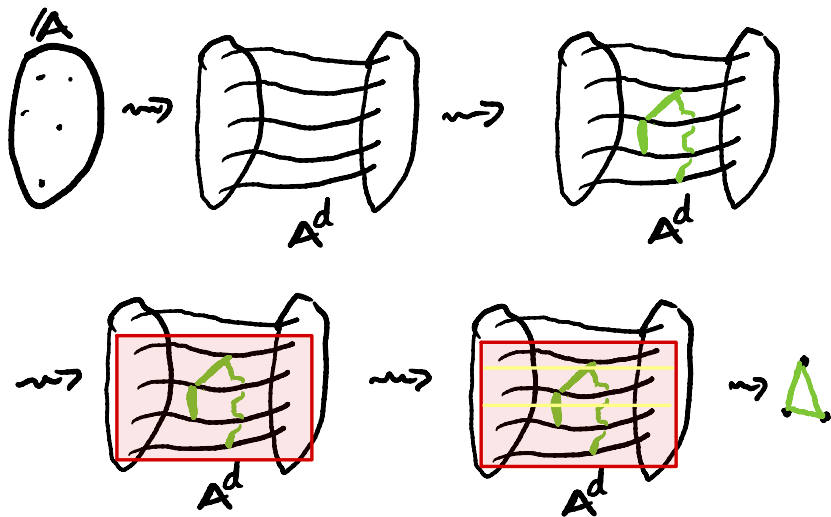


PP-DEFINABILITY REQUIRES EQUAL DOMAINS!

DEFINITION

$\mathcal{A}$ PP-INTERPRETS $\mathcal{B}$ $\Leftrightarrow$

$\mathcal{B}$ IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF $\mathcal{A}$



EXAMPLE

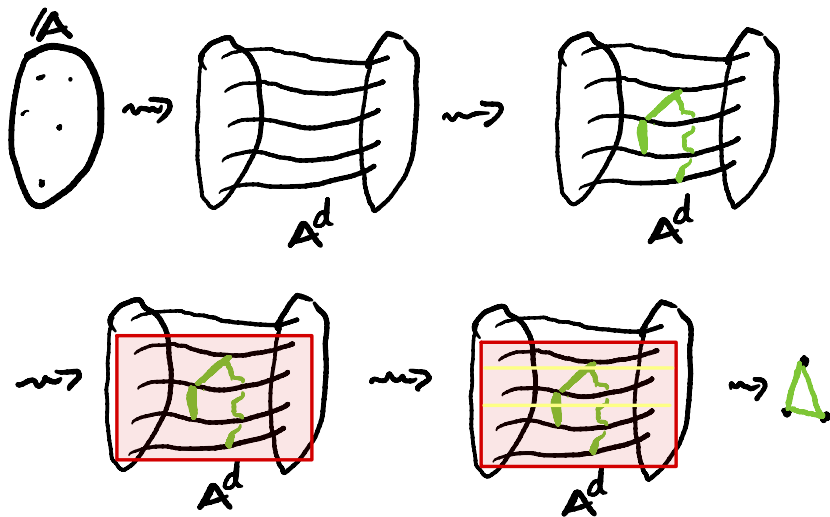
$$\begin{aligned}
 (\mathbb{Z}, +, \cdot, 0, 1) &\rightsquigarrow (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 0), (1, 1)) \rightsquigarrow \\
 &\rightsquigarrow (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\
 &\rightsquigarrow (\mathbb{Q}, +, \cdot, 0, 1)
 \end{aligned}$$

 PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

$\mathcal{A}$ PP-INTERPRETS $\mathcal{B} \Leftrightarrow$

$\mathcal{B}$ IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF $\mathcal{A}$



EXAMPLE

$$\begin{aligned}
 (\mathbb{Z}, +, \cdot, 0, 1) &\rightsquigarrow (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 0), (1, 1)) \rightsquigarrow \\
 &\rightsquigarrow (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\
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 \end{aligned}$$

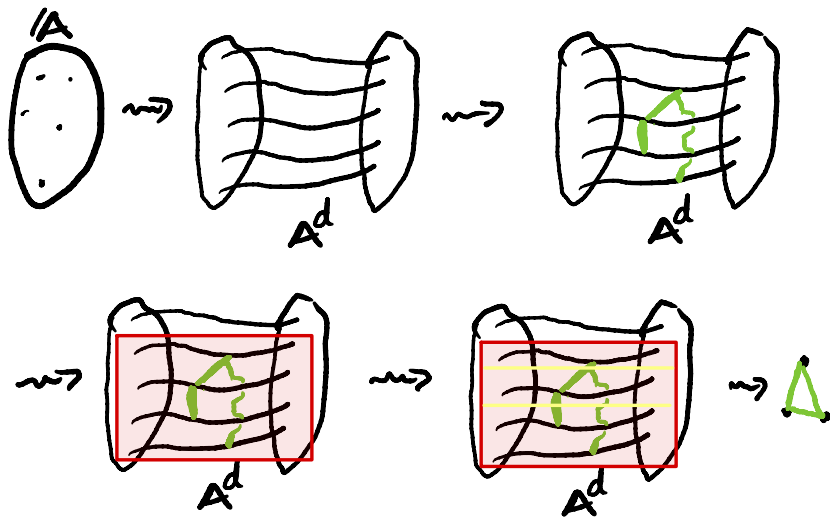
EXERCISE • SHOW THIS IS A PP-INTERPRETATION
• PP-INTERPRET $(\mathbb{Q}, <)$



DEFINITION

1A PP-INTERPRETS 1B $\Leftrightarrow$

B IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF A



EXAMPLE

$$\begin{aligned} (\mathbb{Z}, +, \cdot, 0, 1) &\leadsto (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 1), (1, 1)) \leadsto \\ &\leadsto (\mathbb{Z} \times (\mathbb{Z} \setminus \{0, 1\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\ &\leadsto (\mathbb{Q}, +, \cdot, 0, 1) \end{aligned}$$

EXERCISE • SHOW THIS IS A PP-INTERPRETATION
• PP-INTERPRET (Q. 4)

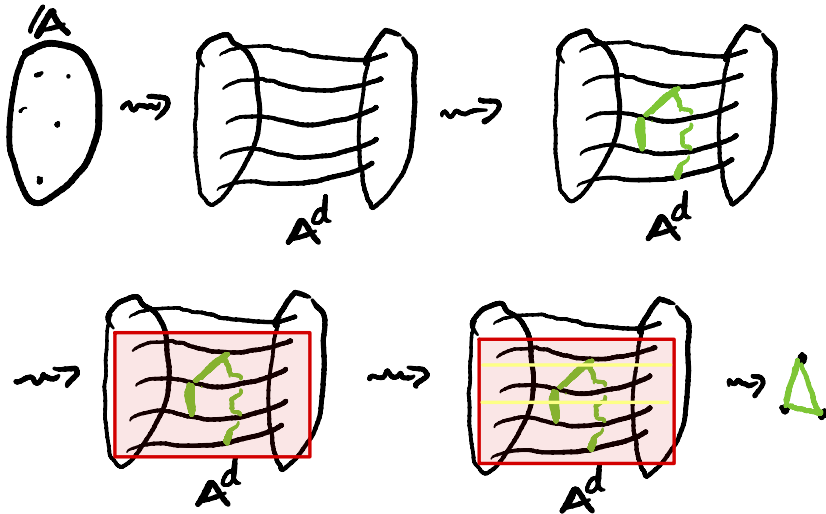
POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

PP-DEFINABILITY REQUIRES EQUAL DOMAINS!

DEFINITION

1A PP-INTERPRETS **1B** $\Leftrightarrow$

1B IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF **A**



EXAMPLE

$$\begin{aligned} (\mathbb{Z}, +, \cdot, 0, 1) &\leadsto (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 0), (1, 1)) \leadsto \\ &\leadsto (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\ &\leadsto (\mathbb{Q}, +, \cdot, 0, 1) \end{aligned}$$

EXERCISE • SHOW THIS IS A PP-INTERPRETATION
• PP-INTERPRET $(\mathbb{Q}, <)$

POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

EQUATIONAL CONDITIONS:

- $\exists f \in \text{Pol}(\mathbb{Q}, <) \forall x, y \quad f(x, y) = f(y, x)$
- $\exists f \in \text{Pol}(\mathbb{Z}, \{0\}, \{1\}, +)$
 $\forall x, y, z \quad f(x, x, y) = f(y, x, x) = y$
- $\exists f \in \text{Pol}(K_3) : \forall x \quad f f f(x) = x$

DEFINITION

- EQUATIONAL (STRONG MAL'CEV) CONDITION:
SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$

$$s_1(\text{variables}) = t_1(\text{variables})$$

$\wedge$

$\vdots$

$$\wedge s_n(\text{variables}) = t_n(\text{variables})$$

WHERE $s_i, t_i \dots$ TERMS OVER $f_1, f_2, \dots$

DEFINITION

- EQUATIONAL (STRONG MAL'CEV) CONDITION:

SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$

$$s_1(\text{variables}) = t_1(\text{variables})$$

$\wedge$

$\vdots$

$$\wedge s_k(\text{variables}) = t_k(\text{variables})$$

WHERE $s_i, t_i, \dots$ TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL: $\Leftrightarrow \text{POL (ANY STRUCTURE)} \models \Psi$
 $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS

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- $\text{EQ}(\text{POL}(A)) := \{\Psi \mid \text{POL}(A) \models \Psi\}$

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(ADOLESCENT) THEOREM

(BULATOV + JEAVONS + KRACHIN 1995)

$$\text{EQ}(\text{Pol}(A)) \subseteq \text{EQ}(\text{Pol}(B))$$

$\stackrel{Q}{\Rightarrow} A$ PP-INTERPRETS B

$\stackrel{Q}{\Rightarrow} \text{CSP}(B)$ REDUCES TO $\text{CSP}(A)$

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Q PROOF:

DEFINITION

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Q PROOF: AS WITH PP-DEFINITIONS

GIVEN INSTANCE OF $\text{CSP}(B)$:

$$\varphi \equiv \exists x_1 \dots x_n C_1(\gamma) \wedge \dots \wedge C_m(\gamma)$$

- FOR EACH OCCURENCE OF x_i CREATE NEW VARIABLES $x_i^1 \dots x_i^d$ ($d = \text{POWER}$)
- ADD CONSTRAINTS THAT ALL TUPLES $(x_i^1 \dots x_i^d)$ BE EQUIVALENT
- ADD CONSTRAINTS RESTRICTING $(x_i^1 \dots x_i^d)$ TO SUBSET
- TRANSLATE CONSTRAINTS ON x_i INTO CONSTRAINTS ON x_i^d

DEFINITION

- EQUATIONAL (STRONG MAL'CEV) CONDITION: SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$

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WHERE $s_i, t_i, \dots$ TERMS OVER $f_1, f_2, \dots$

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 $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS
- $\text{EQ}(\text{POL}(A)) := \{\Psi \mid \text{POL}(A) \models \Psi\}$

(ADOLESCENT) THEOREM (BULATOV + JEAVONS + KRAHIN 100)

$$\text{EQ}(\text{POL}(A)) \subseteq \text{EQ}(\text{POL}(B))$$

$$\stackrel{Q}{\Rightarrow} A \text{ PP-INTERPRETS } B$$

$$\stackrel{Q}{\Rightarrow} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$$

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EXERCISE

CHECK THIS WORKS

① PROOF SKETCH OF:

$$EQ(Pol(A)) \subseteq EQ(Pol(B))$$

$\stackrel{Q}{\Rightarrow}$ A PP-INTERPRETS B

① PROOF SKETCH OF:

$$\text{EQ}(\text{Pol}(A)) \subseteq \text{EQ}(\text{Pol}(B))$$

$\Rightarrow A$ PP-INTERPRETS B

• $\exists \varphi: \text{Pol}(A) \rightarrow \text{Pol}(B)$

PRESERVING EQUATIONS (COMPACTNESS)

0 PROOF SKETCH OF:

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• CREATE SIGNATURE τ ENUMERATING
 $\text{Pol}(A)$

VIEW $\text{Pol}(A)$ AS τ -ALGEBRA A

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VIEW $\text{Pol}(A)$ AS τ -ALGEBRA $\underline{A}$

• THE IMAGE OF φ INDUCES A
 τ -ALGEBRA $\underline{B}$ ON B SATISFYING
ALL EQUATIONS OF $\underline{A}$

BIRKHOFF $\Rightarrow \underline{B} \in \text{HSP}^{\text{fin}} \underline{A}$

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ALL EQUATIONS OF $\underline{A}$

BIRKHOFF $\Rightarrow \underline{B} \in \mathbf{HSP}^{fin} \underline{A}$

• THE INVARIANT RELATIONS OF $\underline{B}$
(IN PARTICULAR $|B|$)
HAVE PP-INTERPRETATION IN A

□

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 τ -ALGEBRA B ON B SATISFYING
ALL EQUATIONS OF A
- BIRKHOFF** $\Rightarrow B \in \text{HSPT}^{\text{fin}} A$
- THE INVARIANT RELATIONS OF B
(IN PARTICULAR B)
HAVE PP-INTERPRETATION IN A

DEFINITION

A, B RELATIONAL STRUCTURES

$$\varphi: \text{Pol}(A) \rightarrow \text{Pol}(B)$$

CLOVE HOMOMORPHISM $\Leftrightarrow$

- φ PRESERVES ARITIES
- φ PROJECTIONS
- φ COMPOSITION

□

PROOF SKETCH OF:

$$\text{EQ}(\text{Pol}(\mathcal{A})) \subseteq \text{EQ}(\text{Pol}(\mathcal{B}))$$

$\Rightarrow \mathcal{A}$ PP-INTERPRETS $\mathcal{B}$

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DEFINITION

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(ADOLESCENT) THEOREM

(BULATOV + JEAVONS + KROKHIN
'00)

$\mathcal{A}, \mathcal{B}$ FINITE

$$\exists \varphi: \text{Pol}(\mathcal{A}) \xrightarrow{\text{hom}} \text{Pol}(\mathcal{B})$$

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PROOF SKETCH OF:

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- CREATE SIGNATURE τ ENUMERATING $\text{Pol}(\mathcal{A})$

VIEW $\text{Pol}(\mathcal{A})$ AS τ -ALGEBRA $\underline{A}$

- THE IMAGE OF φ INDUCES A τ -ALGEBRA $\underline{B}$ ON $\mathcal{B}$ SATISFYING ALL EQUATIONS OF $\underline{A}$

BIRKHOFF $\Rightarrow \underline{B} \in \text{MSPT}^{\text{fin}} \underline{A}$

- THE INVARIANT RELATIONS OF $\underline{B}$ (IN PARTICULAR $\mathcal{B}$) HAVE PP-INTERPRETATION IN $\mathcal{A}$

□

DEFINITION

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EXERCISE: SHOW THE CONVERSE

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY

INTERPRETABILITY



POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY

INTERPRETABILITY



EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$
CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\text{CSP}(\{0,1\}, R)$ IS 1-IN-3SAT
(NP-COMPLETE)

$$\text{Pol}(\{0,1\}, R) = \{f: f \text{ PROJECTION}\}$$

$\downarrow$ HOM

$\text{Pol}(A)$ FOR ANY A !

STRONGEST WRT PP-INTERPRETATIONS!

EXERCISE SHOW IT.

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY
INTERPRETABILITY



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STRONGEST WRT PP-INTERPRETATIONS!

EXAMPLE



$CSP(K_3)$ IS 3-COLORING

EXERCISE SHOW IT.

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY

INTERPRETABILITY



EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$
CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

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STRONGEST WRT PP-INTERPRETATIONS!

EXAMPLE



$\text{CSP}(K_3)$ IS 3-COLORING

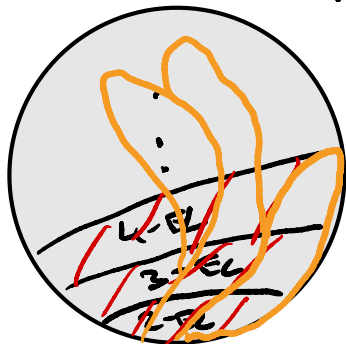
EXERCISE

$$\forall f(x_1, \dots, x_n) \in \text{Pol}(K_3)$$

$$\exists i \exists g \in \text{Aut}(K_3)$$

$$f(x_1, \dots, x_n) = g(x_i)$$

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY
INTERPRETABILITY



EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$
CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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(NP-COMPLETE)

$$\text{Pol}(\{0,1\}, R) = \{f: f \text{ PROJECTION}\}$$

$\downarrow \text{hom}$

$\text{Pol}(A)$ FOR ANY A !

STRONGEST WRT PP-INTERPRETATIONS!

EXAMPLE



$\text{CSP}(K_3)$ IS 3-COLORING

EXERCISE

$$\forall f(x_1, \dots, x_n) \in \text{Pol}(K_3)$$

$$\exists i \exists g \in \text{Aut}(K_3)$$

$$f(x_1, \dots, x_n) = g(x_i)$$

COROLLARY

$$\varphi: \text{Pol}(K_3) \rightarrow \text{Pol}(\{0,1\}, R)$$

$$f \mapsto i\text{-th PROJ}$$

CLONE HOMOMORPHISM

$\Rightarrow K_3$ PP-INTERPRETS $(\{0,1\}, R)$
AND VICE-VERSA

$\Rightarrow$ 3-COLORING IS NP-COMPLETE

SUMMARY

- $CSP(A)$: DECIDING PRIMITIVE POSITIVE THEORY OF A

EXAMPLE: SOLVING EQUATIONS

SAT

n -COLORING

THEORETICAL COMPUTER SCIENCE

- PP-DEFINITIONS (BABY) & PP-INTERPRETATIONS (ADOLESCENT)

$\Rightarrow$ REDUCTIONS

"MODEL THEORY"

- CHARACTERIZED BY :

POLYMORPHISMS / EQUATIONS . THESE ALSO PROVIDE ALGORITHMS.

UNIVERSAL ALGEBRA

PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS

BY 1-IN-3SAT

PP-INTERPRETATIONS INSUFFICIENT

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EXAMPLE

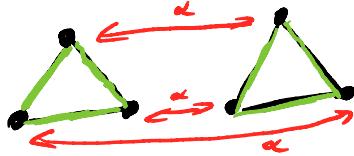
$$A = K_3 \dot{\cup} K_3$$



PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY 1-IN-3SAT

EXAMPLE

$$|A| = K_3 \dot{\cup} K_3$$



$$\text{LET } S(x, y, z) = \begin{cases} z, & x, y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, s \in \text{POL}(A)$

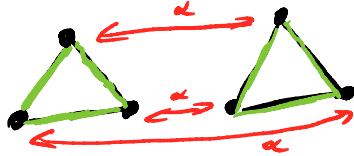
$$\text{AND } S(x, x, y) = y = S(y, \alpha y, x)$$

$$\Rightarrow \text{POL}(A) \not\rightarrow \text{POL}(K_3)$$

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY 1-IN-3SAT

EXAMPLE

$$A = K_3 \dot{\cup} K_3$$



$$\text{LET } S(x, y, z) = \begin{cases} z, & x, y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$$

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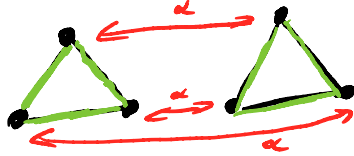
$$\text{BUT } \text{CSP}(A) = \text{CSP}(K_3) !$$

$$\text{SINCE } A \xrightarrow{\text{hom}} K_3, K_3 \xrightarrow{\text{hom}} A$$

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY CSP($\mathcal{U}_3$)

EXAMPLE

$$A = \mathcal{U}_3 \dot{\cup} \mathcal{U}_3$$



$$\exists \alpha, s \in \text{Pol}(A)$$

$$s(x, x, y) = y = s(y, \alpha y, x)$$

$$\Rightarrow \text{Pol}(A) \not\supseteq \text{Pol}(\mathcal{U}_3)$$

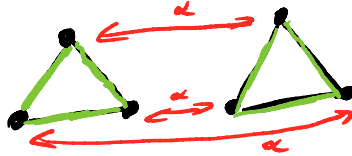
$$\text{BUT } \text{CSP}(A) = \text{CSP}(\mathcal{U}_3) !$$

$$\text{SINCE } A \xrightarrow{\text{hom}} \mathcal{U}_3, \mathcal{U}_3 \xrightarrow{\text{hom}} A$$

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY $CSP(K_3)$

EXAMPLE

$$|A| = K_3 \dot{\cup} K_3$$



$$\exists \alpha, s \in \text{Pol}(A) \\ s(x, x, y) = y = s(y, \alpha y, x)$$

$$\Rightarrow \text{Pol}(A) \not\supseteq \text{Pol}(K_3)$$

$$\text{BUT } CSP(A) = CSP(K_3) !$$

$$\text{SINCE } A \xrightarrow{\text{hom}} K_3, K_3 \xrightarrow{\text{hom}} A$$

DEFINITION

$|A|, |B|$ RELATIONAL STRUCTURES

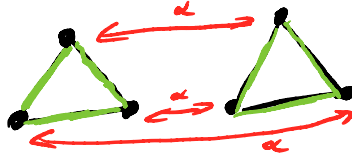
HOMOMORPHICALLY EQUIVALENT $\Leftrightarrow$

$\exists f: A \rightarrow B$ HOMOMORPHISM & VICE-VERSA

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY $CSP(K_3)$

EXAMPLE

$$|A| = K_3 \cup K_3$$



$$\exists \alpha, s \in \text{Pol}(A)$$

$$s(x, x, y) = y = s(y, \alpha y, x)$$

$$\Rightarrow \text{Pol}(A) \neq \text{Pol}(K_3)$$

$$\text{BUT } CSP(A) = CSP(K_3) !$$

$$\text{SINCE } |A| \xrightarrow{\text{hom}} K_3, K_3 \xrightarrow{\text{hom}} |A|$$

DEFINITION

$|A|, |B|$ RELATIONAL STRUCTURES

HOMOMORPHICALLY EQUIVALENT $\Leftrightarrow$

$\exists h: A \rightarrow B$ HOMOMORPHISM & VICE-VERSA

$$\textcircled{1} |A|, |B| \text{ H.E. } \Leftrightarrow CSP(A) = CSP(B)$$

PROOF

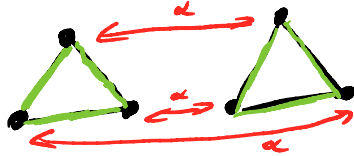
$$\begin{array}{c} \text{X} \\ \nearrow \quad \searrow \\ |A| \leftrightarrow |B| \end{array}$$

$$\Leftarrow |A| \in CSP(A) \Rightarrow |A| \rightarrow |B|$$

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY CSP(K_3)

EXAMPLE

$$|A| = K_3 \dot{\cup} K_3$$



$$\exists \alpha, \sigma \in \text{Pol}(A)$$

$$S(x, x, y) = y = S(y, \alpha y, x)$$

$$\Rightarrow \text{Pol}(A) \not\supseteq \text{Pol}(K_3)$$

$$\text{BUT } \text{CSP}(A) = \text{CSP}(K_3) !$$

$$\text{SINCE } |A| \xrightarrow{\text{hom}} K_3, K_3 \xrightarrow{\text{hom}} |A|$$

DEFINITION

$|A|, |B|$ RELATIONAL STRUCTURES

HOMOMORPHICALLY EQUIVALENT $\Leftrightarrow$

$\exists h: A \rightarrow B$ HOMOMORPHISM & u.c.e.v.f.s.

$$\textcircled{1} |A|, |B| \text{ H.E. } \Leftrightarrow \text{CSP}(A) = \text{CSP}(B)$$

PROOF

$$\begin{array}{c} \text{---} \rightarrow \text{---} \\ \text{---} \nwarrow \text{---} \text{---} \searrow \text{---} \\ |A| \leftrightarrow |B| \end{array}$$

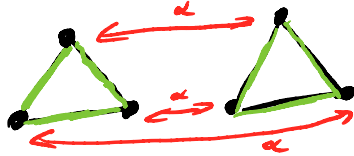
$$\text{---} \leftarrow \text{---} |A| \in \text{CSP}(A) \Rightarrow |A| \rightarrow |B|$$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY $CSP(K_3)$

EXAMPLE

$$|A| = K_3 \dot{\cup} K_3$$



$$\exists \alpha, s \in \text{Pol}(A)$$

$$s(x, x, y) = y = s(y, \alpha y, x)$$

$$\Rightarrow \text{Pol}(A) \neq \text{Pol}(K_3)$$

$$\text{BUT } CSP(A) = CSP(K_3) !$$

$$\text{SINCE } |A| \xrightarrow{\text{hom}} K_3, K_3 \xrightarrow{\text{hom}} |A|$$

DEFINITION

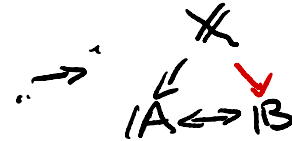
$|A|, |B|$ RELATIONAL STRUCTURES

HOMOMORPHICALLY EQUIVALENT $\Leftrightarrow$

$\exists h: A \rightarrow B$ HOMOMORPHISM & VICE-VERSA

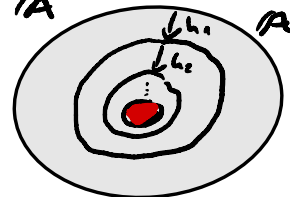
$$\textcircled{1} |A|, |B| \text{ H.E. } \Leftrightarrow CSP(A) = CSP(B)$$

PROOF



$$\Leftarrow |A| \in CSP(A) \Rightarrow |A| \rightarrow |B|$$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD
- EACH EQUIVALENCE CLASS HAS UNIQUE SMALLEST REPRESENTATIVE: THE **CORE** OF $|A|$



DEFINITION

$\text{A PP-CONSTRUCTS B} : \Leftrightarrow$

$$\text{A} \xrightarrow{\text{PP-INT}} C_1 \xrightarrow{\text{H.E}} C_2 \xrightarrow{\text{PP-INT}} C_3 \rightarrow \dots \rightarrow \text{B}$$

DEFINITION

$\nVdash A$ PP-CONSTRUCTS $\nVdash B : \Leftrightarrow$

$$\nVdash A \xrightarrow{\text{PP-INT}} C_1 \xrightarrow{\text{H.E}} C_2 \xrightarrow{\text{PP-INT}} C_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM

(BARTO + OPRŠAL + P. 16)

$\text{EQ}^1 \dots$ HEIGHT 1 - CONDITIONS

$$f_i(\text{variables}) = f_j(\text{variables})$$

~~$\times$~~

~~$\times$~~

$$\text{EQ}^1(\text{POL}(A)) \leq \text{EQ}^1(\text{POL}(B))$$

DEFINITION

$\mathcal{A}$ PP-CONSTRUCTS $\mathcal{B} : \Leftrightarrow$

$$\mathcal{A} \xrightarrow{\text{PP-INT}} \mathcal{C}_1 \xrightarrow{\text{H.E}} \mathcal{C}_2 \xrightarrow{\text{PP-INT}} \mathcal{C}_3 \rightarrow \dots \rightarrow \mathcal{B}$$

(ADULT) THEOREM

(BARTO + OPRŠAL + P. 16)

$\text{EQ}^1 \dots$ HEIGHT 1 - CONDITIONS

$$f_i(\text{variables}) = g_i(\text{variables})$$

~~f_i~~

~~g_i~~

$$\text{EQ}^1(\text{Pol}(\mathcal{A})) \subseteq \text{EQ}^1(\text{Pol}(\mathcal{B}))$$

$$\Leftrightarrow \exists \varphi: \text{Pol}(\mathcal{A}) \rightarrow \text{Pol}(\mathcal{B})$$

PRESERVING H1-EQUATIONS

"MINION HOMOMORPHISM"

$$\Leftrightarrow \mathcal{A} \text{ PP-CONSTRUCTS } \mathcal{B}$$

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} C_1 \xrightarrow{\text{H.E}} C_2 \xrightarrow{\text{PP-INT}} C_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM (BARTO + OPRŠAL + P. 16)

$EQ^1 \dots$ HEIGHT 1 - CONDITIONS

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~~f_i~~

~~g_i~~

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PRESERVING H1-EQUATIONS

"MINION HOMOMORPHISM"

$$\Leftrightarrow A \text{ PP-CONSTRUCTS } B$$

$$\Rightarrow \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$$

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} \mathcal{C}_1 \xrightarrow{\text{H.E}} \mathcal{C}_2 \xrightarrow{\text{PP-INT}} \mathcal{C}_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM

(BARTO + UPRŠAL + P. 16)

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EXAMPLE

HEIGHT 1:

- $m(k, x, y) = m(x, y, x) = m(y, x, x)$
- $c(x_1, \dots, x_n) = c(x_2, \dots, x_n, x_1)$

NOT HEIGHT 1:

- $m(x, x, y) = m(k, y, x) = m(y, k, k) = x$
- $s(x, x, y) = s(y, xy, x)$

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} \mathcal{C}_1 \xrightarrow{\text{H.E}} \mathcal{C}_2 \xrightarrow{\text{PP-INT}} \mathcal{C}_3 \rightarrow \dots \rightarrow B$$

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EXAMPLE

$$A = \mathcal{U}_3 \cup \mathcal{U}_3$$

PP-CONSTRUCTS $\mathcal{U}_3$

~~PP-INTERPRETS~~

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} \mathcal{C}_1 \xrightarrow{\text{H.E}} \mathcal{C}_2 \xrightarrow{\text{PP-INT}} \mathcal{C}_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM

(BARTO + UPRŠAL + P. 16)

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EXAMPLE

$A = \mathcal{U}_3 \cup \mathcal{U}_3$ PP-CONSTRUCTS $\mathcal{U}_3$

~~PP-INTERPRETS~~

$$f \in \text{Pol}(A) \Rightarrow f|_{\mathcal{U}_3} \text{ IS ESSENTIALLY UNARY}$$

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} \mathcal{C}_1 \xrightarrow{\text{H.E}} \mathcal{C}_2 \xrightarrow{\text{PP-INT}} \mathcal{C}_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM

(BARTO + UPRŠAL + P. 16)

$\text{EQ}^1 \dots$ HEIGHT 1-CONDITIONS

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EXAMPLE

HEIGHT 1:

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EXAMPLE

$A = \mathcal{U}_3 \cup \mathcal{U}_3$ PP-CONSTRUCTS $\mathcal{U}_3$

~~PP-INTERPRETS~~

$$f \in \text{Pol}(A) \Rightarrow f|_{\mathcal{U}_3} \text{ IS ESSENTIALLY UNARY}$$

$$\Rightarrow \text{EQ}^1(\text{Pol}(A)) \text{ TRIVIAL}$$

COROLLARY

$EQ^*(P_0(A))$ TRIVIAL

$\Rightarrow A$ PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

COROLLARY

$EQ^*(P_0(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

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THEOREM

$EQ^*(P_0(A))$ NON-TRIVIAL $\Rightarrow$

COROLLARY

$EQ^*(PO(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(PO(A))$ NON-TRIVIAL $\Rightarrow$

$PO(A) \models \exists R \forall x, y, z$

$R(x, y, x, z, y, z) =$

$R(y, x, z, x, z, y)$

6-ARY
SIGGERS

(SIGGERS '0)

COROLLARY

$EQ^*(PO(A))$ TRIVIAL

$\Rightarrow$ $\exists$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(PO(A))$ NON-TRIVIAL $\Rightarrow$

$PO(A) \models \exists f \forall x, y, z$

6-ARY
SIGGERS

$f(x, y, x, z, y, z) =$

$f(y, x, z, x, z, y)$

(SIGGERS '0)

$\models \exists f \forall a, r, e$

4-ARY
SIGGERS

$f(a, r, e, a) =$

$f(r, a, r, e)$

(KEARNES + MARLOWE
+ MCKENZIE '14)

COROLLARY

$EQ^*(Po(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(Po(A))$ NON-TRIVIAL $\Rightarrow$

$$Po(A) \models \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

6-ARY (SIGGERS '0)

SIGGERS

$$\models \exists f \forall a, r, e \quad f(a, r, e, a) = f(r, a, r, e)$$

4-ARY (KEARNES + MARLOWE + MCKENZIE '14)

SIGGERS

$$\models \exists f \forall x_1 \dots x_n \quad f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO & HOLZER '11)

COROLLARY

$EQ^*(Po(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(Po(A))$ NON-TRIVIAL $\Rightarrow$

$$Po(A) \models \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

6-ARY (SIGGERS '0)

SIGGERS

$$\models \exists f \forall a, r, e \quad f(a, r, e, a) = f(r, a, r, e)$$

4-ARY (KEARNES + MARLOWE + MCKENZIE '14)

SIGGERS

$$\models \exists f \forall x_1 \dots x_n \quad f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO & KLEIN '11)

$$\models \exists f \forall x, y \quad f(x \dots x, y) = \dots = f(y, x \dots x)$$

$\forall n \geq |A|$ PRIME

WNU (MARSTEN + MCKENZIE '08)

COROLLARY

$EQ^*(Po(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

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THEOREM

$EQ^*(Po(A))$ NON-TRIVIAL $\Rightarrow$

$$Po(A) \models \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

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SIGGERS

$$\models \exists f \forall a, r, e \quad f(a, r, e, a) = f(r, a, r, e)$$

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SIGGERS

$$\models \exists f \forall x_1 \dots x_n \quad f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO & HODGSON '11)

$$\models \exists f \forall x, y \quad f(x \dots x, y) = \dots = f(y, x \dots x)$$

$\forall n \geq |A|$ PRIME

WNU (MARSTI + MCKENZIE '08)

$\Rightarrow CSP(A) \in P$ (BUKATOV, ZHUK '17)

THEOREM $EQ^*(POL(A))$ NON-TRIVIAL

$\Rightarrow CSP(A) \in P$ (BULATOV, ZHUK '17)

THEOREM $\text{EQ}^*(\text{POL}(A))$ NON-TRIVIAL

$\Rightarrow \text{CSP}(A) \in P$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE
BY SOLVING LINEAR EQUATIONS
OR LOCAL CONSISTENCY CHECKING

SUMMARY

- $CSP(A)$: DECIDING PRIMITIVE POSITIVE THEORY OF A

EXAMPLE: SOLVING EQUATIONS

SAT

n -COLORING

THEORETICAL COMPUTER SCIENCE

- PP-DEFINITIONS (BABY) & PP-INTERPRETATIONS (ADOLESCENT)

$\Rightarrow$ REDUCTIONS

"MODEL THEORY"

- CHARACTERIZED BY :
POLYMORPHISMS / EQUATIONS

UNIVERSAL ALGEBRA

SUMMARY

- $CSP(A)$: DECIDING PRIMITIVE POSITIVE THEORY OF A

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& PP-CONSTRUCTIONS

"MODEL THEORY"

(ADD HOMOMORPHIC
EQUIVALENCE)

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POLYMORPHISMS / EQUATIONS

UNIVERSAL ALGEBRA

$H1$ -EQUATIONS

SUMMARY

- $CSP(A)$: DECIDING PRIMITIVE POSITIVE THEORY OF A

EXAMPLE: SOLVING EQUATIONS

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- PP-DEFINITIONS (BABY) & PP-INTERPRETATIONS (ADOLESCENT)

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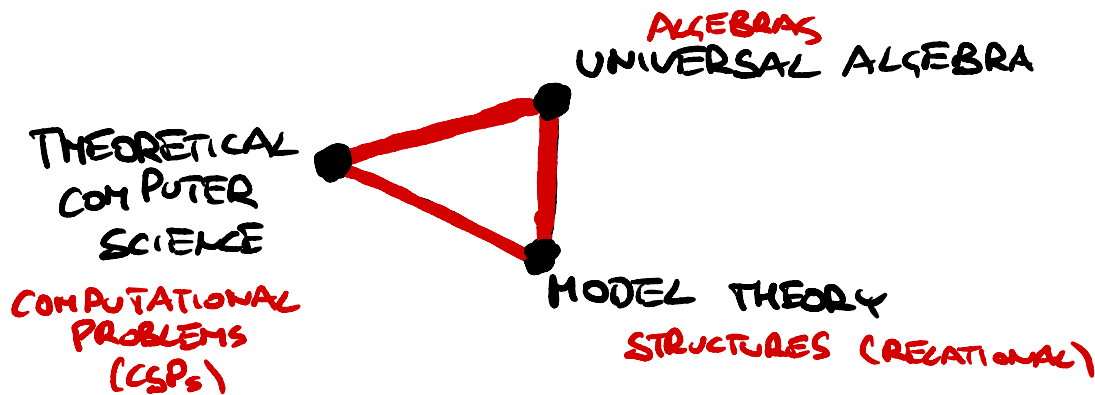
(ADD HOMOMORPHIC
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UNIVERSAL ALGEBRA

$H1$ -EQUATIONS

PP-CONSTRUCT $\mathcal{U}_3$
 $\Rightarrow$ HARD, NO $H1$ -EQUATIONS
OTHERWISE:
EASY, $H1$ -EQUATIONS



PART I

THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II

INFINITE-DOMAIN CSPs MODELLING PROBLEMS
MATHEMATICS

PART II : INFINITE DOMAIN CSP : MODELLING PROBLEMS
MATHEMATICS

FINITE-DOMAIN CSPs : RATHER LIMITED

FINITE-DOMAIN CSPs: RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD

ENP

FINITE-DOMAIN CSPs: RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

SATISFYING ASSIGNMENT?

ENP

FINITE-DOMAIN CSPs: RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

SATISFYING ASSIGNMENT?

- n-COLORING PROBLEM

GENERALIZATION: H-COLORING PROBLEM

H GRAPH, GIVEN GRAPH G, DECIDE IF $G \rightarrow H$ (HELL + NESETRIL '91)

FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

ENP

FINITE-DOMAIN CSPs: RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

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FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

FINITE-DOMAIN CSPs: RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

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- n-COLORING PROBLEM

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FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}/\mathbb{Z}$

FINITE-DOMAIN CSPs: RATHER LIMITED

- ENP {
- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
 - BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT
$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$
SATISFYING ASSIGNMENT?
 - n-COLORING PROBLEM
GENERALIZATION: H-COLORING PROBLEM
H GRAPH, GIVEN GRAPH G, DECIDE IF $G \rightarrow H$ (HELL + NESETRIL '91)
FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}/\mathbb{Z}$
- $\text{CSP}(\mathbb{Z}, <)$ AND FO-DEFINABLE STRUCTURES E.G. $\text{CSP}(\mathbb{Z}, \{(x, y) \mid x - y \in \{1, 3, 5\}\})$

FINITE-DOMAIN CSPs: RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

SATISFYING ASSIGNMENT?

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GENERALIZATION: H-COLORING PROBLEM

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FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}/\mathbb{Z}$
- $\text{CSP}(\mathbb{Z}, <)$ AND FO-DEFINABLE STRUCTURES E.G. $\text{CSP}(\mathbb{Z}, \{(x, y) \mid |x - y| \in \{1, 3\}^3\})$
- $\text{CSP}(\mathbb{Q}, <)$ AND — " — E.G. $\text{CSP}(\mathbb{Q}, \text{BETWEEN}(x, y, z))$

EXERCISE

COMPLEXITY?

FINITE-DOMAIN CSPs: RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

SATISFYING ASSIGNMENT?

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FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}/\mathbb{Z}$
- $\text{CSP}(\mathbb{Z}, <)$ AND FO-DEFINABLE STRUCTURES E.G. $\text{CSP}(\mathbb{Z}, \{(x, y) \mid (x - y) \in \{1, 3\}\})$
- $\text{CSP}(\mathbb{Q}, <)$ AND — " — E.G. $\text{CSP}(\mathbb{Q}, \text{BETWEEN}(x, y, z))$
- "FORBIDDEN PATTERN PROBLEMS"

EXERCISE

COMPLEXITY?

GRAPH ORIENTATION PROBLEM

$\mathcal{T}$... FINITE SET OF TOURNAMENTS

GIVEN UNDIRECTED GRAPH G : CAN IT BE ORIENTED $\mathcal{T}$ -FREE?

GRAPH ORIENTATION PROBLEM

$\mathcal{F}$... FINITE SET OF TOURNAMENTS

GIVEN UNDIRECTED GRAPH G : CAN IT BE ORIENTED $\mathcal{F}$ -FREE?

EXAMPLE

• $\mathcal{F} = \{ \text{triangle}, \text{K}_4 \} \leadsto \text{PROBLEM TRIVIAL}$

• $\mathcal{F} = \{ \text{edge} \}$

• $\mathcal{F} = \{ \text{edge}, \text{triangle} \}$

$\text{EXERCISE: COMPLEXITY?}$

GRAPH ORIENTATION PROBLEM

$\mathcal{F}$... FINITE SET OF TOURNAMENTS

GIVEN UNDIRECTED GRAPH G : CAN IT BE ORIENTED $\mathcal{F}$ -FREE?

EXAMPLE

• $\mathcal{F} = \{ \text{triangle}, \text{K}_4 \} \Rightarrow$ PROBLEM TRIVIAL

• $\mathcal{F} = \{ \text{bidirectional edge} \}$

• $\mathcal{F} = \{ \text{bidirectional edge}, \text{triangle} \}$

$\Rightarrow$ EXERCISE: COMPLEXITY?

$\exists A_3$ "NICE" : $\text{CSP}(A_3) = \{ G \mid G \text{ HAS } \mathcal{F}\text{-FREE ORIENTATION} \}$

GRAPH ORIENTATION PROBLEM

$\mathcal{F}$... FINITE SET OF TOURNAMENTS

GIVEN UNDIRECTED GRAPH G : CAN IT BE ORIENTED $\mathcal{F}$ -FREE?

EXAMPLE

• $\mathcal{F} = \{ \text{triangle}, \text{diamond} \} \Rightarrow$ PROBLEM TRIVIAL

• $\mathcal{F} = \{ \text{edge} \}$

• $\mathcal{F} = \{ \text{edge}, \text{triangle} \}$

$\Rightarrow$ EXERCISE: COMPLEXITY?

$\exists \mathcal{A}_3$ "NICE" : $\text{CSP}(\mathcal{A}_3) = \{ G \mid G \text{ HAS } \mathcal{F}\text{-FREE ORIENTATION} \}$: LATER

HMSNP : τ ... RELATIONAL SIGNATURE, σ ... UNARY PREDICATES

$\mathcal{F}$... FINITE SET OF $\tau \cup \sigma$ -STRUCTURES

- GIVEN τ -STRUCTURE X , CAN IT BE σ -COLORED SO THAT IT IS $\mathcal{F}$ -FREE?

GRAPH ORIENTATION PROBLEM

$\mathcal{F}$... FINITE SET OF TOURNAMENTS

GIVEN UNDIRECTED GRAPH G : CAN IT BE ORIENTED $\mathcal{F}$ -FREE?

EXAMPLE

• $\mathcal{F} = \{ \text{triangle}, \text{K}_4 \} \Rightarrow$ PROBLEM TRIVIAL

• $\mathcal{F} = \{ \text{edge with two arrows} \}$

• $\mathcal{F} = \{ \text{edge with two arrows}, \text{triangle} \}$

$\Rightarrow$ EXERCISE: COMPLEXITY?

$\exists A_3$ "NICE" : $\text{CSP}(A_3) = \{ G \mid G \text{ HAS } \mathcal{F}\text{-FREE ORIENTATION} \}$: LATER

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EXAMPLE

$\mathcal{F} = \{ \text{triangle with 2 green vertices}, \text{triangle with 2 pink vertices} \}$ NO-MONOCROMATIC TRIANGLE

GRAPH ORIENTATION PROBLEM

$\mathcal{F}$... FINITE SET OF TOURNAMENTS

GIVEN UNDIRECTED GRAPH G : CAN IT BE ORIENTED $\mathcal{F}$ -FREE?

EXAMPLE

• $\mathcal{F} = \{ \text{triangle}, \text{K}_4 \} \Rightarrow$ PROBLEM TRIVIAL

• $\mathcal{F} = \{ \text{edge with arrow} \}$

• $\mathcal{F} = \{ \text{edge with arrow}, \text{triangle} \}$

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HMSNP: τ ... RELATIONAL SIGNATURE, σ ... UNARY PREDICATES

$\mathcal{F}$... FINITE SET OF $\tau \cup \sigma$ -STRUCTURES

• GIVEN τ -STRUCTURE X , CAN IT BE σ -COLORED SO THAT IT IS $\mathcal{F}$ -FREE?

EXAMPLE

$\mathcal{F} = \{ \text{green triangle}, \text{pink triangle} \}$

NO-MONOCHROMATIC TRIANGLE

GMSNP: $\mathcal{F} = \{ \text{green triangle}, \text{pink triangle} \}$

—————

(COLORING EDGES)

GRAPH ORIENTATION, HMSNP, GMSNP :

WHY ALL THOSE FORBIDDEN PATTERN PROBLEMS?

GRAPH ORIENTATION, HMSNP, GMSNP :

WHY ALL THOSE FORBIDDEN PATTERN PROBLEMS?

NP = ESO : EXISTENTIAL SECOND ORDER LOGIC (FASIN '73)

$\tau \dots$ SIGNATURE , $K \dots$ ISOMORPHISM-CLOSED CLASS OF
FINITE τ -STRUCTURES

$\Rightarrow K \in \text{NP} \Leftrightarrow K$ DEFINABLE BY ESO-SENTENCE :
 $\exists x_1 \dots x_n (\phi)$ FIRST-ORDER

GRAPH ORIENTATION, HMSNP, GMSNP :

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SNP = STRICT NP : $\exists x_1 \dots x_n \forall \bar{x} \underbrace{\phi(\bar{x}, x_1 \dots x_n)}_{\text{QUANTIFIER-FREE}}$

$\wedge \neg(\dots \wedge \dots \wedge \dots)$ FORBIDDEN PATTERNS

GRAPH ORIENTATION, HMSNP, GMSNP:

WHY ALL THOSE FORBIDDEN PATTERN PROBLEMS?

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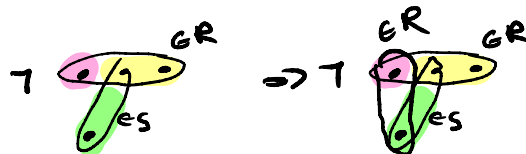
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$\wedge \neg(\dots \wedge \dots \wedge \dots)$ FORBIDDEN PATTERNS

PATTERNS CONNECTED, MONOTONE $\Rightarrow K$ IS A CSP

= CLOSED UNDER HOMOMORPHISMS:
 τ -ATOMS ONLY APPEAR POSITIVELY



THE GRAPH ORIENTATION PROBLEM AS CSP

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{F}$... FINITE SET OF FINITE TOURNAMENTS

$\mathcal{K} = \{ \text{ID} \mid \text{ID } \mathcal{F}\text{-FREE ORIENTED FINITE GRAPH} \}$

 $\mathcal{K}$ CLOSED UNDER SUBSTRUCTURES
AMALGAMATION PROPERTY

THE GRAPH ORIENTATION PROBLEM AS CSP

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 $\mathcal{K}$ CLOSED UNDER SUBSTRUCTURES
AMALGAMATION PROPERTY

$(A = (A, \rightarrow)) := \text{Fr\"{a}ss\'e-Limit}(\mathcal{K})$

$G := (A, E) \quad E(x, y) := \langle \rightarrow \vee y \rightarrow x \rangle$

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{F}$... FINITE SET OF FINITE TOURNAMENTS

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 $\text{IH FINITE GRAPH} \Rightarrow$

$\text{IH} \in \text{CSP}(G) \Leftrightarrow \text{IH} \xrightarrow{\text{hom}} G$

$\xLeftrightarrow{\text{Ex.}} \text{IH} \xrightarrow{\text{emb}} G$

$\Leftrightarrow \text{IH HAS } \mathcal{F}\text{-FREE ORIENTATION}$

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{F}$... FINITE SET OF FINITE TOURNAMENTS

$\mathcal{K} = \{ \text{ID} \mid \text{ID } \mathcal{F}\text{-FREE ORIENTED FINITE GRAPH} \}$

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AMALGAMATION PROPERTY

$(\mathcal{A} = (\mathcal{A}, \rightarrow) := \text{Fraïssé-Limit}(\mathcal{K}))$

$\mathcal{G} := (\mathcal{A}, E) \quad E(\mathcal{K}, \mathcal{G}) := \Leftrightarrow x \rightarrow y \vee y \rightarrow x$

$\mathcal{H}$ FINITE GRAPH $\Rightarrow$

$\mathcal{H} \in \text{CSP}(\mathcal{G}) \Leftrightarrow \mathcal{H} \xrightarrow{\text{hom}} \mathcal{G}$

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$\Leftrightarrow \mathcal{H}$ HAS $\mathcal{F}$ -FREE
ORIENTATION

THEOREM

(BOHRSELY + GUZMÁN - PRO '23)

(FELLER + P. '24)

(BITTER + N. '24)

$\text{CSP}(\mathcal{G}) \in \text{P}$ UNLESS $\mathcal{G}$ PP-CONSTRUCTS $\mathcal{K}_3$

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{F}$... FINITE SET OF FINITE TOURNAMENTS

$\mathcal{K} = \{ \text{ID} \mid \text{ID } \mathcal{F}\text{-FREE ORIENTED FINITE GRAPH} \}$

$\mathcal{K}$ CLOSED UNDER SUBSTRUCTURES
AMALGAMATION PROPERTY

$(\mathcal{A} = (\mathcal{A}, \rightarrow) := \text{Fraïssé-Limit}(\mathcal{K}))$

$G := (\mathcal{A}, E) \quad E(\mathcal{K}, \mathcal{A}) := \{ x \rightarrow y \vee y \rightarrow x \}$

$\mathcal{H}$ FINITE GRAPH $\Rightarrow$

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THEOREM

(BODIRSKY + GUZMÁN - PRO '23)
(FELLER + P. '24)
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$\text{CSP}(G) \in \text{P}$ UNLESS G PP-CONSTRUCTS K_3

WHAT IF THE CLASS OF SOLUTIONS
TO A PROBLEM DOES NOT HAVE
AMALGAMATION?

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{K} \dots$ FINITE SET OF FINITE TOURNAMENTS

$\mathcal{K} = \{ \text{ID} \mid \text{ID } \mathcal{J}\text{-FREE ORIENTED FINITE GRAPH} \}$

$\mathcal{K}$ CLOSED UNDER SUBSTRUCTURES
AMALGAMATION PROPERTY

$(\mathcal{A} = (\mathcal{A}, \rightarrow) := \text{Fraïssé-Limit}(\mathcal{K}))$

$\mathcal{G} := (\mathcal{A}, E)$ $E(\mathcal{K}, \mathcal{G}) := \mathcal{K} \rightarrow \mathcal{G} \cup \mathcal{G} \rightarrow \mathcal{K}$

$\mathcal{H}$ FINITE GRAPH $\Rightarrow$

$\mathcal{H} \in \text{CSP}(\mathcal{G}) \Leftrightarrow \mathcal{H} \xrightarrow{\text{hom}} \mathcal{G}$

$\xLeftrightarrow{\text{EX.}} \mathcal{H} \xrightarrow{\text{emb}} \mathcal{G}$

$\Leftrightarrow \mathcal{H}$ HAS $\mathcal{J}$ -FREE ORIENTATION

THEOREM

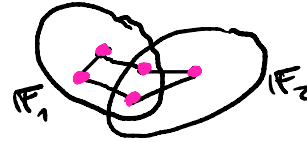
(BODIRSKY + GUZMÁN - PRO '23)
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WHAT IF THE CLASS OF SOLUTIONS TO A PROBLEM DOES NOT HAVE AMALGAMATION?

EXAMPLE

COLOURED GRAPHS WITHOUT PINK 5-CYCLE



(C5)

OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AMALGAMATION

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{F} \dots$ FINITE SET OF FINITE TOURNAMENTS

$\mathcal{K} = \{ \text{ID} \mid \text{ID } \mathcal{F}\text{-FREE ORIENTED FINITE GRAPH} \}$

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AMALGAMATION PROPERTY

$(A = (A, \rightarrow)) := \text{Fraïssé-Limit}(\mathcal{K})$

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$\mathcal{H}$ FINITE GRAPH $\Rightarrow$

$\mathcal{H} \in \text{CSP}(G) \Leftrightarrow \mathcal{H} \xrightarrow{\text{hom}} G$

EX. $\Leftrightarrow \mathcal{H} \xrightarrow{\text{emb}} G$

$\Leftrightarrow \mathcal{H}$ HAS $\mathcal{F}$ -FREE ORIENTATION

THEOREM

(BODIRSKY + GUZMAN - PRO '23)

(FELLER + P. '24)

(BITTER + NIEBERG '24)

$\text{CSP}(G) \in \text{P}$ UNLESS G PP-CONSTRUCTS K_3

WHAT IF THE CLASS OF SOLUTIONS TO A PROBLEM DOES NOT HAVE AMALGAMATION?

EXAMPLE

COLOURED GRAPHS WITHOUT PINK 5-CYCLE



(CG)

OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AMALGAMATION

THEOREM

(CHERAN-SHELAH-SH.)

$\mathcal{F} \dots$ FINITE SET OF STRUCTURES, CONNECTED, NON-CLOSED

$\Rightarrow \exists A$ REDUCT OF HOMOGENEOUS STRUCTURE GIVEN BY FINITE SET OF FORBIDDEN STRUCTURES
 $\text{CSP}(A) = \{ X \mid \exists X^* \text{ EXPANSION AVOIDING } \mathcal{F} \}$

CONJECTURE

(BOOIREMY + P. 111)

A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
STRUCTURE IS GIVEN

BY FINITE SET OF FORBIDDEN PATTERNS

$\Rightarrow \text{CSP}(A)$ IN P UNLESS

A PP-CONSTRUCTS $\mathbb{Q}_3$

CONJECTURE

(BOGIRESH + P. 111)

A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
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EXAMPLE

- TRUE FOR GRAPH ORIENTATION
w/ FORBIDDEN TOURNAMENTS

CONJECTURE

(BOGUSKY + P. 111)

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EXAMPLE

- TRUE FOR GRAPH ORIENTATION
w/ FORBIDDEN TOURNAMENTS

- TRUE FOR MMSNP

$$\mathcal{F} = \left\{ \begin{array}{c} \text{green triangle} \\ \text{pink triangle} \end{array} \right\}$$

NO-MONOCROMATIC
TRIANGLE

CONJECTURE

(BOGUSKY + P. '11)

A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
STRUCTURE IS GIVEN

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EXAMPLE

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NO-MONOCROMATIC
TRIANGLE

- OPEN FOR QMSNP

$$\mathcal{F} = \left\{ \begin{array}{c} \text{green triangle} \\ \text{pink triangle} \end{array} \right\}$$

CONJECTURE

(BOGART + P. 111)

A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
STRUCTURE IS GIVEN

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$\Rightarrow \text{CSP}(A)$ IN P UNLESS

A PP-CONSTRUCTS K_3

EXAMPLE

- TRUE FOR GRAPH ORIENTATION
w/ FORBIDDEN TOURNAMENTS

- TRUE FOR MMSNP

$\mathcal{F} = \{ \text{triangle with green edges}, \text{triangle with pink edges} \}$ NO-MONOCROMATIC TRIANGLE

- OPEN FOR QMSNP

$\mathcal{F} = \{ \text{triangle with green edges}, \text{triangle with pink edges} \}$

FIRST-ORDER REDUCTS OF $(Q, \triangleleft)$

CONJECTURE

(BOGOSKY + P. 111)

1/A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
STRUCTURE IS GIVEN

BY FINITE SET OF FORBIDDEN PATTERNS

$\Rightarrow \text{CSP}(A)$ IN P UNLESS

1/A PP-CONSTRUCTS $\mathbb{Q}_3$

EXAMPLE

- TRUE FOR GRAPH ORIENTATION
✓/ FORBIDDEN TOURNAMENTS

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- OPEN FOR QMSNP

$\mathcal{F} = \{ \text{triangle with green edges}, \text{triangle with pink edges} \}$

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

- $\text{CSP}(\mathbb{Q}, <)$

GIVEN FINITE DIGRAPH D , $D \rightarrow^? (\mathbb{Q}, <)$

CONJECTURE

(BOGOSKY + P. 111)

A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
STRUCTURE IS GIVEN

BY FINITE SET OF FORBIDDEN PATTERNS

$\Rightarrow \text{CSP}(A)$ IN P UNLESS

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EXAMPLE

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FIRST-ORDER REDUCTS OF $(Q, <)$

- $\text{CSP}(Q, <)$

GIVEN FINITE DIGRAPH D , $D \xrightarrow{?} (Q, <)$

- $\text{CSP}(Q, B(x, y, z))$

$B(x, y, z) : \Leftrightarrow (x < y \wedge y < z) \vee (z < y \wedge y < x)$

CONJECTURE

(BOGOSKY + P. 111)

1A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
STRUCTURE IS GIVEN

BY FINITE SET OF FORBIDDEN PATTERNS

$\Rightarrow \text{CSP}(A)$ IN P UNLESS

1A PP-CONSTRUCTS $\mathbb{Q}_3$

EXAMPLE

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- TRUE FOR MMSNP

$$\mathcal{F} = \left\{ \begin{array}{c} \text{green triangle} \\ \text{pink triangle} \end{array} \right\}$$

NO-MONOCROMATIC
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- $\text{CSP}(Q, <)$

GIVEN FINITE DIGRAPH D , $D \xrightarrow{?} (Q, <)$

- $\text{CSP}(Q, B(x, y, z))$

$$B(x, y, z) : \Leftrightarrow (x < y \wedge y < z) \vee \\ (z < y \wedge y < x)$$

- $\text{CSP}(Q, R)$

$$R(x, y, z) \Leftrightarrow (x = y < z) \vee (x = z < y) \\ \vee (y = z < x)$$

EXERCISE: COMPLEXITY?

CONJECTURE

(BODIRSKY + P. 111)

1A (FIRST-ORDER) REDUCT OF HOMOGENEOUS
STRUCTURE IS GIVEN

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FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

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GIVEN FINITE DIGRAPH D , $D \xrightarrow{?} (\mathbb{Q}, <)$

- $\text{CSP}(\mathbb{Q}, B(x, y, z))$

$$B(x, y, z) \Leftrightarrow (x < y \wedge y < z) \vee \\ (z < y \wedge y < x)$$

- $\text{CSP}(\mathbb{Q}, R)$

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EXERCISE: COMPLEXITY?

CSPs OF FO-REDUCTS OF $(\mathbb{Q}, <)$

"LINEAR-ORDER-SAT"

"TEMPORAL CSPs"

THEOREM

- LINEAR-ORDER-SAT (1A) IN P UNLESS
A FP-CONSTRUCTS K_3 (BODIRSKY + KARA '07)
- GRAPH-SAT (BODIRSKY + P. '11)
- POSET-SAT (LAN PHAM + KOMPATSCHER '16)
- TOURNAMENT-SAT (HOTTEY + P. '21)
- HYPERGRAPH-SAT (HOTTEY + MAGY + P. '23)
- $\vdots$

THEOREM

- LINEAR-ORDER-SAT (1A) IN P UNLESS
A FP-CONSTRUCTS K_3 (BODIRSKY + KA'ARA '07)
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- TOURNAMENT-SAT (HOTTEY + P. '21)
- HYPERGRAPH-SAT (HOTTEY + MAGY + P. '23)
- $\vdots$

METHOD: POLYMORPHISMS!

IN A NUTSHELL: FOR W-CAT. A:

BASIC THEORY ... AS FOR FINITE STRUCTURES

ADVANCED THEORY ... MUCH HARDER / FALSE

IN A NUTSHELL: FOR ω -CAT. $\mathcal{A}$:

BASIC THEORY ... AS FOR FINITE STRUCTURES

ADVANCED THEORY ... MUCH HARDER / FALSE

(BASKY) THEOREM (BOODISKY + NEŠETŘIL '03)

$\mathcal{A}$ ω -CATEGORICAL, $R \subseteq A^m$ RELATION

$\Rightarrow R$ PP-DEFINABLE IN $\mathcal{A} \Leftrightarrow$

R INVARIANT UNDER $\text{Po}(\mathcal{A})$

IN A NUTSHELL: FOR ω -CAT. $\mathcal{A}$:

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PROOF

$\Rightarrow$

TRIVIAL

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PROOF

$\rightarrow$

TRIVIAL

$\leftarrow$

R IS A UNION OF ORBITS OF
 $\text{Aut}(\mathcal{A}) \curvearrowright A^m$

LET $\bar{r}_1, \dots, \bar{r}_l \in R$ BE REPRESENTATIVES
OF THESE ORBITS

LET $(\bar{a}_0, \bar{a}_1, \dots)$ ENUMERATION
OF A^l

IN A NUTSHELL: FOR ω -CAT. $\mathcal{A}$:

BASIC THEORY ... AS FOR FINITE STRUCTURES
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PROOF

$\rightarrow$ TRIVIAL

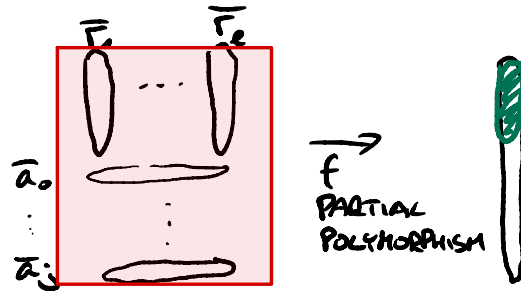
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OF A^ℓ

PP-DEFINE R_i THIS WAY:



IN A NUTSHELL: FOR ω -CAT. A :

BASIC THEORY ... AS FOR FINITE STRUCTURES
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A ω -CATEGORICAL, $R \subseteq A^m$ RELATION
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PROOF

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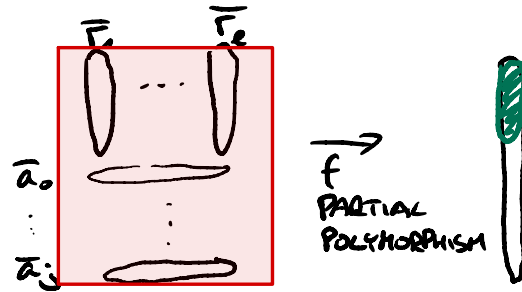
$\leftarrow$

R IS A UNION OF ORBITS OF
 $\text{Aut}(A) \curvearrowright A^m$

LET $\bar{r}_1, \dots, \bar{r}_l \in R$ BE REPRESENTATIVES
OF THESE ORBITS

LET $(\bar{a}_0, \bar{a}_1, \dots)$ ENUMERATION
OF A^l

PP-DEFINE R_j THIS WAY:



• $R_j \supseteq R$ (PROJECTIONS)

IN A NUTSHELL: FOR ω -CAT. $\mathcal{A}$:

BASIC THEORY ... AS FOR FINITE STRUCTURES
ADVANCED THEORY ... MUCH HARDER / FALSE

(BACH) THEOREM (BOODIRSKY + NEŠETŘIL '03)

$\mathcal{A}$ ω -CATEGORICAL, $R \subseteq A^m$ RELATION
 $\Rightarrow R$ PP-DEFINABLE IN $\mathcal{A} \Leftrightarrow$
 R INVARIANT UNDER $\text{Po}(\mathcal{A})$

PROOF

$\rightarrow$ TRIVIAL

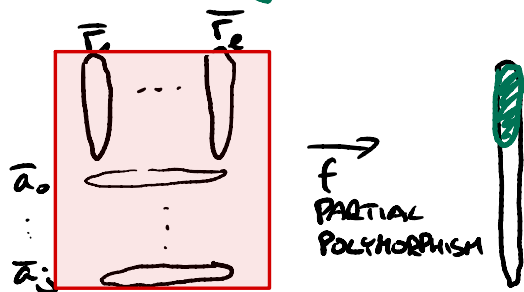
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- $R_j \supseteq R_{j+1}$, EVENTUALLY CONSTANT
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$\Rightarrow \bigcap_j R_j =: R_\infty$ PP-DEFINABLE
P.S.

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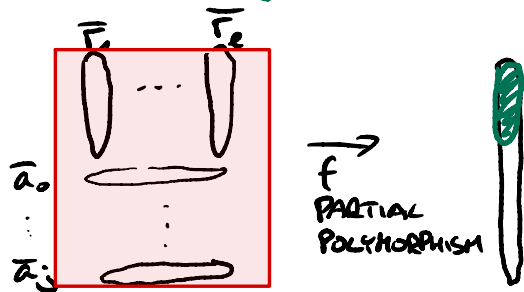
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 $R \subseteq R_\infty$

- $\bar{r} \in R_\infty$
 $\Rightarrow \exists f_j$ PART. POLYMORPHISMS
WITNESSES

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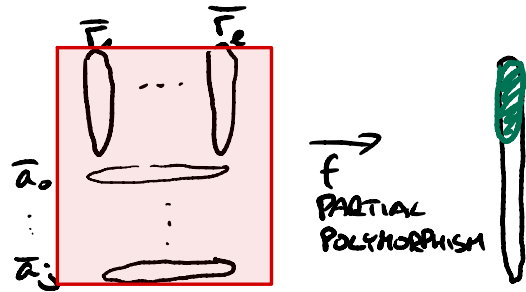
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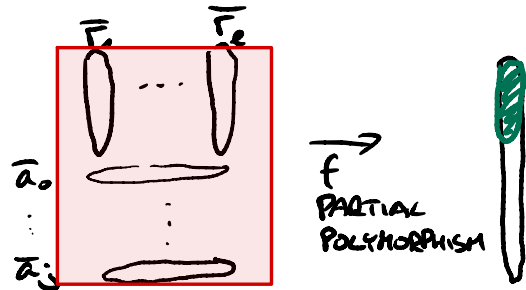
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.. $\exists$ SUBSEQUENCE SUCH THAT
 f_j EXTENDS f_i , $i < j$ MOD. ORBIT

.. COMPOSING WITH AUTOMORPHISMS,
 f_j EXTENDS f_i $f := \cup f_i$
 $\Rightarrow \bar{r} = f(\bar{r}_1, \dots, \bar{r}_j) \in R$

ADOLESCENT / ADULT THEOREMS:

CHARACTERISE

- PP-INTERPRETATIONS BY EQUATIONS/
CLONE
HOMOMORPHISMS
- PP-CONSTRUCTIONS BY MU EQUATIONS,
MINION
HOMOMORPHISMS

2.

ADOLESCENT / ADULT THEOREMS:

CHARACTERISE

- PP-INTERPRETATIONS BY EQUATIONS/
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2.

TOPOLOGY

ADOLESCENT / ADULT THEOREMS:

CHARACTERISE

- PP-INTERPRETATIONS BY EQUATIONS/
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- PP-CONSTRUCTIONS BY MU EQUATIONS,
MINION
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2

.

TOPOLOGY

$$A^{\omega} = \{f: A^{\omega} \rightarrow A\} \text{ CARRIES}$$

TOPOLOGY OF POINTWISE CONVERGENCE:-

$$(f_i)_{i \in \omega} \rightarrow f \Leftrightarrow$$

$$\forall F \subseteq A^{\omega} \text{ FINITE}$$

$$f_i|_F = f|_F \text{ EVENTUALLY}$$

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COMPLETE METRIC SPACE /

POLISH SPACE

$\bigcup_n A_n^{\omega} \dots$ SUM SPACE
(EACH A_n^{ω} CLOPEN)

ADOLESCENT / ADULT THEOREMS:

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COMPLETE METRIC SPACE /

POLISH SPACE

$$\bigcup_n A^{\omega} \dots \text{ SUM SPACE} \\ (\text{EACH } A^{\omega} \text{ CLOSED})$$

$\text{Pol}(A)$ CLOSED SUBSPACE:

$$f \notin \text{Pol}(A) \Rightarrow \exists F \subseteq A \text{ FINITE:}$$

$$O_{f|_F} = \{g \mid g|_F = f|_F\} \cap \text{Pol}(A) = \emptyset$$

THEOREM (ADOLESCENT + ADULT)

A ω -CATEGORICAL, B FINITE $\Rightarrow$

BODIRSKY + P.
15
BARTO + UPADHYAY
+ P.
17

THEOREM (ADOLESCENT + ADULT)

BODIRSKY + P.
15
BARTO + DRAŽIĆ
+ P.
17

A ω -CATEGORICAL, B FINITE $\Rightarrow$

$$\textcircled{1} \text{ Pol}(A) \xrightarrow[\text{CLONE HOMOMORPHISM}]{\text{u.c.}} \text{Pol}(B) \Leftrightarrow A \text{ PP-INTERPRETS } B$$

$\Leftrightarrow \exists F \subseteq A$ FINITE :

$$\text{EQ}(\text{Pol}(A)) \underset{F}{=} \text{EQ}(\text{Pol}(B))$$

$\rightarrow \text{CSP}(B)$ REDUCES TO $\text{CSP}(A)$

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$$\textcircled{2} \text{ Pol}(A) \xrightarrow[\text{MINION HOMOMORPHISM}]{\text{u.c.}} \text{Pol}(B) \Leftrightarrow A \text{ PP-CONSTRUCTS } B$$

$$\Rightarrow \exists F \subseteq A \text{ FINITE:}$$

$$\text{EQ}'(\text{Pol}(A) \upharpoonright_F) \subseteq \text{EQ}'(\text{Pol}(B))$$

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WHAT IF $\text{Pol}(A) \xrightarrow[\text{MINION}]{\text{u.c.}} \text{Pol}(U_3)$?

THEOREM (ADOLESCENT + ADULT)

BODIRSKY + P.
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WHAT IF $\text{Pol}(A) \xrightarrow[\text{MINION}]{\text{u.c.}} \text{Pol}(U_3)$?

RECALL A FINITE $\Rightarrow$

$\text{Pol}(A)$ HAS

- CYCLIC OPERATION $C(x_1 \dots x_n) = C(x_2 \dots x_n x_1)$
- WNU OPERATION $W(x \dots x_4) = W(yx \dots x)$
- 4-ARY SIGGERS OPERATION $S(a, r, a) = S(r, a, r)$

• 6-ARY SIGGERS OPERATION

$$S(x y x z y z) = S(y x z x z y)$$

...

FALSE FOR w -CATEGORICAL $\mathcal{A}$!

FALSE FOR ω -CATEGORICAL $\mathcal{A}$!

EXAMPLE

LET $\mathcal{A} = \mathcal{K}_\omega = (\mathbb{N}, +)$ ω -CATEGORICAL
 $\text{CSP}(\mathcal{A}) \in \mathcal{P}$

FALSE FOR ω -CATEGORICAL $\mathcal{A}$!

EXAMPLE

LET $\mathcal{A} = \mathcal{K}_\omega = (\mathbb{N}, +)$ ω -CATEGORICAL
 $\text{CSP}(\mathcal{A}) \in \text{P}$

- DOES $\text{POL}(\mathcal{A})$ HAVE CYCLIC $C(x_1 \dots x_n)$?

FALSE FOR ω -CATEGORICAL $\mathcal{A}$!

EXAMPLE

LET $\mathcal{A} = \mathcal{K}_\omega = (\mathbb{N}, \neq)$ ω -CATEGORICAL
 $\text{CSP}(\mathcal{A}) \in \mathcal{P}$

- DOES $\text{Pol}(\mathcal{A})$ HAVE CYCLIC $C(x_1 \dots x_n)$?

PICK $a_1, \dots, a_n$ DISTINCT

$$C(a_1 \dots a_n) = a$$

$$C(a_2 \dots a_{n-1} a_1) = a$$

⚡

FALSE FOR ω -CATEGORICAL $\mathcal{A}$!

EXAMPLE

LET $\mathcal{A} = \mathcal{K}_\omega = (\mathbb{N}, +)$ ω -CATEGORICAL
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⋮

- $S(x, y, x, z, y, z) = S(y, x, z, x, z, y)$ ⋮

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- $S(x, y, x, z, y, z) = S(y, x, z, x, z, y)$ $\downarrow$

- ANY SINGLE NON-TRIVIAL H_1 -EQUATION $\downarrow$

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EXERCISE

WHY DOES $\mathcal{K}_\omega$ NOT PR-CONSTRUCT $\mathcal{K}_3$?

PSEUDO-H1-EQUATIONS

EQUIVALENCE RELATION ON AA^* :

$$f \sim g \Leftrightarrow g \in \overline{\{\alpha f \mid \alpha \in \text{Aut}(A)\}}$$

$$\Leftrightarrow \forall F \subseteq A \text{ FINITE}$$

$$\exists \alpha \in \text{Aut } A$$

$$g|_F = \alpha f|_F$$

" f, g ARE EQUAL MODULO ORBITS"

PSEUDO-H1-EQUATIONS

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EXERCISE

$$A^{A^*} / \sim =: A^{A^*} / \text{Aut } A$$

COMPACT (POLISH)

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COMPACT (POLISH)

IDEA: TRY TO MIMIC PROOFS FOR
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THEOREM (BARTO + P. 17)

A ω -CATEGORICAL, $\text{Pol}(A)$ NON-TRIVIAL:

$$\text{Pol}(A) \stackrel{\text{u.c.}}{\not\cong} \text{Pol}(\mathbb{Z}_3)$$

NON MONOMORPHISM

$$\Rightarrow \text{Pol}(A) \neq$$

$$\exists u, v, s : \forall x, y, z$$

$$u \circ s(x, y, x, z, y, z) = v \circ s(y, x, z, x, z, y)$$

PSEUDO-N1-EQUATIONS

EQUIVALENCE RELATION ON A^{A^*} :

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THEOREM (BARTO + P. 17)

$|A|$ ω -CATEGORICAL, $\text{Pol}(A)$ NON-TRIVIAL:

$$\text{Pol}(A) \stackrel{\text{IS}}{\neq} \text{Pol}(\mathbb{N}_3)$$

~~MONOMORPHISM~~

$$\Rightarrow \text{Pol}(A) \neq$$

$$\exists u, v, s : \forall x, y, z$$

$$u \circ s(x, y, x, z, y, z) = v \circ s(y, x, z, x, z, y)$$

EXAMPLE

$|A| = (\mathbb{Q}, <, x=y \rightarrow u=v)$: ω -CATEGORICAL

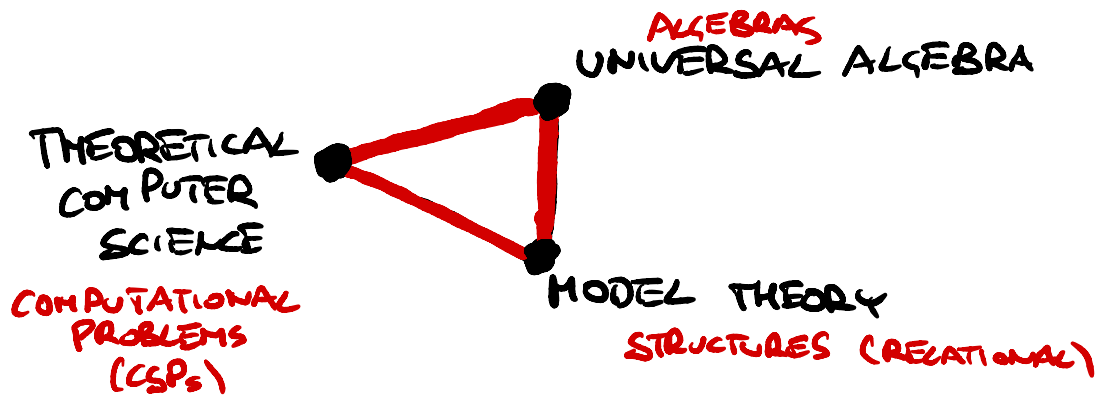
$\text{CSP}(A)$ IN P

NO PSEUDO-CYCLIC POLYMORPHISM

EXERCISE: SHOW IT!

SUMMARY

- PP-CONSTRUCTIONS COMBINE PP-INTERPRETATIONS + HOMOMORPHIC EQUIVALENCE
- CHARACTERIZED BY H1-EQUATIONS OF POLYMORPHISMS
- WORKS ALSO FOR ω -CATEGORICAL STRUCTURES (ω /TOPOLOGY)
- $EQ'(POL(A))$ TRIVIAL $\Rightarrow$ NP-HARD $CSP(A)$
- - NON-TRIVIAL $\Rightarrow$ MANY CONCRETE H1-EQUATIONS FOR FINITE A
 - SOME (BUT NOT ALL) PSEUDO-EQUATIONS FOR ω -CATEGORICAL A
- FORBIDDEN PATTERN PROBLEMS $\Rightarrow$ NATURAL ω -CATEGORICAL CSP_s IN NP
- DICHOTOMY CONJECTURE FOR SUCH PROBLEMS

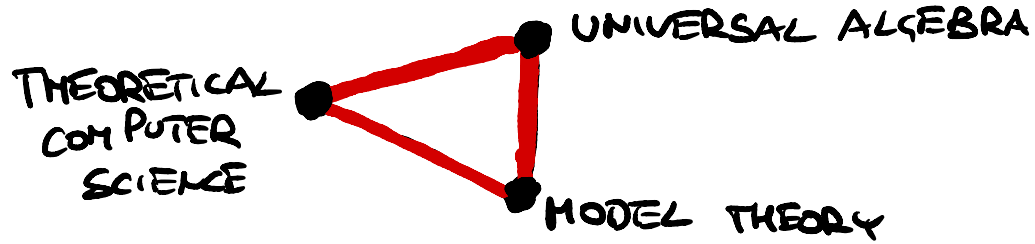


PART I

THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II

INFINITE-DOMAIN CSPs MODELLING PROBLEMS
MATHEMATICS



Thank you!