

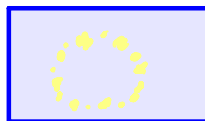
CONSTRAINT SATISFACTION PROBLEMS

WITH INFINITELY MANY VALUES

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OUTLINE

- 1 CONSTRAINT SATISFACTION PROBLEMS
⇒ RELATIONAL STRUCTURES
- 2 ALGEBRAIC INVARIANTS
(FOR FINITE STRUCTURES)
- 3 GOING INFINITE
- 4 GOING FINITE

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES x_1, \dots, x_n
- CONSTRAINTS ON THEM

QUESTION

- \exists VALUES FOR $x_1 \dots x_n$
SATISFYING ALL CONSTRAINTS?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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- EXAMPLE**
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MODEL:

- SET V OF POSSIBLE VALUES (FIXED)
E.G. $\{0,1\}$, $\{0,1,2\}$, \mathbb{Q} , \mathbb{Z} , \mathbb{N} , ...
- ALLOWED CONSTRAINTS (FIXED)
 C_1, \dots, C_m RELATIONS ON V
 $C_i \subseteq V^{d_i}$

SO $A := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE
= "TEMPLATE"

CSP(A)

- GIVEN**
- VARIABLES $x_1 \dots x_n$
 - PP-SENTENCE
 $\varphi \equiv \exists x_1 \dots \exists x_n C_{i_1}(\text{VARIABLES}) \wedge C_{i_2}(\text{VARIABLES}) \wedge \dots C_{i_k}(\text{VARIABLES})$

QUESTION

$A \models \varphi$?

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QUESTION

$A \models \varphi$?

META-QUESTION

FOR WHAT A IS CSP(A) EASY/HARD?

- P**: \exists ALGORITHM PROVIDING ANSWER IN $p(\# \text{ VARIABLES})$ STEPS, p POLYNOMIAL
- #** $p(\# \text{ VARIABLES})$ STEPS, p POLYNOMIAL
- NP**: NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLES

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

TERNARY RELATIONS

CSP(A)

GIVEN

• VARIABLES x_1, \dots, x_n

• CONSTRAINTS $x_1^2 + x_2^3 = x_3^4$

⋮

QUESTION

• SOLUTION IN \mathbb{Z} ?

UNDECIDABLE (MATYASEVIC '77)

HILBERT 10

EXAMPLES

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

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• SOLUTION IN \mathbb{Z} ?

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-
- SAME OVER \mathbb{Z}_p :

NP-COMPLETE

↳ IF IN P \Rightarrow P = NP

EXAMPLES

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

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-
- SAME WITHOUT • :

IN P (GAUSS)

EXAMPLES

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

TERNARY RELATIONS

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- SAME OVER \mathbb{Z}_p :

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- SAME WITHOUT \cdot :

IN P (GAUSS)

- $A = (\mathbb{Q}, <)$

CSP(A)

GIVEN

• VARIABLES x_1, \dots, x_n

• CONSTRAINTS $x_1 < x_2$

$x_2 < x_3$

$x_3 < x_1$

⋮

QUESTION

• SOLUTION IN \mathbb{Q} ?



IN P

EXAMPLES

• $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$ TERMINARY RELATIONS

CSP(A)

GIVEN

- VARIABLES x_1, \dots, x_n
- CONSTRAINTS $x_1^2 + x_2^3 = x_3^4$

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- VARIABLES x_1, \dots, x_n
- CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
⋮

QUESTION

- SOLUTION IN \mathbb{Q} ?



IN P

• $A = \mathbb{R}_3 = \triangle^0 = (0, 1, 2, \mathbb{E})$

CSP(A)

GIVEN

- VARIABLES x_1, \dots, x_n
- CONSTRAINTS $E(x_1, x_2)$
 $E(x_2, x_3)$
⋮

QUESTION

- SOLUTION IN $\{0, 1, 2\}$?

3-COLORING PROBLEM = NP-COMPLETE

THEOREM

(BOLATOV, ZHUK '17)

(CONJECTURE FEDER + VARDI '93)

\mathbb{A} FINITE

$\Rightarrow \text{CSP}(\mathbb{A}) \in \text{P}$ OR

NP-COMPLETE

THEOREM

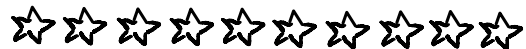
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NP-COMPLETE



$(\text{P} \neq \text{NP})$

$\text{CSP}(\mathbb{A}) \in \text{P} \Leftrightarrow \mathbb{A}$ HAS ALGEBRAIC
INVARIANT

$$S(x, y, x, z, y, z) =$$

$$S(y, x, z, x, z, y)$$

THEOREM

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A FINITE

$\Rightarrow \text{CSP}(A) \in P$ OR
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$(P \neq NP)$

$\text{CSP}(A) \in P \Leftrightarrow A$ HAS ALGEBRAIC
 INVARIANT
 $S(x, y, x, z, y, z) =$
 $S(y, x, z, x, z, y)$



ALGEBRAIC INVARIANTS = POLYMORPHISMS

$A = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^r \rightarrow A$ POLYMORPHISM: \Leftrightarrow

f HOMOMORPHISM $A^r \rightarrow A \Leftrightarrow$

$\forall: \forall r_1, \dots, r_2 \in R_i$

$$f\left(\begin{array}{c} \vdots \\ \vdots \end{array}\right) \dots \begin{array}{c} \vdots \\ \vdots \end{array} \in R_i$$

$\text{POL}(A) := \{f \mid f \text{ POLYMORPHISM OF } A\}$

- CONTAINS PROJECTIONS $(x_1, \dots, x_r) \mapsto x_i$
- COMPOSITION-CLOSED

\Rightarrow ESSENTIALLY TERM FUNCTIONS
 OF AN ALGEBRA ON A !

EXAMPLES

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$
- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
- $? \in \text{Pol}(K_3)$

EXAMPLES

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- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

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(BABY) THEOREM

(BODNARČEVK + KALWIŠNIN + USTOJVT
ROMOV '09)
GEISER '08

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \Rightarrow A$ PP-DEFINES B

$\Rightarrow \text{CSP}(A)$ HARDER THAN $\text{CSP}(B)$

EXAMPLES

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- GOOD FOR CLASSIFYING CSPs ON $\{0, 1\}$
(SCHAEFER '78)

• BETTER: EQUATIONAL CONDITIONS:

- $\exists f \in \text{Pol}(\mathbb{Q}, <) \forall x, y \quad f(x, y) = f(y, x)$

- $\exists f \in \text{Pol}(\mathbb{Z}_p, +, \cdot, <)$

$\forall x, y, z \quad f(x, x, y) = f(y, x, x) = y$

- $\exists f \in \text{Pol}(K_3) : \forall x \quad f(f(x)) = x$ ☹️

EXAMPLES

- $\text{min}(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$
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- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

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GEISER '68

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 $\forall x, y, z \quad f(x, x, y) = f(y, x, x) = y$

- $\exists f \in \text{Pol}(K_3) : \forall x \quad f(f(x)) = x$ 😞

DEFINITION

- EQUATIONAL (STRONG MAL'CEV) CONDITION: SENTENCE

$$\Psi = \exists f_1, \exists f_2 \dots \forall x_1, \forall x_2, \dots$$

$$\begin{array}{l} S_1(\text{variables}) = t_1(\text{variables}) \\ \wedge \\ \vdots \\ \wedge S_k(\text{variables}) = t_k(\text{variables}) \end{array}$$

WHERE S_i, t_i, \dots TERMS OVER f_1, f_2, \dots

- Ψ TRIVIAL: $\Leftrightarrow \text{Pol}(ANY \text{ STRUCTURE}) \models \Psi$
 $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS
- $\text{EQ}(\text{Pol}(A)) := \{\Psi \mid \text{Pol}(A) \models \Psi\}$

(ADOLESCENT) THEOREM (BULATOV + JEAVONS + KRACHIN
1994)

$EQ(POL(A)) \subseteq EQ(POL(B))$

\Rightarrow A PP-INTERPRETS B

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(ADULT) THEOREM (BARTO + OPRŠAL + P. 16)

$EQ^1 \dots$ HEIGHT 1 - CONDITIONS

$$f_i(\text{variables}) = f_j(\text{variables})$$

~~x~~ ~~x~~

$$EQ^1(Pol(A)) \subseteq EQ^1(Pol(B))$$

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COROLLARY

$EQ^1(Pol(A))$ TRIVIAL

\Rightarrow A PP-CONSTRUCTS EVERYTHING

\Rightarrow CSP(A) NP-COMPLETE

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⇒ CSP(A) NP-COMPLETE

THEOREM $EQ^1(Pol(A))$ NON-TRIVIAL

⇒

$$Pol(A) \neq \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

(SIGGERS '0)

$$\neq \exists f \forall a, r, e \quad f(a, r, e, a) = f(r, a, r, e)$$

(KEARNES + MARLOWE + MCKENZIE '14)

$$\neq \exists f \forall x_1 \dots x_n$$

$$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

∀ n ≥ |A| PRIME

(BARTO + UPRÁL '11)

$$\neq \exists f \forall x, y \quad f(x \dots x, y) = \dots = f(y, x \dots x)$$

∀ n ≥ |A| PRIME

(MARŠIĆ + MCKENZIE '08)




⇒ CSP(A) ∈ P (BULATOV, ŽUK '17)

GOING INFINITE

FINITE-DOMAIN CSPs : SAT
K-COLORING
LIN. EQUATIONS / \mathbb{Z}_p

INFINITE-DOMAIN CSPs :

EXAMPLES

- $(\mathbb{Q}, <)$
 - $(\mathbb{Q}, \text{Betw}(\cdot, \cdot, \cdot))$
 - GENERAL :
UNBOUNDED SCHEDULING
 - NO MONOCHROMATIC Δ :
 - COLORING VERTICES OF GRAPH AVOIDING
- 
- COLORING EDGES AVOIDING
- 
- GRAPH ORIENTATION AVOIDING
- 




- $(\mathbb{Z}, +, 1, <)$
 - $(\mathbb{Z}, +, 1, \cdot)$
 - $(\mathbb{Q}, +, 1, \cdot)$
-
- NUMERIC PROBLEMS

GOING INFINITE

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- $(\mathbb{Z}, +, 1, <)$
 - $(\mathbb{Z}, +, 1, \cdot)$
 - $(\mathbb{Q}, +, 1, \cdot)$
-
- NUMERIC PROBLEMS

"FORBIDDEN PATTERN PROBLEMS"

Σ ... SIGNATURE E.G. $\{<\}$

\mathcal{F} ... FINITE SET OF "FORBIDDEN" Σ -STRUCTURES
 E.G. $\mathcal{F} = \{ \emptyset, \cdot, \odot, \rightarrow, \triangle \}$




- GIVEN VARIABLES,
 CONSTRAINTS DEFINABLE OVER Σ
 E.G. $\text{Betw}(x, y, z) = x < y < z \vee z < y < x$
- \exists SOLUTION IN SOME Σ -STRUCTURE
 AVOIDING \mathcal{F} ?

GOING INFINITE

FINITE-DOMAIN CSPs: SAT
 k-COLORING
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INFINITE-DOMAIN CSPs:

EXAMPLES

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, \text{Betw}(x, y, z))$
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 UNBOUNDED SCHEDULING
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 - COLORING EDGES AVOIDING

- GRAPH ORIENTATION AVOIDING 

- $(\mathbb{Z}, +, 1, <)$
 - $(\mathbb{Z}, +, 1, \cdot)$
 - $(\mathbb{Q}, +, 1, \cdot)$
- NUMERIC PROBLEMS

"FORBIDDEN PATTERN PROBLEMS"

$\mathcal{I} \dots$ SIGNATURE E.G. $\{<\}$

$\mathcal{F} \dots$ FINITE SET OF "FORBIDDEN" \mathcal{I} -STRUCTURES
 E.G. $\mathcal{F} = \{ \emptyset, \dots, \odot, \rightarrow, \Delta \}$

- GIVEN VARIABLES, CONSTRAINTS DEFINABLE OVER \mathcal{I}
 E.G. $\text{Betw}(x, y, z) = x < y < z \vee z < y < x$
- \exists SOLUTION IN SOME \mathcal{I} -STRUCTURE AVOIDING \mathcal{F} ?




OFTEN THERE IS "NICE" \mathcal{I} A
 WHOSE CSP IS THIS PROBLEM:

GOING INFINITE

FINITE-DOMAIN CSPs: SAT
 K-COLORING
 LIN. EQUATIONS / \mathbb{Z}_k

INFINITE-DOMAIN CSPs:

EXAMPLES

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, \text{Betw}(x, y, z))$
- GENERAL:
 UNBOUNDED SCHEDULING
- NO MONOCHROMATIC Δ :
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- COLORING EDGES AVOIDING

- GRAPH ORIENTATION AVOIDING 

- $(\mathbb{Z}, +, 1, <)$
 - $(\mathbb{Z}, +, 1, \cdot)$
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"FORBIDDEN PATTERN PROBLEMS"

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 E.G. $\mathcal{F} = \{ \emptyset, \dots, \text{circle}, \rightarrow\rightarrow, \Delta \}$

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- \exists SOLUTION IN SOME \mathcal{I} -STRUCTURE AVOIDING \mathcal{F} ?

OFTEN THERE IS "NICE" \mathcal{I} A
 WHOSE CSP IS THIS PROBLEM:

- RELATIONS OF \mathcal{I} A DEFINABLE IN \mathcal{B} S.T.
- $\text{Aut } \mathcal{B} \curvearrowright \mathcal{B}^n$ HAS FINITELY MANY ORBITS:
 - \mathcal{B} ω -CATEGORICAL
 - $\exists d \forall r \exists d$ u -ORBITS UNIQUELY DETERMINED BY d -SUBORBITS
 - \mathcal{B} d -HOMOGENEOUS
 - d -BOUNDED

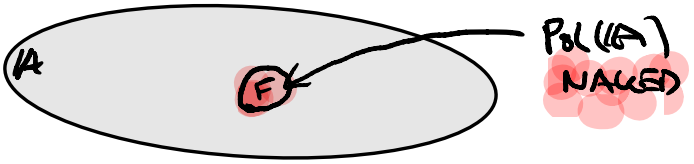
(ADULT) THEOREM (BARTO + OPSAAL + P. '16)

\mathbb{A} W-CATEGORICAL

$EQ^{\text{LOCAL}}(POL(\mathbb{A}))$ TRIVIAL $\Rightarrow \mathbb{A}$ FP-CONSTRUCTS EVERYTHING

$\Rightarrow CSP(\mathbb{A})$ NP-HARD

$\rightarrow \exists$ FSA FINITE: $EQ^{\text{LOCAL}}(POL(\mathbb{A})|_F)$ TRIVIAL



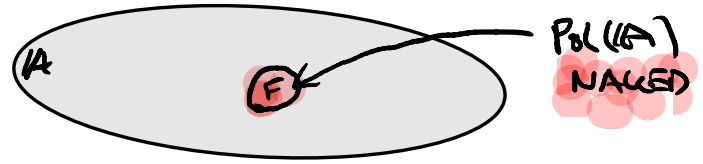
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\mathbb{A} ω -CATEGORICAL

$EQ^{\text{LOCAL}}(POL(\mathbb{A}))$ TRIVIAL $\Rightarrow \mathbb{A}$ FP-CONSTRUCTS EVERYTHING

$\Rightarrow CSP(\mathbb{A})$ NP-HARD

\exists FSA FINITE: $EQ^{\text{LOCAL}}(POL(\mathbb{A})|_F)$ TRIVIAL



THEOREM \mathbb{A} ω -CATEGORICAL
 $EQ^{\text{LOCAL}}(POL(\mathbb{A}))$ NON-TRIVIAL

\Rightarrow

$$POL(\mathbb{A}) \neq \exists u, v, f \forall x, y, z$$

$$u \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$$

(BARTO + P. '17)

F ... MANY MORE
(BARTO + BODOR + UOZIK + MOTTE + P. '23)

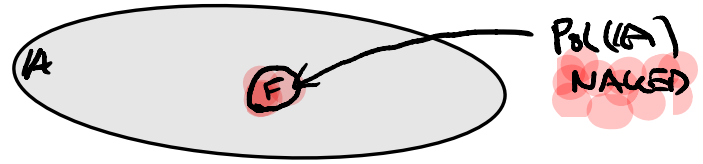
(ADULT) THEOREM (BARTO + OPSAAL + P. '16)

1A ω -CATEGORICAL

$EQ^{\text{LOCAL}}(PoL(A))$ TRIVIAL \Rightarrow 1A FP-CONSTRUCTS EVERYTHING

\Rightarrow CSP(A) NP-HARD

\rightarrow \exists FSA FINITE: $EQ^{\text{LOCAL}}(PoL(A)|_F)$ TRIVIAL



THEOREM 1A ω -CATEGORICAL
 $EQ^{\text{LOCAL}}(PoL(A))$ NON-TRIVIAL

\Rightarrow

$$PoL(A) = \exists u, v, f \forall x, y, z \\ w \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y) \\ \text{(BARTO + P. '17)}$$

F ... MANY MORE
(BARTO + BODOR + UOZIK + MOTTE + P. '23)

GOING ~~X~~INFINITE

$$PoL(A) = \exists u, v, f \forall x, y, z \\ w \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$$

... SO WHAT?

PROBLEM: f MIGHT BE UCLL!

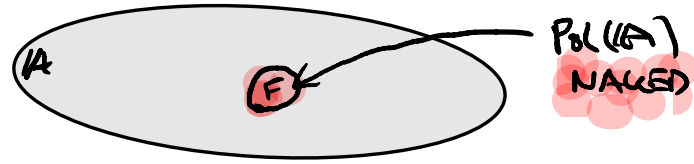
(ADULT) THEOREM (BARTO + OPSAAL + P. '16)

\mathcal{A} ω -CATEGORICAL

$EQ^*(Po(\mathcal{A}))$ TRIVIAL $\Rightarrow \mathcal{A}$ FP-CONSTRUCTS EVERYTHING

$\Rightarrow CSP(\mathcal{A})$ NP-HARD

\exists FSA FINITE: $EQ^*(Po(\mathcal{A})|_F)$ TRIVIAL



THEOREM \mathcal{A} ω -CATEGORICAL
 $EQ^*(Po(\mathcal{A}))$ NON-TRIVIAL

\Rightarrow

$$Po(\mathcal{A}) = \exists u, v, f \forall x, y, z \\ \omega \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y) \\ \text{(BARTO + P. '17)}$$

F ... MANY MORE
(BARTO + BODOR + UOZIK + MOTTE + P. '23)

GOING ~~X~~ INFINITE

$$Po(\mathcal{A}) = \exists u, v, f \forall x, y, z \\ \omega \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$$

... SO WHAT ?

PROBLEM: f MIGHT BE UGLY !

UGLINESS: IF \mathcal{A} IS DEFINABLE IN d -HOMOGENEOUS \mathcal{B}

$CSP(\mathcal{A})$: GIVEN x_1, \dots, x_n
WANT TO ASSIGN AUT \mathcal{B} -ORBITS TO d -TUPLES OF VARIABLES
... BUT f MIGHT NOT ACT ON ORBITS

EXAMPLE $CSP(\mathcal{Q}, \text{Betw}(x, y, z))$

GIVEN VARIABLES x_1, \dots, x_n
WANT TO ASSIGN " $<$ ", " $>$ ", " $=$ " TO EVERY PAIR (x_i, x_j)

DEFINITION B d -HOMOGENEOUS

$f: B^d \rightarrow B$ CANONICAL WRT $B \Leftrightarrow$
 f PRESERVES EQUIVALENCE RELATION
INDUCED BY $\text{Aut}(B) \curvearrowright B^d$

EXAMPLE

• $f: \mathbb{Q} \rightarrow \mathbb{Q}$ CANONICAL WRT
 $x \mapsto -x$ $(\mathbb{Q}, <)$

• $\text{min}: \mathbb{Q}^2 \rightarrow \mathbb{Q}$ NOT CANONICAL

$$\text{min}(a, b) = u$$

$$\text{min}(c, d) = v$$

DEFINITION \mathcal{B} d -HOMOGENEOUS

$f: \mathcal{B}^d \rightarrow \mathcal{B}$ CANONICAL WRT $\mathcal{B} \Leftrightarrow$

f PRESERVES EQUIVALENCE RELATION
INDUCED BY $\text{Aut}(\mathcal{B}) \curvearrowright \mathcal{B}^d$

EXAMPLE

• $f: \mathcal{Q} \rightarrow \mathcal{Q}$ CANONICAL WRT
 $x \mapsto -x$ $(\mathcal{Q}, <)$

• $\text{min}: \mathcal{Q}^2 \rightarrow \mathcal{Q}$ NOT CANONICAL

$$\text{min}(a, b) = a$$

$$\text{min}(c, d) = c$$

THEOREM (MOTTET + BODIRSKY '18)

\mathcal{A} DEFINABLE IN \mathcal{B} , \mathcal{B} d -HOMOGENEOUS
 d -BOUNDED

$f \in \text{Pol}(\mathcal{A})$ CANONICAL WRT \mathcal{B} ,

$$u \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$$

$\Rightarrow \text{CSP}(\mathcal{A}) \in \text{P}$

DEFINITION \mathcal{B} d -HOMOGENEOUS

$f: \mathcal{B}^d \rightarrow \mathcal{B}$ CANONICAL WRT $\mathcal{B} \Leftrightarrow$
 f PRESERVES EQUIVALENCE RELATION
INDUCED BY $\text{Aut}(\mathcal{B}) \curvearrowright \mathcal{B}^d$

EXAMPLE

• $f: \mathcal{Q} \rightarrow \mathcal{Q}$ CANONICAL WRT
 $x \mapsto -x$ $(\mathcal{Q}, <)$

• $\min: \mathcal{Q}^2 \rightarrow \mathcal{Q}$ NOT CANONICAL

$$\begin{array}{l} \min(a, b) = u \\ \quad \vee \wedge \quad ? \\ \min(c, d) = v \end{array}$$

THEOREM (MOTTET + BODIRSKY '18)

\mathcal{A} DEFINABLE IN \mathcal{B} , \mathcal{B} d -HOMOGENEOUS
 d -BOUNDED

$f \in \text{Pol}(\mathcal{A})$ CANONICAL WRT \mathcal{B} ,
 $u \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, y, z, y)$
 $\Rightarrow \text{CSP}(\mathcal{A}) \in \mathcal{P}$

• BODIRSKY + ISANOV + P. '13
USE RAMSEY THEORY TO FIND
CANONICAL POLYMORPHISMS
(THAT MIGHT NOT SATISFY IDENTITIES)

• MOTTET + P. '22
USE ALGEBRA TO FIND
CANONICAL POLYMORPHISMS
SATISFYING IDENTITIES

THEOREM

LET A BE DEFINABLE IN:

- \mathbb{Q} (BODIRSKY + UKA'RA '07)
 - THE RANDOM GRAPH (BODIRSKY + P. '11)
 - ANY HOMOGENEOUS GRAPH (BODIRSKY + MARTIN + PONGRÁČ + P. '16)
 - THE UNIVERSAL HOMOGENEOUS TOURNAMENT (MOTTET + P. '20)
 - PARTIAL ORDER (KOMPATSCHER + VAN PHAM '17)
 - 2-BRANCHING C-RELATION (BODIRSKY + JOHNSON + VAN PHAM '16)
 - ANY FINITELY BOUNDED HOMOGENEOUS HYPERGRAPH WITH RAMSEY EXPANSION BY GENERIC TOTAL ORDER (MOTTET + NAGY + P. '23)
- IF $\text{POL}(A) = \exists u, v, f \forall x, y, z \quad u \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$
THEN $\text{CSP}(A) \in \text{P}$
- OTHERWISE $\text{CSP}(A)$ NP-COMPLETE

THEOREM

LET A BE DEFINABLE IN:

- \mathbb{Q} (BODIRSKY + UKA'RA '07)
 - THE RANDOM GRAPH (BODIRSKY + P. '11)
 - ANY HOMOGENEOUS GRAPH (BODIRSKY + MARTIN + PONGRÁČ + P. '16)
 - THE UNIVERSAL HOMOGENEOUS TOURNAMENT (MOTTET + P. '20)
 - PARTIAL ORDER (KOMPATSCHER + VAN PHAM '17)
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 - ANY FINITELY BOUNDED HOMOGENEOUS HYPERGRAPH WITH RAMSEY EXPANSION BY GENERIC TOTAL ORDER (MOTTET + NAGY + P. '23)
- IF $\text{POL}(A) = \exists u, v, f \forall x, y, z \quad u \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$
THEN $\text{CSP}(A) \in P$
- OTHERWISE $\text{CSP}(A)$ NP-COMPLETE
-

CONJECTURE (BODIRSKY + P. '11, BARTO + P. '16)

TRUE $\forall A$ DEFINABLE IN d -BOUNDED HOMOGENEOUS \mathbb{R}

THANK YOU !

NO!

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