A HIDDEN SYMMETRY ON LEAF LABELLED UNORDERED ROOTED BINARY TREES

MARTIN RUBEY

Abstract. In this note we exhibit a simple bijection between leaf labelled unordered rooted binary trees and perfect matchings to show that two natural combinatorial statistics on the former set of objects enjoy symmetric joint distribution. A similar result is obtained for leaf labelled unordered rooted trees and set partitions.

1. Introduction

For an unordered rooted binary tree T with labelled leaves, let $s(T)$ be the smallest label minus one of a leaf, whose sibling is also a leaf and has smaller label. Furthermore, let $d(T)$ be the distance of the leaf labelled 1 from the root. For an example, see Figure 1.a.

Theorem 1. *There is a bijection σ on leaf labelled unordered rooted binary trees such that*

$$\big(s(T), d(T)\big) = \big(d(\sigma(T)), s(\sigma(T))\big).$$

It is well known that leaf labelled unordered rooted binary trees with $r+1$ leaves are in bijection with perfect matchings of $\{1, \dots, 2r\}$:

Theorem 2 ([2, Example 5.2.6])**.** *There is a bijection ϕ between leaf labelled unordered rooted binary trees and perfect matchings such that*

$$s(T) = i(\phi(T)), \tag{1}$$

where $i(M)$ is the number of initial openers of the perfect matching M.

We prove Theorem 1 by slightly modifying the bijection ϕ to show:

Theorem 3. *There is a bijection ψ between leaf labelled unordered rooted binary trees and perfect matchings such that*

$$\big(s(T), d(T)\big) = \big(i(\psi(T)), t(\psi(T))\big),$$

where $t(M)$ is the number of terminal closers of the perfect matching M, which are additionally larger than the partner of 1.

More than the result itself we would like to advertise the following doctrine:

> Whenever two combinatorial statistics have the same distribution,
> try to refine the result to uncover hidden symmetries.

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When the objects involved are of general interest, such as perfect matchings or set partitions, chances are good that the online database FindStat [1] is helpful. In the case at hand, we can transform the statistics s and d into statistics on perfect matchings, which comprise one of the supported combinatorial collections[1].

The statistic d was unknown to the database[2]. However, it was detected that it is equidistributed with the statistics i and s, which have the same distribution by Theorem 2. Following the doctrine, and using a small script that checks all statistics in the database, and their transformations under some natural bijections on perfect matchings such as reversal and rotation[3], Theorem 3 was discovered. As will be evident shortly, the proof is quite straightforward.

2. Bijections

Let us first recall the classical bijection proving Theorem 2. Let T be a leaf labelled unordered rooted binary tree. We inductively label the remaining notes (with the exception of the root itself) as follows, such that labels decrease along every path from the root to a node:

Let m be the largest label used so far. Consider the set of nodes which are not yet labelled, but have both children labelled. Among those, label the node having the child with smallest label with $m + 1$.

Once all nodes are labelled, the perfect matching corresponding to T is the set of pairs consisting of the labels of the siblings.

For example, performing this algorithm on the tree in Figure 1.a produces the labelling in Figure 1.b.

To prove Theorem 3, the only necessary modification is to insist that the labels of the nodes on the path from the root to 1 are maximal, that is, $2r, 2r-1, \dots, 2r-d(T)+2$.

References

[1] Martin Rubey, Christian Stump, et al. *FindStat - The combinatorial statistics database*. 2017. URL: www.findstat.org.

[2] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge: Cambridge University Press, 1999, pp. xii+581. ISBN: 0-521-56069-1; 0-521-78987-7. DOI: 10.1017/CBO9780511609589.

E-mail address: Martin.Rubey@tuwien.ac.at

Fakultät für Mathematik und Geoinformation, TU Wien, Austria

[1] See www.findstat.org/CollectionsDatabase for a list of the currently supported combinatorial collections.

[2] The statistic d is now www.findstat.org/St001041.

[3] See www.findstat.org/MapsDatabase/PerfectMatchings/PerfectMatchings for the very short list.

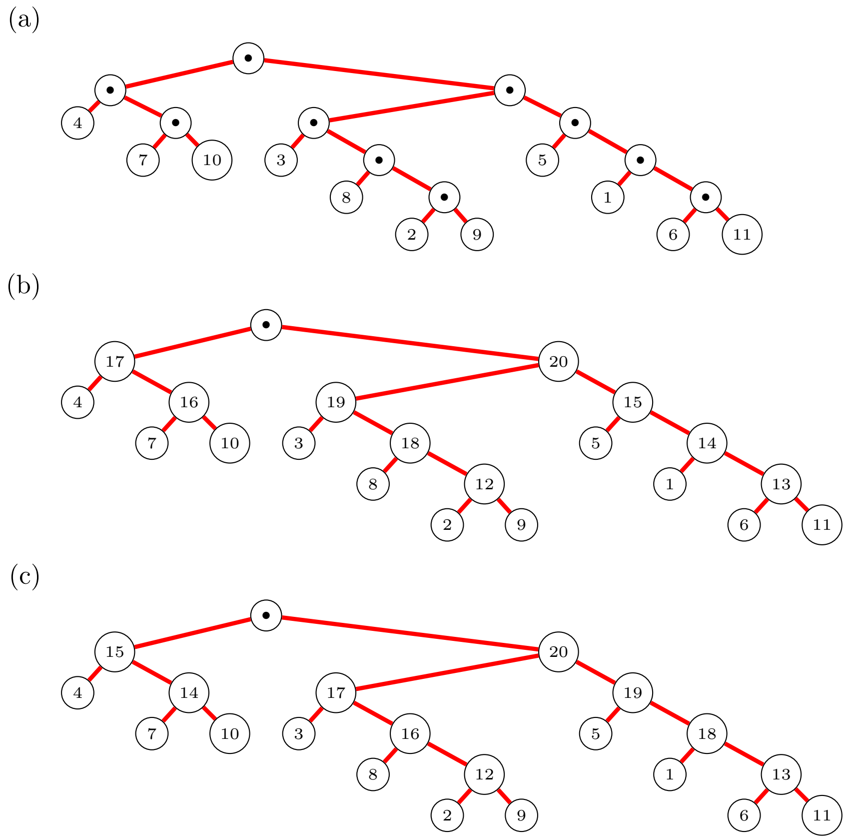

FIGURE 1. (a) A leaf labelled unordered rooted binary tree T with $r = 10$ $d(T) = 4$ and $s(T) = 8$. (b) The classical labelling proving Theorem 2. (c) The labelling proving Theorem 3.