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Diploma Thesis

Coxeter groupoids

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Contents

Introduction

In their publication [HY08], I. Heckenberger and H. Yamane provide a framework for a generalization of root systems and Coxeter groups. These notions were adapted to the language of category theory by M. Cuntz and I. Heckenberger in [CH09]. The aim of this diploma thesis is to give a self-contained account of certain results and proofs given in [HY08] using solely the terminology introduced in [CH09] and [HS09]. Our main goal is to provide a proof of [HS09, Thm. 1.6], that is, given a Cartan scheme

$$\mathcal{C} = \mathcal{C}(I, \mathcal{M}, (r_i)_{i\in I}, (A^N)_{N\in\mathcal{M}})$$

and a generalized root system of type $\mathcal{C}$

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^N)_{N\in\mathcal{M}}),$$

then the corresponding Weyl groupoid $\mathcal{W}(\mathcal{R})$ is a Coxeter groupoid with respect to its associated morphisms $(s_i^N)_{i\in I, N\in\mathcal{M}}$ and the numbers

$$m_{i,j}^N = |\boldsymbol{\Delta}_+^N \cap \mathbb{N}_0\{e_i, e_j\}|$$

where $i, j \in I$, $N \in \mathcal{M}$. This goal is achieved in Chapter 4, Theorem 4.25.

Thesis Overview

In the first chapter, we shall review some basic category theoretic background material related to the construction of Weyl and Coxeter groupoids.

In the second chapter, we will introduce the notion of a Weyl groupoid associated to a Cartan scheme and give an outline of concepts and definitions related to generalized root systems that will be used in this thesis.

In the third chapter, we shall briefly discuss Coxeter groupoids and their length functions.

Chapter four forms the core of this thesis, and is concerned with the study of the structure of generalized root systems. The first section deals with several observations that will extensively be used throughout the remainder of this chapter. In the second section, we focus on the study of generalized root systems of rank two, and give a characterisation of the sets of real roots associated to a generalized root system of such rank. In the third section, we shall prove the main goal of this thesis, that is we show that the Weyl groupoid $\mathcal{W}$ of a Cartan scheme $\mathcal{C}(I, \mathcal{M}, (r_i)_{i\in I}, (A^N)_{N\in\mathcal{M}})$ that admits a generalized root system $\mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^N)_{N\in\mathcal{M}})$ is a Coxeter groupoid with respect to its corresponding morphisms $(s_i^N)_{i\in I, N\in\mathcal{M}}$. We begin by applying the results of the previous section to generalized root systems of arbitrary rank by passing to restrictions. This allows us to show that Weyl groupoids associated to generalized root systems satisfy the Coxeter relations. This implies that there is a Coxeter groupoid $(\mathcal{G}, (t_i^N)_{i\in I, N\in\mathcal{M}})$ with objects $\mathrm{Ob}(\mathcal{G}) = \mathrm{Ob}(\mathcal{W})$ and a functor $\rho : \mathcal{G} \to \mathcal{W}$ that is the identity on the objects and that maps each morphism t_i^N to the morphism s_i^N. We then show that the length of a morphism $\omega \in \mathrm{Hom}(\mathcal{G})$ is equal to the number of negative roots contained in the set $\rho(\omega)(\boldsymbol{\Delta}_+^N)$, where N denotes the domain of ω. As a consequence, we obtain that the functor ρ is fully faithful, hence $\mathcal{W}$ is a Coxeter groupoid. We then proceed into listing further implications and, in the fourth section, focus on finite generalized root systems.

Lastly, in the fifth chapter, we will briefly discuss the connections between the terminology in [CH09], [HS09] and the framework introduced in [HY08].

1. Category Theoretic Preliminaries

The purpose of this chapter is to recall some basic constructions on categories given in [Mac71, Chpt. 2], mainly the notions of a quotient category and the free category generated by a graph, which will be used in Chapter 2 and Chapter 3 in the construction of the Weyl groupoid of a Cartan scheme and Coxeter groupoids by means of generators and relations.

First we fix some notations. Given a category $\mathcal{C}$ we denote by $\mathrm{Ob}(\mathcal{C})$ its class of objects and by $\mathrm{Hom}(\mathcal{C})$ its class of morphisms. The category $\mathcal{C}$ is called a small category, if both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}(\mathcal{C})$ are proper sets. For every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(N, N')$ we denote by $\mathrm{dom}(f) = N$ its domain and by $\mathrm{cod}(f) = N'$ its codomain. Finally, for every object $N \in \mathrm{Ob}(\mathcal{C})$ let

$$\mathrm{Hom}(N, \mathcal{C}) := \{f \in \mathrm{Hom}(\mathcal{C}) \mid \mathrm{dom}(f) = N\},$$
$$\mathrm{Hom}(\mathcal{C}, N) := \{f \in \mathrm{Hom}(\mathcal{C}) \mid \mathrm{cod}(f) = N\}.$$

1.1. **The Free Category Generated by a Graph.**

We define graphs in a way such that edges are directed and that loops and multiple edges are permitted.

Definition 1.1 (Graph)**.** A *graph* G consists of a class $\mathrm{O}(G)$ of objects and a class $\mathrm{A}(G)$ of arrows f together with two maps

$$\partial_0 : \mathrm{A}(G) \to \mathrm{O}(G),\ \partial_0(f) = \textit{domain } f,$$
$$\partial_1 : \mathrm{A}(G) \to \mathrm{O}(G),\ \partial_1(f) = \textit{codomain } f.$$

If $\mathrm{O}(G)$ and $\mathrm{A}(G)$ are proper sets, we call G a small graph.

Definition 1.2 (Morphism of Graphs)**.** Let G and G' be graphs. A *morphism of graphs* $D : G \to G'$ is a pair of maps $D_O : \mathrm{O}(G) \to \mathrm{O}(G')$ and $D_A : A(G) \to A(G')$ such that

$$D_{\mathrm{O}}\partial_0 = \partial_0' D_{\mathrm{A}} \quad \text{and} \quad D_{\mathrm{O}}\partial_1 = \partial_1' D_{\mathrm{A}}.$$

A category $\mathcal{C}$ determines a graph $\mathrm{U}(\mathcal{C})$ with objects $\mathrm{O}(\mathrm{U}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$ and arrows $\mathrm{A}(\mathrm{U}(\mathcal{C})) = \mathrm{Hom}(\mathcal{C})$, such that a morphism $f \in \mathrm{Hom}(\mathcal{C})$ is interpreted as an arrow with domain $\partial_0(f) = \mathrm{dom}(f)$ and codomain $\partial_1(f) = \mathrm{cod}(f)$. A functor $F : \mathcal{B} \to \mathcal{C}$ determines a morphism of graphs $\mathrm{U}(F) : \mathrm{U}(\mathcal{B}) \to \mathrm{U}(\mathcal{C})$ that maps every object $B \in \mathrm{O}(\mathrm{U}(\mathcal{C}))$ to $F(B)$ and every arrow $f \in \mathrm{A}(\mathrm{U}(\mathcal{C}))$ to $F(f)$.

This allows us to define the free category generated by a graph.

Definition 1.3 (Free Category)**.** Given a small graph G, there is a small category $\mathcal{C}_G$ with objects $\mathrm{Ob}(\mathcal{C}_G) = \mathrm{O}(G)$ and a morphism $\mathrm{can} : G \to \mathrm{U}(\mathcal{C}_G)$ of graphs with the following universal property. For every category $\mathcal{B}$ and every morphism $D : G \to \mathrm{U}(\mathcal{B})$ of graphs there is a unique functor $D' : \mathcal{C}_G \to \mathcal{B}$ such that $D = \mathrm{U}(D') \circ \mathrm{can}$. The category $\mathcal{C}_G$ is called the *free category generated by the graph G*.

Proof. We define the category $\mathcal{C}_G$ in the following way. Let $\mathrm{Ob}(\mathcal{C}_G) := \mathrm{O}(G)$ and let $\mathrm{Hom}(\mathcal{C}_G)$ be the set of all tuples $(b, f_n, \ldots, f_1, a)$ with $n \in \mathbb{N}_0$, $a, b \in \mathrm{O}(G)$ such that $\partial_1(f_k) = \partial_0(f_{k+1})$ for all integers $1 \leq k \leq n-1$ and, in case $n > 0$, $a = \partial_0(f_1)$, $b = \partial_1(f_n)$. Given two such tuples $(d, g_m, \ldots, g_1, c), (b, f_n, \ldots, f_1, a)$ let

$$\mathrm{dom}((b, f_n, \ldots, f_1, a)) := a \quad \text{and} \quad \mathrm{cod}((b, f_n, \ldots, f_1, a)) := b.$$

If $b = c$, then we define their composition by

$$(d, g_m, \ldots, g_1, c) \circ (b, f_n, \ldots, f_1, a) := (d, g_m, \ldots, g_1, f_n, \ldots, f_1, a).$$

For all objects a, let $\mathrm{id}_a := (a, a)$. It is easily checked, that $\mathcal{C}_G$ is a well-defined category. Define the graph morphism $\mathrm{can} : G \to \mathrm{U}(\mathcal{C}_G)$ by

$$\mathrm{can}_{\mathrm{O}}(a) := a \quad \text{and} \quad \mathrm{can}_{\mathrm{A}}(f) := (\partial_1(f), f, \partial_0(f))$$

for all objects $a \in \mathrm{O}(G)$ and arrows $f \in \mathrm{A}(G)$. Now let $\mathcal{B}$ be an arbitrary category and $D : G \to \mathrm{U}(\mathcal{B})$ a morphism of graphs. Suppose, there is a functor $D' : \mathcal{C}_G \to \mathcal{B}$ such that $D = \mathrm{U}(D') \circ \mathrm{can}$. Then for all objects $a \in \mathrm{Ob}(\mathcal{C}_G)$ and all morphisms $(b, f_n, \ldots, f_1, a) \in \mathrm{Hom}(\mathcal{C}_G)$ we have

$$D'(a) = D(a) \quad \text{and} \quad D'((b, f_n, \ldots, f_1, a)) = D(f_n) \cdots D(f_1)\, \mathrm{id}_a.$$

Thus the functor D' is uniquely determined. On the other hand, a functor $D' : \mathcal{C}_G \to \mathcal{B}$ can be defined in exactly this way and one trivially obtains $D = \mathrm{U}(D') \circ \mathrm{can}$. Hence the category $\mathcal{C}_G$ is indeed freely generated by the graph G. □

Note that if two pairs $(\mathcal{C}_G, \mathrm{can})$ and $(\mathcal{C}_G{}', \mathrm{can}')$ both satisfy Definition 1.3, then there is a fully faithful functor $F : \mathcal{C}_G \to \mathcal{C}_G{}'$ such that F is the identity on the objects and $\mathrm{can}' = \mathrm{U}(F) \circ \mathrm{can}$. By the construction given in the proof above, this implies the following result.

Proposition 1.4. *Let G be a small graph. Given a morphism $s \in \mathrm{Hom}(\mathcal{C}_G)$ there is a unique tuple of arrows $(s'_1, \ldots, s'_n) \in \mathrm{A}(G)^n$, $n \in \mathbb{N}_0$ such that the composition of morphisms $s_n \cdots s_1$, where $s_i :=$ can(s'_i) for all i, makes sense and*

$$s = s_n \ldots s_1 \mathrm{id}_N.$$

1.2. **Quotient Categories.**

Throughout this chapter, let $\mathcal{C}$ be a category and let R be a map that assigns to each pair of objects $(a, b) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$ a binary relation $R_{a,b}$ on the hom-set $\mathrm{Hom}_{\mathcal{C}}(a, b)$.

Definition 1.5 (Congruence)**.** The map R is called a congruence, if for all objects $a, b \in \mathrm{Ob}(\mathcal{C})$ the relation $R_{a,b}$ is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(a, b)$ and for all objects $a, b, c, d \in \mathrm{Ob}(\mathcal{C})$ and morphisms $g \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, $f, f' \in \mathrm{Hom}_{\mathcal{C}}(b, c)$, $h \in \mathrm{Hom}_{\mathcal{C}}(c, d)$ we have that

$$f\, R_{b,c}\, f' \quad \text{implies} \quad hfg\; R_{a,d}\; hf'g.$$

Example 1.6. Let $\mathcal{D}, \mathcal{E}$ be categories and $F : \mathcal{D} \to \mathcal{E}$ a functor. For all objects $a, b \in \mathrm{Ob}(\mathcal{D})$ and morphisms $f, f' \in \mathrm{Hom}_{\mathcal{D}}(a, b)$ define $fS_{a,b}f'$ if and only if $F(f) = F(f')$. Then S is a congruence on the category $\mathcal{D}$.

Proposition 1.7. *There is a unique congruence R' on the category $\mathcal{C}$ with $R_{a,b} \subset R'_{a,b}$ for all $a, b \in \mathrm{Ob}(\mathcal{C})$, such that for each congruence S on $\mathcal{C}$ with $R_{a,b} \subset S_{a,b}$ for all $a, b \in \mathrm{Ob}(\mathcal{C})$, one has $R'_{a,b} \subset S_{a,b}$ for all $a, b \in \mathrm{Ob}(\mathcal{C})$.*

Proof. It is obvious that such a congruence R' is unique, so we start right away with showing its existence. For all objects $b, c \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f, f' \in \mathrm{Hom}_{\mathcal{C}}(b, c)$ define $fR^1_{b,c}f'$ if and only if $f = f'$ or there are objects $a, d \in \mathrm{Ob}(\mathcal{C})$ and morphisms $g \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, $h \in \mathrm{Hom}_{\mathcal{C}}(c, d)$, $s, s' \in \mathrm{Hom}_{\mathcal{C}}(b, c)$ such that $f = hsg$, $f' = hs'g$ and $sR_{b,c}s'$ or $s'R_{b,c}s$. Thus $R_{b,c}$ is a subrelation of $R^1_{b,c}$ and $R^1_{b,c}$ is symmetric, reflexive and for all objects $a, d \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f, f' \in \mathrm{Hom}_{\mathcal{C}}(b, c)$, $g \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, $h \in \mathrm{Hom}_{\mathcal{C}}(c, d)$ we have that

$$f\, R^1_{b,c}\, f' \quad \text{implies} \quad hfg\; R^1_{a,d}\; hf'g.$$

For all objects $b, c \in \mathrm{Ob}(\mathcal{C})$ let $R'_{b,c}$ be the transitive closure of $R^1_{b,c}$, that is for all morphisms $f, f' \in \mathrm{Hom}_{\mathcal{C}}(b, c)$ define $fR'_{b,c}f'$ if and only if there are morphisms $h_1, \ldots, h_n \in \mathrm{Hom}_{\mathcal{C}}(b, c)$, $n \geq 2$ such that

$$f = h_1\, R^1_{b,c}\, h_2\, R^1_{b,c} \ldots R^1_{b,c}\, h_n = f'.$$

Then R' is a congruence on the category $\mathcal{C}$ such that for all objects $a, b \in \mathrm{Ob}(\mathcal{C})$ the relation $R_{a,b}$ is a subrelation of $R'_{a,b}$. Now, let S be an arbitrary congruence on the category $\mathcal{C}$ such that for all $a, b \in \mathrm{Ob}(\mathcal{C})$ we have that $R_{a,b} \subset S_{a,b}$. Since S is a congruence, this implies that $R^1_{a,b} \subset S_{a,b}$ and thus $R'_{a,b} \subset S_{a,b}$. □

Definition 1.8 (Quotient Category)**.** There is a category $\mathcal{C}/R$ with objects $\mathrm{Ob}(\mathcal{C}/R) = \mathrm{Ob}(\mathcal{C})$ and a functor $\mathrm{can} : \mathcal{C} \to \mathcal{C}/R$ with $\mathrm{can}(a) = a$ for all $a \in \mathrm{Ob}(\mathcal{C})$ such that:

(1) For all objects $a, b \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(a, b)$: $f\, R_{a,b}\, g$ implies $\mathrm{can}(f) = \mathrm{can}(g)$.

(2) Given a functor $H : \mathcal{C} \to \mathcal{B}$ such that ($f\, R_{a,b}\, g$ implies $H(f) = H(g)$) for all objects $a, b \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, there is a unique functor $\overline{H} : \mathcal{C}/R \to \mathcal{B}$ with $H = \overline{H} \circ \mathrm{can}$.

Proof. By Proposition 1.7 there is congruence R' on the category $\mathcal{C}$ with $R_{a,b} \subset R'_{a,b}$ for all $a, b \in \mathrm{Ob}(\mathcal{C})$, such that for each congruence S on $\mathcal{C}$ with $R_{a,b} \subset S_{a,b}$ for all $a, b \in \mathrm{Ob}(\mathcal{C})$, one has $R'_{a,b} \subset S_{a,b}$ for all $a, b \in \mathrm{Ob}(\mathcal{C})$. We define the category $\mathcal{C}/R$ as follows. Let $\mathrm{Ob}(\mathcal{C}/R) := \mathrm{Ob}(\mathcal{C})$ and for all objects $a, b \in \mathrm{Ob}(\mathcal{C}/R)$ let $\mathrm{Hom}_{\mathcal{C}/R}(a, b)$ be the set of all equivalence classes of the relation

$R'_{a,b}$. Given two equivalence classes $[f] \in \mathrm{Hom}_{\mathcal{C}/R}(a,b)$ and $[g] \in \mathrm{Hom}_{\mathcal{C}/R}(b,c)$, define their composition by

$$[g] \circ [f] := [gf].$$

For all objects a, let $\mathrm{id}_a := [\mathrm{id}_a]$. It is easily checked, that $\mathcal{C}/R$ is a well-defined category. Define the functor $\mathrm{can} : \mathcal{C} \to \mathcal{C}/R$ by

$$\mathrm{can}(a) := a \quad \text{and} \quad \mathrm{can}(f) := [f]$$

for all objects $a \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f \in \mathrm{Hom}(\mathcal{C})$. The pair $(\mathcal{C}/R, \mathrm{can})$ satisfies condition (1). Now let $\mathcal{B}$ be an arbitrary category and $H : \mathcal{C} \to \mathcal{B}$ a functor such that ($f\,R\,g$ implies $H(f) = H(g)$) for all objects $a, b \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(a,b)$. Suppose that there is a functor $\overline{H} : \mathcal{C}/R \to \mathcal{B}$ such that $H = \overline{H} \circ \mathrm{can}$. Then for all objects $a \in \mathrm{Ob}(\mathcal{C})$ we have $\overline{H}(a) = H(a)$ and for all morphisms $[f]$ we have $\overline{H}([f]) = H(f)$. Thus the functor $\overline{H}$ is uniquely determined. Furthermore, a functor $\overline{H} : \mathcal{C}/R \to \mathcal{B}$ can be defined in exactly this way and one immediately obtains $H = \overline{H} \circ \mathrm{can}$. In order to see that this is well-defined, it suffices to show that for all objects $a, b \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f, f' \in \mathrm{Hom}_{\mathcal{C}}(a,b)$ we have that $f R'_{a,b} f'$ implies $H(f) = H(f')$. But this is obvious, since the sets $S_{c,d} := \{(g,g') | g, g' \in \mathrm{Hom}_{\mathcal{C}}(c,d), H(g) = H(g')\}$ form a congruence by Example 1.6 and, by assumption, we have that $R_{c,d} \subset S_{c,d}$ for all objects c and d, hence $R'_{c,d} \subset S_{c,d}$. □

Note that if two pairs $(\mathcal{C}/R, \mathrm{can})$ and $(\mathcal{C}/R', \mathrm{can}')$ both satisfy conditions (1) and (2), then there is a fully faithful functor $F : \mathcal{C}/R \to \mathcal{C}/R'$ such that F is the identity on the objects and $\mathrm{can}' = F \circ \mathrm{can}$.

Proposition 1.9. *The functor* $\mathrm{can} : \mathcal{C} \to \mathcal{C}/R$ *is full. In particular, if* $\mathrm{Hom}(\mathcal{C})$ *is generated by a family of morphisms* $(f_i)_{i \in I}$ *then* $\mathrm{Hom}(\mathcal{C}/R)$ *is generated by* $(\mathrm{can}(f_i))_{i \in I}$.

2. Generalized Root Systems and their Weyl Groupoids

In this chapter we will recall the definition of Cartan Schemes, Weyl groupoids and generalized root systems given in [CH09].

Definition 2.1 (Cartan Matrix)**.** Let I be a non-empty finite set. A matrix $A = (a_{ij})_{i,j\in I} \in \mathbb{Z}^{I\times I}$ is called a *generalized Cartan matrix*, if

(M1) $a_{ii} = 2$ for all $i \in I$,
(M2) $a_{ij} \leq 0$ and ($a_{ij} = 0$ implies $a_{ji} = 0$) for all $i, j \in I$ with $i \neq j$.

Definition 2.2 (Cartan Scheme)**.** Let I be a non-empty finite set and $\mathcal{M}$ a non-empty set. For all $i \in I$ let $r_i : \mathcal{M} \to \mathcal{M}$ be a map and for all $N \in \mathcal{M}$ let $A^N = (a^N_{ij})_{i,j\in I}$ be a generalized Cartan matrix. The quadrupel

$$\mathcal{C} = \mathcal{C}(I, \mathcal{M}, (r_i)_{i\in I}, (A^N)_{N\in\mathcal{M}})$$

is called a *Cartan scheme*, if

(C1) $r_i^2 = id$ for all $i \in I$,
(C2) $a^N_{ij} = a^{r_i(N)}_{ij}$ for all $N \in \mathcal{M}$ and $i, j \in I$.

Definition 2.3 (Weyl Groupoid)**.** Let $\mathcal{C} = \mathcal{C}(I, \mathcal{M}, (r_i)_{i\in I}, (A^N)_{N\in\mathcal{M}})$ be Cartan scheme. For all $i \in I$ and $N \in \mathcal{M}$ define an endomorphism $s^N_i \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^I)$ by

$$s^N_i(e_j) = e_j - a^N_{ij}e_i \qquad \text{for all } j \in I \tag{2.1}$$

where $(e_i)_{i\in I}$ denotes the standard basis of $\mathbb{Z}^I$. The *Weyl groupoid* $\mathcal{W}(\mathcal{C})$ of the Cartan scheme $\mathcal{C}$ is the category with objects $\mathcal{M}$, where the morphisms are generated by all s^N_i interpreted as morphisms $s^N_i \in \mathrm{Hom}_{\mathcal{W}(C)}(N, r_i(N))$. The cardinality of I is termed the *rank* of the Weyl groupoid $\mathcal{W}(\mathcal{C})$.

Remark 2.4. Formally, the Weyl groupoid $\mathcal{W}(C)$ can be defined as follows. Consider the graph G with objects $\mathcal{M}$ and arrows $I\times\mathcal{M}$, where a pair $(i, N) \in I \times \mathcal{M}$ is interpreted as an arrow from the object N to the object $r_i(N)$. Let $(\mathcal{C}_G, \mathrm{can})$ be the free category generated by the graph G. The monoid $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^I)$ may be be considered as a category with a single object. Thus there is a unique functor $F : \mathcal{C}_G \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^I)$ such that for every $i \in I$ and $N \in \mathcal{M}$ the pair $(i, N) \in \mathrm{Hom}(\mathcal{C}_G)$ is mapped to the endomorphism s^N_i. Given two objects $N, N' \in Ob(\mathcal{C}_G)$, we define the relation $R_{N,N'}$ on $\mathrm{Hom}_{\mathcal{C}_G}(N, N')$ by $fR_{N,N'}g$ if and only if $F(f) = F(g)$. The Weyl groupoid $\mathcal{W}(\mathcal{C})$ of the Cartan scheme $\mathcal{C}$ is defined to be the quotient category $(\mathcal{W}(\mathcal{C}), \mathrm{can}')$ with generators G and relations $(R_{N,N'})_{N,N'\in\mathcal{M}}$. Note that, by Example 1.6, R is a congruence on the category $\mathcal{C}$ and hence we have $\mathrm{can}'(f) = \mathrm{can}'(g)$ if and only if $F(f) = F(g)$. By the universal property of the quotient category, the functor F factors through $\mathcal{W}(\mathcal{C})$, i.e. there is a unique functor $\bar{F} : \mathcal{W}(\mathcal{C}) \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^I)$ with $\bar{F} \circ \mathrm{can}' = F$. Thus $\mathcal{W}(\mathcal{C})$ is a category with objects $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = \mathcal{M}$ and, by Propositions 1.4 and 1.9, with $\mathrm{Hom}(\mathcal{W})$ generated by the morphisms

$$t^N_i := \mathrm{can}'(\mathrm{can}((i, N))) \in \mathrm{Hom}_{\mathcal{W}}(N, r_i(N)), i \in I, N \in \mathcal{M}$$

such that two arbitrary morphisms $\omega,\ \omega' \in \mathrm{Hom}_{\mathcal{W}}(N, N')$, $N, N' \in \mathcal{M}$,

$$\omega = t_{i_n}^{r_{i_{n-1}}\cdots r_{i_1}(N)} \cdots t_{i_1}^{N}\mathrm{id}_N \ \text{ with } n \in \mathbb{N}_0, i_1, \dots i_n \in I,$$
$$\omega' = t_{j_m}^{r_{i_{m-1}}\cdots r_{j_1}(N)} \cdots t_{j_1}^{N}\mathrm{id}_N \ \text{ with } m \in \mathbb{N}_0, j_1, \dots j_m \in I,$$

are equal if and only if $\bar{F}(\omega) = \bar{F}(\omega')$, i.e. if in $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^I)$ equation

$$s_{i_n}^{r_{i_{n-1}}\cdots r_{i_1}(N)} \cdots s_{i_1}^{N} = s_{j_m}^{r_{i_{m-1}}\cdots r_{j_1}(N)} \cdots s_{j_1}^{N}$$

holds.

Remark 2.5. The Weyl groupoid is indeed a groupoid, i.e. a category where every morphism is an isomorphism.

Proof. Given $i \in I$ and $N \in \mathcal{M}$, the $\mathbb{Z}$-endomorphism s_i^N has an inverse, since for each $j \in I$ we have

$$\begin{aligned} s_i^{r_i(N)} s_i^N(e_j) &= s_i^{r_i(N)}(e_j) - a_{ij}^N s_i^{r_i(N)}(e_i) && \text{by (2.1)} \\ &= e_j - a_{ij}^{r_i(N)} e_i - a_{ij}^N(e_i - a_{ii}^N e_i) && \text{by (2.1) and (C1)} \\ &= e_j + a_{ij}^N(a_{ii}^N - 2)e_i && \text{by (C2)} \\ &= e_j && \text{by (M1).} \end{aligned}$$

and thus, by (C1), we have that

$$s_i^{r_i(N)} s_i^N = \mathrm{id}_N \text{ and } s_i^N s_i^{r_i(N)} = \mathrm{id}_{r_i(N)} \text{ in } \mathcal{W}(\mathcal{C}). \tag{2.2}$$

Since the morphisms of $\mathcal{W}(\mathcal{C})$ are generated by the morphisms $(s_i^N)_{i\in I, N\in\mathcal{M}}$, this proves that the Weyl groupoid is a groupoid. □

Recall that a groupoid $\mathcal{G}$ is termed *connected*, if for all objects $N, N' \in \mathrm{Ob}(\mathcal{G})$ the hom-set $\mathrm{Hom}_{\mathcal{G}}(N, N')$ is non-empty. Thus, by (C1), the Weyl groupoid $\mathcal{W}(\mathcal{C})$ is connected, if and only if the subgroup $\langle\{r_i | i \in I\}\rangle \subset \mathrm{Aut}_{Set}(\mathcal{M})$ acts transitively on the set $\mathcal{M}$. An arbitrary category $\mathcal{G}$ is called *finite*, if $\mathrm{Hom}(\mathcal{G})$ is finite. This implies that $\mathrm{Ob}(\mathcal{G})$ is finite.

Convention 2.6. In order to simplify notation, for all $i \in I$, $N, N' \in \mathcal{M}$ and $\omega \in \mathrm{Hom}_{\mathcal{W}}(\mathcal{C})(N, N')$ we will write

$$s_i\omega = s_i^{N'}\omega \quad \in \mathrm{Hom}_{\mathcal{W}(\mathcal{C})}(N, r_i(N')),$$
$$\omega s_i = \omega s_i^{r_i(N)} \in \mathrm{Hom}_{\mathcal{W}(\mathcal{C})}(r_i(N), N').$$

Hence, given $n \in \mathbb{N}_0$ and $i_1, \dots, i_n \in I$, this implies

$$s_{i_n} \cdots s_{i_1}\mathrm{id}_N = s_{i_n}^{r_{i_{n-1}}\cdots r_{i_1}(N)} \cdots s_{i_2}^{r_{i_1}(N)} s_{i_1}^{N} \quad \in \mathrm{Hom}(N, r_{i_n}\cdots r_{i_1}(N)),$$
$$\mathrm{id}_N s_{i_n} \cdots s_{i_1} = s_{i_n}^{r_{i_n}(N)} s_{i_{n-1}}^{r_{i_{n-1}} r_{i_n}(N)} \cdots s_{i_1}^{r_{i_1}\cdots r_{i_n}(N)} \in \mathrm{Hom}(r_{i_1}\cdots r_{i_n}(N), N).$$

Note that any given morphism of $\mathcal{W}$ can be written in this way.

Remark 2.7. Let $\omega, \omega' \in \mathrm{Hom}(\mathcal{W})$ and $i \in I$. Then Equations (2.2) yield that

$$s_i s_i \omega = \omega$$

and therefore

$$s_i \omega = \omega' \quad \text{implies} \quad \omega = s_i \omega'.$$

Thus for all $i_1, \dots, i_n \in I$, $n \in \mathbb{N}_0$, $N \in \mathcal{M}$ we have

$$(\mathrm{id}_X s_{i_n} \cdots s_{i_1} \mathrm{id}_N)^{-1} = \mathrm{id}_N s_{i_1} \cdots s_{i_n} \mathrm{id}_X$$

where $X := r_{i_n} \cdots r_{i_1}(N)$.

Definition 2.8 (Generalized Root System)**.** Let $\mathcal{C} = \mathcal{C}(I, \mathcal{M}, (r_i)_{i \in I}, (A^N)_{N \in \mathcal{M}})$ be a Cartan scheme and $(\boldsymbol{\Delta}^N)_{N \in \mathcal{M}}$ be a family of subsets $\boldsymbol{\Delta}^N \subset \mathbb{Z}^I$. For all $N \in \mathcal{M}$ let

$$\boldsymbol{\Delta}^N_+ := \boldsymbol{\Delta}^N \cap \mathbb{N}_0^I \quad \text{and} \quad \boldsymbol{\Delta}^N_- := -\boldsymbol{\Delta}^N_+.$$

For all $i, j \in I$ and $N \in \mathcal{M}$ define

$$m^N_{i,j} := |\boldsymbol{\Delta}^N_+ \cap \mathbb{N}_0\{e_i, e_j\}| \in \mathbb{N} \cup \{\infty\}.$$

The pair

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^N)_{N \in \mathcal{M}})$$

is called a *root system of type* $\mathcal{C}$, if

(R1) $\boldsymbol{\Delta}^N = \boldsymbol{\Delta}^N_+ \cup \boldsymbol{\Delta}^N_-$ for all $N \in \mathcal{M}$,
(R2) $\boldsymbol{\Delta}^N \cap \mathbb{Z}e_i = \{e_i, -e_i\}$ for all $i \in I$, $N \in \mathcal{M}$,
(R3) $s_i^N(\boldsymbol{\Delta}^N) = \boldsymbol{\Delta}^{r_i(N)}$ for all $i \in I$, $N \in \mathcal{M}$,
(R4) $(r_i r_j)^{m^N_{i,j}}(N) = N$ for all $i, j \in I$ and $N \in \mathcal{M}$ such that the number m^N_{ij} is finite.

The cardinality of I is termed the *rank* of the root system $\mathcal{R}$. The elements of the sets $\boldsymbol{\Delta}^N_+$ and $\boldsymbol{\Delta}^N_-$ are called *positive* and *negative roots*, respectively. We say that $\mathcal{W}(\mathcal{R}) := \mathcal{W}(\mathcal{C})$ is the *Weyl groupoid of* $\mathcal{R}$.

Note that (R1) and (R2) imply

(R1') $\boldsymbol{\Delta}^N = \boldsymbol{\Delta}^N_+ \dot\cup \boldsymbol{\Delta}^N_-$ for all $N \in \mathcal{M}$. (disjoint union)

and that (R3) and the fact, that the morphisms of $\mathcal{W}(\mathcal{R})$ are generated by the family of morphisms $(s_i^N)_{i \in I, N \in \mathcal{M}}$, yield that

(R3') $\omega(\boldsymbol{\Delta}^N) = \boldsymbol{\Delta}^X$ for all $N, X \in \mathcal{M}$ and $\omega \in \mathrm{Hom}_{\mathcal{W}}(N, X)$.

For each $N \in \mathcal{M}$ the set of *real roots* of N is defined to be

$$\boldsymbol{\Delta}^{N\,\mathrm{re}} := \{\omega(e_i) \,|\, \omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), N), i \in I\} \subset \boldsymbol{\Delta}^N$$

A real root is positive, if it lies in the set $\boldsymbol{\Delta}^{N\,\mathrm{re}}_+ := \boldsymbol{\Delta}^{N\,\mathrm{re}} \cap \mathbb{N}_0^I \subset \boldsymbol{\Delta}^N_+$. A root system is termed connected, if its Weyl groupoid is connected. It is called finite, if $\boldsymbol{\Delta}^N$ is finite for all $N \in \mathcal{M}$.

Remark 2.9. Note that requirement (1) of [HY08, Def. 2] implies that the generalization of root systems given there is the equivalent of a connected root system as defined above. (For further remarks on this, see Chapter 5.) However, there is no need to restrict ourselves to connected root systems in the following chapters.

Remark 2.10. In order to comply with the definitions given in the literature, we have so far always required the set I to be finite. However, if we drop this requirement and substitute $\mathbb{Z}^I$ by $\mathbb{Z}^{(I)}$ in the definitions of Cartan schemes, Weyl groupoids and generalized root systems, then all proofs in the following chapters will work just as well. This is no surprise, since this finiteness requirement is not made in [HY08] either.

3. Coxeter Groupoids

In this chapter we will introduce the notion of a Coxeter groupoid and its length function as given in [HS09].

Definition 3.1 (Coxeter Groupoid)**.** Let I and $\mathcal{M}$ be non-empty sets, and $(r_i)_{i\in I}$ a family of maps $r_i : \mathcal{M} \to \mathcal{M}$. Let $(m_{i,j}^N)_{i,j\in I, N\in\mathcal{M}}$ be a family of numbers $m_{i,j}^N \in \mathbb{N} \cup \{\infty\}$ such that $m_{i,i}^N = 1$ for all $i \in I$, $N \in \mathcal{M}$ and $(r_i r_j)^{m_{i,j}^N}(N) = N$ for all $i, j \in I$ and $N \in \mathcal{M}$ with $m_{i,j}^N < \infty$. Let $\mathcal{W}$ be a groupoid with objects $\mathrm{Ob}(\mathcal{W}) = \mathcal{M}$ and $(s_i^N)_{i\in I, N\in\mathcal{M}}$ a family of morphisms $s_i^N \in \mathrm{Hom}_{\mathcal{W}}(N, r_i(N))$. We say that the pair $(\mathcal{W}, (s_i^N)_{i\in I,N\in\mathcal{M}})$ *satisfies the Coxeter relations* (with respect to the quadruple $(I, \mathcal{M}, (r_i)_{i\in I}, (m_{i,j}^N)_{i,j\in I,N\in\mathcal{M}})$), if for all $N \in \mathcal{M}$ and $i, j \in I$ with $m_{i,j}^N < \infty$ we have that

$$(3.1) \qquad \underbrace{s_i^{r_j(r_ir_j)^{m_{i,j}^N-1}(N)} s_j^{(r_ir_j)^{m_{i,j}^N-1}(N)} \cdots s_j^{r_ir_j(N)} s_i^{r_j(N)} s_j^N}_{2m_{i,j}^N factors} = \mathrm{id}_N.$$

The pair $(\mathcal{W}, (s_i^N)_{i\in I,N\in\mathcal{M}})$ is called a *Coxeter groupoid*, if it satisfies the Coxeter relations and for each pair $(\mathcal{G}, (t_i^N)_{i\in I,N\in\mathcal{M}})$ satisfying the Coxeter relations with respect to the same quadruple $(I, \mathcal{M}, (r_i)_{i\in I}, (m_{i,j}^N)_{i,j\in I,N\in\mathcal{M}})$ there is a unique functor $F : \mathcal{W} \to \mathcal{G}$ such that $F(N) = N$ for all objects $N \in \mathcal{M}$ and $F(s_i^N) = t_i^N$ for all $i \in I$ and $N \in \mathcal{M}$.

Proposition 3.2. *Given $I, \mathcal{M}, (r_i)_{i\in I}$, and $(m_{i,j}^N)_{i,j\in I,N\in\mathcal{M}}$ as above, there is a Coxeter groupoid $(\mathcal{W}, (s_i^N)_{i\in I,N\in\mathcal{M}})$ with respect to the quadruple $(I, \mathcal{M}, (r_i)_{i\in I}, (m_{i,j}^N)_{i,j\in I,N\in\mathcal{M}})$.*

Proof. Consider the graph G with objects $\mathcal{M}$ and arrows $I \times \mathcal{M}$, where a pair $(i, N) \in I \times \mathcal{M}$ is interpreted as an arrow from the object N to the object $r_i(N)$. Let $(\mathcal{C}_G, \mathrm{can})$ be the free category generated by the graph G. For each two objects $N, N' \in Ob(\mathcal{C}_G)$, we define a relation $R_{N,N'}$ on $\mathrm{Hom}_{\mathcal{C}_G}(N, N')$ by $R_{N,N'} := \emptyset$ if $N \neq N'$ and otherwise

$$R_{N,N} := \{(i, r_j(r_ir_j)^{m_{i,j}^N-1}(N)) \cdots (i, r_j(N))(j, N), \mathrm{id}_N) \mid i, j \in I \text{ with } m_{i,j}^N < \infty\}.$$

Let the pair $(\mathcal{W}, \mathrm{can}')$ be the quotient category with generators G and relations $(R_{N,N'})_{N,N'\in\mathcal{M}}$. For all $i \in I$ and $N \in \mathcal{M}$ let $s_i^N := \mathrm{can}'(\mathrm{can}((i, N)))$. Then the pair $(\mathcal{W}, (s_i^N)_{i\in I,N\in\mathcal{M}})$ is a Coxeter groupoid. □

Remark 3.3. Given two Coxeter groupoids $(\mathcal{W}, (s_i^N)_{i\in I,N\in\mathcal{M}})$ and $(\mathcal{G}, (t_i^N)_{i\in I,N\in\mathcal{M}})$ with respect to the same quadruple $(I, \mathcal{M}, (r_i)_{i\in I}, (m_{i,j}^N)_{i,j\in I,N\in\mathcal{M}})$, there is a fully faithful functor $F : \mathcal{W} \to \mathcal{G}$ such that $F(N) = N$ for all objects $N \in \mathcal{M}$ and $F(s_i^N) = t_i^N$ for all $i \in I$ and $N \in \mathcal{M}$.

Throughout the rest of this chapter, let $(\mathcal{W}, (s_i^N)_{i\in I,N\in\mathcal{M}})$ be a fixed Coxeter groupoid with respect ot the quadruple $(I, \mathcal{M}, (r_i)_{i\in I}, (m_{i,j}^N)_{i,j\in I,N\in\mathcal{M}})$.

Proposition 3.4. *The morphisms of the Coxeter groupid $\mathcal{W}$ are generated by the family of morphisms $(s_i^N)_{i\in I,N\in\mathcal{M}}$.*

Proof. This follows immediately from Propositions 1.9, 3.3 and the construction given in the proof of Proposition 3.2. □

Remark 3.5. Note that for each $i \in I$ we have $m_{i,i} = 1$ and thus $r_i^2 = \mathrm{id}$. Given an object $N \in \mathcal{M}$, Equation (3.1) (applied to N and to $r_i(N)$ respectively) implies that $s_i^{r_i(N)} s_i^N = \mathrm{id}_N$ and $s_i^N s_i^{r_i(N)} = \mathrm{id}_{r_i(N)}$. Thus we may use the same notation as in Convention 2.6 and by Proposition 3.4, it follows that $\mathcal{W}$ is indeed a groupoid and any given morphism of $\omega \in \mathrm{Hom}(\mathcal{W})$ may be written in the form $\omega = s_{i_1}\cdots s_{i_n}\mathrm{id}_N$ and $\omega = \mathrm{id}_X s_{i_1}\cdots s_{i_n}$ with $i_1,\dots,i_n \in I$, $n \in \mathbb{N}_0$ and $N, X \in \mathcal{M}$. Moreover, Remark 2.7 also holds for Coxeter groupoids.

Definition 3.6. We define the *length function* $l : \mathrm{Hom}(\mathcal{W}) \to \mathbb{N}_0$ as follows. For each $N \in \mathrm{Ob}(\mathcal{W})$ and $\omega \in \mathrm{Hom}(N, \mathcal{W})$ let

$$l(\omega) = \min\{n \in \mathbb{N}_0 | \text{ there are } i_1,\dots i_n \in I \text{ with } \omega = s_{i_n}\cdots s_{i_1}\mathrm{id}_N\}.$$

Proposition 3.7. *Given morphisms $\omega, \omega' \in \mathrm{Hom}(\mathcal{W})$ such that the composition $\omega\omega'$ makes sense, we have*

$$l(\omega\omega') \le l(\omega) + l(\omega') \tag{3.2}$$

and

$$l(\omega^{-1}) = l(\omega). \tag{3.3}$$

Proof. Equation (3.2) trivially follows from the definition of the map l. Let $N, N' \in \mathrm{Ob}(\mathcal{W})$ and $\omega \in \mathrm{Hom}_{\mathcal{W}}(N, N')$. In order to prove equation (3.3) it suffices to show, that $l(\omega^{-1}) \le l(\omega)$. Let $n := l(\omega)$. By definition of l there are $i_1,\dots i_n \in I$ such that $\omega = s_{i_n}\cdots s_{i_1}\mathrm{id}_N$. By Remark 3.5 it follows from $\omega\omega^{-1} = id_{N'}$ that $\omega^{-1} = s_{i_1}\cdots s_{i_n}\mathrm{id}_{N'}$, hence $l(\omega^{-1}) \le n$. □

Remark 3.8. Likewise we may define the length function l of a Weyl groupoid. Then Proposition 3.7 still holds in this case, since the proof given above works just as fine, we only have to make use Remark 2.7 instead of Remark 3.5.

Following [HY08], we additionally show that there is a functor $\mathrm{sgn} : \mathcal{W} \to \mathbb{Z}^\times$ (where we consider $\mathbb{Z}^\times$ as a groupoid with exactly one object) which maps a morphism $s_{i_1}\cdots s_{i_n}\mathrm{id}_N$ to $(-1)^n$.

Proposition 3.9. *There is a unique functor* $\mathrm{sgn} : \mathcal{W} \to \mathbb{Z}^\times$ *such that*

$$\mathrm{sgn}(s_i^N) = -1$$

for all $i \in I$ and $N \in \mathcal{M}$.

Proof. By Proposition 3.4 such a functor must be uniquely determined. In order to prove its existence, consider the graph G with objects $\mathcal{M}$ and arrows $\mathcal{M} \times \mathbb{Z}^\times \times \mathcal{M}$, where a triple $(N, k, N') \in \mathcal{M} \times I \times \mathcal{M}$ is interpreted as an arrow from the object N to the object N'. Let $(\mathcal{C}_G, \mathrm{can})$ be the free category generated by the graph G. Given two objects $N, N' \in Ob(\mathcal{C}_G)$, we define a relation $R_{N,N'}$ on $\mathrm{Hom}_{\mathcal{C}_G}(N, N')$ by

$$R_{N,N'} := \{((N,k,X)(X,k',N'), (N,kk',N')) \mid X \in \mathcal{M}, k, k' \in \mathbb{Z}^\times\}$$

Let the pair $(\mathcal{C}_G/R, \mathrm{can})$ be the quotient category with generators G and relations $(R_{N,N'})_{N,N' \in \mathcal{M}}$. By the universal property of the Coxeter groupoid, there is a unique functor $s\tilde{g}n : \mathcal{W} \to \mathcal{C}/R$ such that $s\tilde{g}n(N) = N$ for all objects $N \in \mathcal{M}$ and $s\tilde{g}n(s_i^N) = (N, -1, r_i(N))$ for all $i \in I$ and $N \in \mathcal{M}$. Furthermore, by the universal property of the quotient category $\mathcal{C}_G/R$, there is a unique functor $F : \mathcal{C}_G/R \to \mathbb{Z}^\times$ such that $F((N,k,N')) = k$ for all $N, N' \in \mathcal{M}$ and $k \in \mathbb{Z}^\times$. Define $sgn := F \circ s\tilde{g}n$. Then for all $i \in I$ and $N \in \mathcal{M}$ we have $\mathrm{sgn}(s_i \, \mathrm{id}_N) = -1$. □

Remark 3.10. The construction above works for arbitrary Coxeter groupoids. In order to prove Theorem 4.25, it suffices to use Corollary 4.20 instead, which gives a shorter, less general construction of the sgn functor.

4. Some Properties of Generalized Root Systems

Our main goal in this chapter is to provide a proof of [HS09, Thm. 1.6], i.e., we show that the Weyl groupoid of a generalized root system together with its associated morphisms $(s_i^N)_{i\in I,N\in\mathcal{M}}$ is a Coxeter groupoid with respect to the numbers $m_{i,j}^N := |\mathbf{\Delta}_+^N \cap \mathbb{N}_0\{e_i, e_j\}|$. We proceed by adapting certain results and proofs given in [HY08].

Troughout this chapter, let $\mathcal{C} = \mathcal{C}(I, \mathcal{M}, (r_i)_{i\in I}, (A^N)_{N\in\mathcal{M}})$ be a fixed Cartan scheme and $A^N = (a_{ij}^N)_{i,j\in I}$ for all $N \in \mathcal{M}$. Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\mathbf{\Delta}^N)_{N\in\mathcal{M}})$ be a fixed root system of type $\mathcal{C}$ and $\mathcal{W} = \mathcal{W}(\mathcal{R})$ its Weyl groupoid with its associated morphisms $(s_i^N)_{i\in I,N\in\mathcal{M}}$.

4.1. Counting Negative Roots.

Proposition 4.1. *We will make frequent use of the fact that for all $i \in I$ and $N \in \mathcal{M}$ one has*

$$s_i^N(e_i) = -e_i, \tag{4.1}$$

$$s_i^N(e_j) \in e_j + \mathbb{N}_0 e_i \text{ for all } j \in I \setminus \{i\}. \tag{4.2}$$

Proof. This follows immediately from Definition 2.1 and Equation (2.1). □

Lemma 4.2. *Given $N \in \mathcal{M}$ and $i \in I$, one has*

$$s_i^N(\mathbf{\Delta}_+^N \setminus \{e_i\}) = \mathbf{\Delta}_+^{r_i(N)} \setminus \{e_i\}.$$

Proof. By (C1) and (2.2) it suffices to show that

$$s_i^N(\mathbf{\Delta}_+^N \setminus \{e_i\}) \subset \mathbf{\Delta}_+^{r_i(N)} \setminus \{e_i\}.$$

So, let $x \in \mathbf{\Delta}_+^N \setminus \{e_i\}$. By the definition of the set of positive roots $\mathbf{\Delta}_+^N$, there are nonnegative integers $\lambda_j \in \mathbb{N}_0$, $j \in I$, such that $x = \sum_{j\in I} \lambda_j e_j$. Since $x \neq e_i$, we have by (R2) $x \notin \mathbb{Z}e_i$. Thus there is a $k \in I \setminus \{i\}$ such that $\lambda_k \geq 1$. We have

$$s_i^N(x) = \lambda_i s_i^N(e_i) + \sum_{j\in I\setminus\{i\}} \lambda_j s_i^N(e_j).$$

By Proposition 4.1 we have $s_i^N(e_i) = -e_i$ and $s_i^N(e_j) \in e_j + \mathbb{N}_0 e_i$ for all $j \in I \setminus \{i\}$, hence it follows that

$$s_i^N(x) \in \sum_{j\in I\setminus\{i\}} \lambda_j e_j \,+\, \mathbb{Z}e_i.$$

Since $\lambda_k \geq 1$ and $k \neq i$, it follows that $s_i(x) \notin -\mathbf{\Delta}_+^{r_i(N)}$ and $s_i(x) \neq e_i$. By Equation (R3) we have thus $s_i^N(x) \in \mathbf{\Delta}_+^{r_i(N)} \setminus \{e_i\}$.

□

Corollary 4.3. *Let $N, X \in \mathcal{M}$, $\omega \in \mathrm{Hom}_{\mathcal{W}}(N, X)$ and $i \in I$.*

(1) *If $e_i \in \omega(\boldsymbol{\Delta}^N_+)$, then one has*
$$s_i\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^{r_i(X)}_- = s^X_i(\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-) \cup \{-e_i\}$$
and the union in the right-hand side of this equation is disjoint.

(2) *If $e_i \notin \omega(\boldsymbol{\Delta}^N_+)$, equivalently, if $e_i \in \omega(\boldsymbol{\Delta}^N_-)$, we have*
$$s_i\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^{r_i(X)}_- = s^X_i(\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-) \setminus \{e_i\}$$
and $e_i \in s^X_i(\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_+)$.

Proof. By (R1') and (R3') we have $\omega^{-1}(e_i) \in \boldsymbol{\Delta}^N_+$ if and only if $\omega^{-1}(e_i) \in \boldsymbol{\Delta}^N_-$, hence
$$e_i \in \omega(\boldsymbol{\Delta}^N_+) \quad \text{if and only if} \quad -e_i \notin \omega(\boldsymbol{\Delta}^N_+). \tag{4.3}$$
We obviously have
$$s_i\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^{r_i(X)}_- = s^X_i(\underbrace{\omega(\boldsymbol{\Delta}^N_+) \cap -s^{r_i(X)}_i(\boldsymbol{\Delta}^N_+)}_{=:\Gamma}).$$
By Lemma 4.2 the equation
$$s_i(\boldsymbol{\Delta}^{r_i(X)}_+) = (\boldsymbol{\Delta}^X_+ \setminus \{e_i\}) \cup \{-e_i\}$$
holds and it follows that
$$\Gamma = \omega(\boldsymbol{\Delta}^N_+) \cap ((\boldsymbol{\Delta}^X_- \setminus \{-e_i\}) \cup \{e_i\}). \tag{4.4}$$
Now, if $e_i \in \omega(\boldsymbol{\Delta}^N_+)$, then Equations (4.3) and (4.4) imply that
$$\Gamma = (\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-) \dot\cup \{e_i\} \quad \text{(disjoint union)}$$
and by Equation (4.1) it follows that
$$s^X_i(\Gamma) = s^X_i(\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-) \dot\cup \{-e_i\} \quad \text{(disjoint union)}$$
This proves (1). If $e_i \notin \omega(\boldsymbol{\Delta}^N_+)$, then Equations (4.3) and (4.4) imply that
$$\Gamma = (\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-) \setminus \{-e_i\}$$
and $-e_i \in \omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-$. By Equation (4.1) it follows that
$$s^X_i(\Gamma) = s^X_i(\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-) \setminus \{e_i\}$$
and $e_i \in s^X_i(\omega(\boldsymbol{\Delta}^N_+) \cap \boldsymbol{\Delta}^X_-)$. Thus we have proved (2). □

Corollary 4.4. *Let $N, X \in \mathcal{M}$, $\omega \in \mathrm{Hom}_{\mathcal{W}}(N, X)$ and $n \in \mathbb{N}_0$, $i_1, \ldots, i_n \in I$ such that $\omega = s_{i_1} \cdots s_{i_n} \mathrm{id}_N$.*

(1) *One has*

$$|\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X)| \in n - 2\mathbb{N}_0.$$

(2) *If $i_1, \ldots, i_n$ are pairwise distinct, then*

$$|\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| = n$$

(3) *If $|\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X)| = n$, then*

$$\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X = \{-\mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}(e_{i_k}) \mid 1 \le k \le n\}.$$

Proof. Throughout the proof (1), (2) and (3) let $\nu := s_{i_2} \cdots s_{i_n} \mathrm{id}_N$.
(1) The proof is by induction on n. If $n = 0$, then $\omega = \mathrm{id}_N$ and (1) holds by Equation (R1'). Now, assume that $n \ge 1$ and that (1) holds for $n - 1$. Then we have $\omega = s_{i_1}\nu$ and Corollary 4.3 implies that

$$|\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| = |\nu(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^{r_{i_1}(X)}| \pm 1$$

By induction hypothesis we have

$$|\nu(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^{r_{i_1}(X)}| \in n - 1 - 2\mathbb{N}_0.$$

and thus it follows that

$$|\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| \in n - 2\mathbb{N}_0.$$

This proves (1).
(2) We prove this by by induction on n. The base step $n = 0$ holds by Equation (R1'). Now, assume that $n \ge 1$ and that (2) holds for $n - 1$. First, we show that $e_{i_1} \in \nu(\boldsymbol{\Delta}_+^N)$. By Remark 2.7 we have

$$\nu^{-1}(e_{i_1}) = \mathrm{id}_N s_{i_n} \cdots s_{i_2}(e_{i_1}) \tag{4.5}$$

Because $i_1, \ldots, i_n$ are pairwise distinct, Equation (4.2) implies that for all integers $2 \le k \le n$ we have

$$s_{i_k}(\mathbb{N}_0\{e_{i_1}, \ldots, e_{i_{k-1}}\}) \subset \mathbb{N}_0\{e_{i_1}, \ldots, e_{i_k}\},$$

and, by another induction, it follows that

$$\mathrm{id}_N s_{i_n} \cdots s_{i_2}(e_{i_1}) \in \mathbb{N}_0\{e_{i_1}, \ldots, e_{i_n}\}. \tag{4.6}$$

By (R2) and (R3') we have $\nu^{-1}(e_{i_1}) \in \boldsymbol{\Delta}^N$ and therefore, by Equations (4.5) and (4.6), the root $\nu^{-1}(e_{i_1})$ is positive, i.e. $e_{i_1} \in \nu(\boldsymbol{\Delta}_+^N)$. This implies

$$\begin{aligned} |s_{i_1}\nu(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| &= |s_{i_1}^{r_{i_1}(X)}(\nu(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^{r_{i_1}(X)}) \,\dot\cup\, \{e_{i_1}\}| && \text{by Corollary 4.3 (1)} \\ &= |\nu(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^{r_{i_1}(X)}| + 1 \\ &= n && \text{by induction hypothesis.} \end{aligned}$$

Hence we have proved (2).

(3) We proceed again by induction on n. If $n = 0$, then we have $\omega = \mathrm{id}_N$ and thus (3) holds by Equation (R1'). Now, assume that $n \geq 1$ and that (3) holds for $n-1$. Corollary 4.3 yields that

$$|\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^X| = |\nu(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{r_{i_1}(X)}| \pm 1.$$

By assumption, we have

$$|\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^X| = n, \tag{4.7}$$

thus it follows that

$$|\nu(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{r_{i_1}(X)}| = n \pm 1.$$

We have already proved (1), so we know that

$$|\nu(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{r_{i_1}(X)}| \leq n.$$

Hence we have

$$|\nu(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{r_{i_1}(X)}| = n - 1 \tag{4.8}$$

and thus, by induction hypothesis, it follows that

$$\nu(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{r_{i_1}(X)} = \{-\mathrm{id}_X s_{i_2} \cdots s_{i_{k-1}}(e_{i_k}) \,|\, 2 \leq k \leq n\}. \tag{4.9}$$

By Corollary 4.3 and Equations (4.7) and (4.8), we have

$$\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^X = s_{i_1}^{r_{i_1}(X)}(\nu(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{r_{i_1}(X)}) \cup \{-e_{i_1}\}$$

and thus Equation (4.9) yields that

$$\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^X = \{-\mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}(e_{i_k}) \,|\, 1 \leq k \leq n\}.$$

This completes the proof of (3). □

Corollary 4.5. *Given pairwise distinct $i_1, \dots, i_n \in I$, $n \in \mathbb{N}_0$ and $N \in \mathcal{M}$ we have that*

$$l(s_{i_1} \cdots s_{i_n} \mathrm{id}_N) = n.$$

Proof. Let $\omega := s_{i_1} \cdots s_{i_n} \mathrm{id}_N$ and let X be the codomain of the morphism ω. By definition of l we have $l(\omega) \leq n$ and Corollary 4.4 (1) yields that

$$|\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^X| \leq l(\omega).$$

On the other hand, we know by Corollary 4.4 (2) that

$$|\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^X| = n,$$

hence it follows that $l(\omega) = n$. □

4.2. **The Rank Two Case.**

Throughout this section let $I = \{i, j\}$ with $i \neq j$. Fix an element $N \in \mathcal{M}$ and let $d := |\boldsymbol{\Delta}_+^N| \in \mathbb{N} \cup \{\infty\}$. Further, let $i_{2k} := j$ and $i_{2k+1} := i$ for all $k \in \mathbb{Z}$.

Remark 4.6. For all $i \in I$ we have by Lemma 4.2 that $d = |\boldsymbol{\Delta}_+^{r_i(N)}|$. Thus, by (C1), for each $X \in \mathcal{M}$ that lies in the orbit of N under the group action of $\langle \{r_i | i \in I\} \rangle \subset \mathrm{Aut}_{Set}(\mathcal{M})$ on the set $\mathcal{M}$, we have $d = |\boldsymbol{\Delta}_+^X|$. Since the morphisms of the Weyl groupoid are generated by the family of morphisms $(s_i^X)_{i \in I, X \in \mathcal{M}}$, it follows that

$$d = |\boldsymbol{\Delta}_+^X| \text{ for all } X \in \mathcal{M} \text{ with } \mathrm{Hom}_{\mathcal{W}}(N, X) \neq \emptyset.$$

Lemma 4.7. *Given an integer* $0 \leq n < d$ *and* $k_1, \ldots, k_n \in I$ *let* $\omega := s_{k_1} \cdots s_{k_n} \mathrm{id}_N$. *Then*

$$\omega(e_i) \in \boldsymbol{\Delta}_-^X \textit{ implies } \omega(e_j) \in \boldsymbol{\Delta}_+^X.$$

Proof. Let $\omega(e_i) \in \boldsymbol{\Delta}_-^X$ and suppose $\omega(e_j) \notin \boldsymbol{\Delta}_+^X$. Then by (R1') we have $\omega(e_j) \in \boldsymbol{\Delta}_-^X$. By assumption we have $I = \{i, j\}$, thus it follows that

$$\omega(\boldsymbol{\Delta}_+^N) \subset \omega(\mathbb{N}_0\{e_i, e_j\}) \subset \boldsymbol{\Delta}_-^X.$$

This implies that

$$|\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| = |\omega(\boldsymbol{\Delta}_+^N)| = d.$$

However, Corollary 4.4 (1) yields

$$|\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| \leq n.$$

This contradicts the assumption $n < d$. Thus we have $\omega(e_j) \in \boldsymbol{\Delta}_+^X$ □

Lemma 4.8. *The roots*

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \in \boldsymbol{\Delta}_+^N, \quad 0 \leq n < d$$

are pairwise distinct and for all $0 \leq n < d - 1$ *(:=* ∞ *if* d *is not finite) we have*

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \neq e_{i_0}.$$

Proof. For all integers $n \in \mathbb{N}_0$ let

$$\begin{aligned} X_n &:= r_{i_n} \cdots r_{i_1}(N), \\ \omega_n &:= \mathrm{id}_N s_{i_1} \cdots s_{i_n} \mathrm{id}_{X_n}, \\ \Gamma_n &:= \{\omega_t(e_{i_{t+1}}) \mid 0 \leq t < n\}. \end{aligned}$$

We proceed by showing that for all integers $1 \leq n < d+1$ (:= ∞ if d is not finite) we have

(*) $$\Gamma_n = \boldsymbol{\Delta}_+^N \cap \omega_n(\boldsymbol{\Delta}_-^{X_n})$$

(**) $$|\Gamma_n| = n$$

We prove (*) and (**) simultaneously by induction on n. If $n=1$ then (**) holds trivially and (*) holds by Lemma 4.2. Now, assume that $1 \le n < d$ and that (*) and (**) hold for n. We have

$$\begin{aligned}\omega_n(e_{i_n}) &= \omega_{n-1}s_{i_n}(e_{i_n})\\ &= -\omega_{n-1}(e_{i_n}) \qquad \text{by Equation (4.1).}\end{aligned}$$

In particular we have $\omega_n(e_{i_n}) \in -\Gamma_n$. By induction hypothesis (*) we have $\Gamma_n \subset \mathbf{\Delta}_+^N$, hence $\omega_n(e_{i_n}) \in \mathbf{\Delta}_-^N$. Since $n < d$ by assumption, it follows from Lemma (4.7) that

$$e_{i_{n+1}} \in \omega_n^{-1}(\mathbf{\Delta}_+^N). \tag{4.10}$$

We have that

$$\mathbf{\Delta}_+^N \cap \omega_{n+1}(\mathbf{\Delta}_-^{X_{n+1}}) = \omega_{n+1}(\omega_{n+1}^{-1}(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{X_{n+1}}),$$

and Relation (4.10) and Corollary 4.3 (1) yield that

$$\begin{aligned}\omega_{n+1}^{-1}(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{X_{n+1}} &= s_{i_{n+1}}\omega_n^{-1}(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{X_{n+1}}\\ &= s_{i_{n+1}}^{X_n}(\omega_n^{-1}(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{X_n}) \cupdot \{-e_{i_{n+1}}\}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\omega_{n+1}(\omega_{n+1}^{-1}(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{X_{n+1}}) &= \omega_{n+1}s_{i_{n+1}}(\omega_n^{-1}(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{X_n}) \cupdot \{-\omega_{n+1}(e_{i_{n+1}})\}\\ &= (\mathbf{\Delta}_+^N \cap \omega_n(\mathbf{\Delta}_-^{X_n})) \cupdot \{-\omega_{n+1}(e_{i_{n+1}})\}\\ &= \Gamma_n \cupdot \{-\omega_{n+1}(e_{i_{n+1}})\} \qquad \text{by induction hypothesis (*).}\end{aligned}$$

Moreover we have that

$$\begin{aligned}-\omega_{n+1}(e_{i_{n+1}}) &= -\omega_n s_{i_{n+1}}(e_{i_{n+1}})\\ &= \omega_n(e_{i_{n+1}}) \qquad \text{by Equation (4.1).}\end{aligned}$$

Thus, in summary, we have that

$$\begin{aligned}\mathbf{\Delta}_+^N \cap \omega_{n+1}(\mathbf{\Delta}_-^{X_{n+1}}) &= \Gamma_n \cupdot \{\omega_n(e_{i_{n+1}})\}\\ &= \Gamma_{n+1} \qquad \text{by definition of } \Gamma_{n+1}.\end{aligned}$$

Hence (*) holds for $n+1$. Since $|\Gamma_n| = n$ by induction hypothesis (**) and the union in the equation above disjoint, it follows that (**) holds for $n+1$. This proves that (*) and (**) hold for all integers $1 \le n < d+1$.
It follows that the roots

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \in \mathbf{\Delta}_+^N, \quad 0 \le n < d$$

are pairwise distinct. Given an integer $0 \le n < d-1$ we have by definition of Γ_{n+1} that $e_{i_1} \in \Gamma_{n+1}$. Thus (*) implies $\omega_{n+1}^{-1}(e_{i_1}) \in \mathbf{\Delta}_-^{X_{n+1}}$. Since $n+1 < d$, Lemma 4.7 yields that $\omega_{n+1}^{-1}(e_{i_0}) \in \mathbf{\Delta}_+^{X_{n+1}}$. This means $e_{i_0} \notin \omega_{n+1}(\mathbf{\Delta}_-^{X_{n+1}})$ and hence Equation (*) yields that $e_{i_0} \notin \Gamma_{n+1}$. In particular, we have $\omega_n(e_{i_{n+1}}) \neq e_{i_0}$, i.e.

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \neq e_{i_0}.$$

This completes the proof. □

Remark 4.9. Note that we may interchange the roles of i and j. Thus Lemma 4.8 yields analogous results about the roots $\mathrm{id}_N s_{i_0} \cdots s_{i_{n-1}}(e_{i_n})$.

Corollary 4.10. *In particular:*

(1) *If d is finite, then*

$$\boldsymbol{\Delta}_+^N = \{\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \,|\, 0 \leq n < d\}$$

and

$$e_{i_0} = \mathrm{id}_N s_{i_1} \cdots s_{i_{d-1}}(e_{i_d}).$$

(2) *If d is not finite, then*

$$\begin{aligned}\boldsymbol{\Delta}_+^{N\,\mathrm{re}} = &\{\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \,|\, n \in \mathbb{N}_0\} \\ &\cup \{\mathrm{id}_N s_{i_0} \cdots s_{i_{n-1}}(e_{i_n}) \,|\, n \in \mathbb{N}_0\}\end{aligned}$$

and the elements in the above equation are pairwise distinct.

Proof. (1) Let d be finite. Using Lemma 4.8 the first equation of (1) follows from the defition of d and the second from the fact that $e_{i_0} \in \boldsymbol{\Delta}_+^N$.

(2) Let d be infinite. By Lemma 4.8 each element in the right hand side of the equation in (2) is a positive real root. Conversely, given an arbitrary positive real root $x \in \boldsymbol{\Delta}_+^{N\,\mathrm{re}}$ there are $h, h_1, \ldots, h_n \in I$, $n \in \mathbb{N}_0$ such that

$$x = s_{h_1} \cdots s_{h_n}(e_h).$$

By Remark 2.7 we may assume that $h_k \neq h_{k+1}$ for all integers $1 \leq k < n$. Since $I = \{i, j\}$, this implies that there is an integer $k \in \{0, 1\}$ such that

$$x = \mathrm{id}_N s_{i_{1+k}} \cdots s_{i_{n+k}}(e_h).$$

If $n = 0$ then we are already done, so assume $n \geq 1$. Suppose $h \neq i_{n+k+1}$. Then $h = i_{n+k}$ and since $n \geq 1$ by assumption, Equation (4.1) yields

$$x \;=\; -\mathrm{id}_N s_{i_{1+k}} \cdots s_{i_{n+k-1}}(e_{n+k}).$$

Thus, by Lemma 4.8, it follows that x is a negative root, a contradiction the assumption $x \in \boldsymbol{\Delta}_+^{N\,\mathrm{re}}$. Hence we have $h = i_{n+k+1}$ and therefore

$$x = \mathrm{id}_N s_{i_{1+k}} \cdots s_{i_{n+k}}(e_{n+k+1}).$$

This proves the equation in (2). By Lemma 4.8 the roots

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}), n \in \mathbb{N}_0$$

are pairwise distinct and the roots

$$\mathrm{id}_N s_{i_0} \cdots s_{i_{n-1}}(e_{i_n}), n \in \mathbb{N}_0$$

are pairwise distinct as well. Thus it suffices to show, that the union in the equation of (2) is disjoint. Suppose that there are integers $n, m \in \mathbb{N}_0$ such that

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) = \mathrm{id}_N s_{i_0} \cdots s_{i_{m-1}}(e_{i_m}).$$

Remark 2.7 yields that

$$e_{i_m} = \mathrm{id}_X s_{i_{m-1}} \cdots s_{i_0} s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}).$$

where $X := r_{i_{m-1}} \cdots r_{i_0}(N)$. Since we are only interested in indices modulo 2, this equation may be rewritten as

$$e_{i_{0+k}} = \mathrm{id}_X s_{i_{1+k}} \cdots s_{i_{l+k}}(e_{i_{l+k+1}}), \tag{4.11}$$

where $l := m + n$ and $k \in \{0, 1\}$ such that $k \equiv m \mod 2$. By Remark 4.6 we have $|\mathbf{\Delta}^X_+| = |\mathbf{\Delta}^N_+| = d = \infty$, hence Lemma 4.8 (applied to X instead of N and with interchanging the roles of i and j if $k \neq 0$) yields that

$$e_{i_{0+k}} \neq \mathrm{id}_X s_{i_{1+k}} \cdots s_{i_{l+k}}(e_{i_{l+k+1}}),$$

a contradiction to Equation (4.11). We have thus proved the second part of (2). □

Remark 4.11. Note that again interchanging the roles of i and j yields analogous results about the roots $\mathrm{id}_N s_{i_0} \cdots s_{i_{n-1}}(e_{i_n})$.

Remark 4.12. The inclined reader may observe that part (2) of the Corollary above is not required throughout the proof of Theorem 4.25 in the following chapter.

Lemma 4.13. *If d is finite, then*

$$(s_i s_j)^d \mathrm{id}_N = \mathrm{id}_N$$

Proof. By Remark 2.7, this is equivalent to

$$s_{i_d} \cdots s_{i_1} \mathrm{id}_N = s_{i_{d-1}} \cdots s_{i_0} \mathrm{id}_N. \tag{4.12}$$

We have $I = \{i, j\}$ by assumption, hence by (R4) and the definition of d we have that $(r_i r_j)^d(N) = N$. Therefore the two morphisms above have the same codomain. Thus, in order to prove Equation (4.12), it suffices to show that for all integers $k \in \{0, 1\}$ we have that

$$s_{i_{d+k}} \cdots s_{i_{1+k}} \mathrm{id}_N(e_{i_k}) = s_{i_{d-1+k}} \cdots s_{i_k} \mathrm{id}_N(e_{i_k}). \tag{4.13}$$

(Recall that we are only interested in indices modulo 2.) Now, let $k \in \{0, 1\}$ be arbitrary. We prove the equation above by showing that both of its sides equal to $-e_{i_{d+k}}$. We begin with the left-hand side. Corollary 4.10 (1) (with interchanging the roles of i and j if $k \neq 0$) yields that

$$e_{i_k} = \mathrm{id}_N s_{i_{1+k}} \cdots s_{i_{d-1+k}}(e_{i_{d+k}}),$$

and, by Remark 2.7, we may rewrite this as

$$s_{i_{d-1+k}} \cdots s_{i_{1+k}} \mathrm{id}_N(e_{i_k}) = e_{i_{d+k}}.$$

Let Y be the codomain of the morphism above. Then, by Equation (4.1), applying $s_{i_{d+k}} \mathrm{id}_Y$ to both sides yields that

$$s_{i_{d+k}} \cdots s_{i_{1+k}} \mathrm{id}_N(e_{i_k}) = -e_{i_{d+k}}. \tag{4.14}$$

Now, let us look at the right-hand side of Equation (4.13). Let X be the codomain of the morphism $s_{i_{d-1+k}} \cdots s_{i_k} \mathrm{id}_N$. Then, by Equation (4.1), it follows that

$$\mathrm{id}_X s_{i_{d-1+k}} \cdots s_{i_k} \mathrm{id}_N(e_{i_k}) = -\mathrm{id}_X s_{i_{d-1+k}} \cdots s_{i_{1+k}} \mathrm{id}_{r_{i_k}(N)}(e_{i_k}).$$

Since we are only interested in indices modulo 2, we have that

$$\mathrm{id}_X s_{i_{d-1+k}} \cdots s_{i_{1+k}} \mathrm{id}_{r_{i_k}(N)} = \mathrm{id}_X s_{i_{d+1+k}} \cdots s_{i_{2d-1+k}} \mathrm{id}_{r_{i_k}(N)}.$$

By Remark 4.6 we have that $d = |\mathbf{\Delta}_+^N| = |\mathbf{\Delta}_+^X|$, thus by Corollary 4.10 (1) (applied to X and with interchanging the roles of i and j if $d + k \neq 0$) it follows that

$$e_{i_{d+k}} = \mathrm{id}_X s_{i_{d+1+k}} \cdots s_{i_{2d-1+k}} \mathrm{id}_{r_{i_k}(N)}(\underbrace{e_{i_{2d+k}}}_{=e_{i_k}}).$$

Combining the last three equations yields that

$$s_{i_{d-1+k}} \cdots s_{i_k} \mathrm{id}_N(e_{i_k}) = -e_{i_{d+k}}. \tag{4.15}$$

Hence both sides of Equation (4.13) equal to $-e_{i_{d+k}}$. This concludes the proof. $\square$

Remark 4.14. Let d be finite. We have shown in the proof above, that

$$s_{i_d} \cdots s_{i_1} \mathrm{id}_N = s_{i_{d-1}} \cdots s_{i_0} \mathrm{id}_N$$

and that, if $d \equiv 0 \mod 2$, this morphism is the identity on $\mathbb{Z}^I$, and that, if $d \equiv 1 \mod 2$, it is equal to the reflection $\tau \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$ with $\tau(e_i) = e_j$ and $\tau(e_j) = e_i$.

4.3. **The General Case.**

Throughout this section let

$$m_{i,j}^N := |\mathbf{\Delta}_+^N \cap \mathbb{N}_0\{e_i, e_j\}| \in \mathbb{N} \cup \{\infty\}$$

for all $i, j \in I$ and $N \in \mathcal{M}$. For all $i, j \in I$ we have by (R2) that $m_{i,i} = 1$ and by (R4) that $(r_i r_j)^{m_{i,j}^N}(N) = N$ whenever the number $m_{i,j}^N$ is finite. Thus, by Proposition 3.2, there is a Coxeter groupoid $(\mathcal{G}, (t_i^N)_{i \in I, N \in \mathcal{M}})$ with respect to the quadruple $(I, \mathcal{M}, (r_i)_{i \in I}, (m_{i,j}^N)_{i,j \in I, N \in \mathcal{M}})$.

Proposition 4.15. *For all $i, j \in I$, $N \in \mathcal{M}$ and $r \in \langle r_i, r_j \rangle \subset \mathrm{Aut}_{Set}(\mathcal{M})$ we have*

$$m_{i,j}^N = m_{i,j}^{r(N)}. \tag{4.16}$$

Proof. Since s_i^N is an isomorphism, we have that

$$\begin{aligned} m_{i,j}^N &= |\mathbf{\Delta}_+^N \cap \mathbb{Z}\{e_i, e_j\}| \\ &= |s_i^N(\mathbf{\Delta}_+^N \cap \mathbb{Z}\{e_i, e_j\})| \\ &= |s_i^N(\mathbf{\Delta}_+^N) \cap s_i^N(\mathbb{Z}\{e_i, e_j\})|. \end{aligned}$$

By Proposition 4.1 we have that

$$s_i^N(\mathbb{Z}\{e_i, e_j\}) = \mathbb{Z}\{e_i, e_j\}$$

and Lemma 4.2 yields that

$$s_i^N(\mathbf{\Delta}_+^N) = (\mathbf{\Delta}_+^{r_i(N)} \setminus \{e_i\}) \dot\cup \{-e_i\}.$$

Hence we have

$$\begin{aligned} |s_i^N(\mathbf{\Delta}_+^N) \cap s_i^N(\mathbb{Z}\{e_i, e_j\})| &= |((\mathbf{\Delta}_+^{r_i(N)} \setminus \{e_i\}) \dot\cup \{-e_i\}) \cap \mathbb{Z}\{e_i, e_j\}| \\ &= |\mathbf{\Delta}_+^{r_i(N)} \cap \mathbb{Z}\{e_i, e_j\}| \qquad \text{by (R2)} \\ &= m_{i,j}^{r_i(N)}. \end{aligned}$$

□

The following lemma will allow us to make use of the results given in Section 4.2.

Lemma 4.16. *Let $\tilde{I}$ be a non-empty subset of I. For all $N \in \mathcal{M}$ let $\tilde{A}^N := (a_{ij}^N)_{i,j \in \tilde{I}}$. Then the quadruple*

$$\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(\tilde{I}, \mathcal{M}, (r_i)_{i \in \tilde{I}}, (\tilde{A}^N)_{N \in \mathcal{M}})$$

is a Cartan scheme. Let $\tilde{s}_i^N \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{\tilde{I}})$, $i \in \tilde{I}$, $N \in \mathcal{M}$ be the associated automorphisms. We have that

$$s_i^N(\mathbb{Z}^{\tilde{I}}) = \mathbb{Z}^{\tilde{I}} \text{ and } \tilde{s}_i^N = s_i^N|_{\mathbb{Z}^{\tilde{I}}} \text{ for all } i \in \tilde{I}, N \in \mathcal{M},$$

where we regard the module $\mathbb{Z}^{\tilde{I}}$ as a submodule of $\mathbb{Z}^I$. For all $N \in \mathcal{M}$ let $\tilde{\mathbf{\Delta}}^N := \mathbf{\Delta}^N \cap \mathbb{Z}^{\tilde{I}}$. Then the pair

$$\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\tilde{\mathcal{C}}, (\tilde{\mathbf{\Delta}}^N)_{N \in \mathcal{M}})$$

is a root system of type $\tilde{\mathcal{C}}$.

Proof. For all $N \in \mathcal{M}$ the matrix $\tilde{A}^N$ satisfies (M1) and (M2), hence $\tilde{A}^N$ is a generalized Cartan matrix. The quadruple $\tilde{\mathcal{C}}$ satisfies (C1) and (C2), hence $\tilde{\mathcal{C}}$ is a Cartan scheme. The module $\mathbb{Z}^{\tilde{I}}$ may be considered as a submodule of $\mathbb{Z}^I$. Then by Equation (2.1) we have for all $N \in \mathcal{M}$ and all $i, j \in I$ that $\tilde{s}_i^N(e_j) = s_i^N(e_j)$, thus $s_i^N(\mathbb{Z}^{\tilde{I}}) = \mathbb{Z}^{\tilde{I}}$ and $\tilde{s}_i^N = s_i^N|_{\mathbb{Z}^{\tilde{I}}}$. The pair $\tilde{\mathcal{R}}$ obviosly satisfies (R1) and (R2).

For all $N \in \mathcal{M}$ and $i, j \in \tilde{I}$ we have $m_{i,j}^N = \tilde{m}_{i,j}^N (:= \tilde{\boldsymbol{\Delta}}_+^N \cap \mathbb{N}_0\{e_i, e_j\})$, thus (R4) holds. For all $N \in \mathcal{M}$ and $i \in \tilde{I}$ we have

$$\begin{aligned}
\tilde{s}_i^N(\tilde{\boldsymbol{\Delta}}^N) &= s_i^N(\boldsymbol{\Delta}^N \cap \mathbb{Z}^{\tilde{I}}) \\
&= \boldsymbol{\Delta}^{r_i(N)} \cap s_i^N(\mathbb{Z}^{\tilde{I}}) \\
&= \boldsymbol{\Delta}^{r_i(N)} \cap \mathbb{Z}^{\tilde{I}} && \text{by Equation (2.1)} \\
&= \tilde{\boldsymbol{\Delta}}^{r_i(N)},
\end{aligned}$$

hence (R3) holds. Thus the $\tilde{\mathcal{R}}$ is a root system of type $\tilde{\mathcal{C}}$. □

Proposition 4.17. *The pair $(\mathcal{W}(\mathcal{C}), (s_i^N)_{i \in I, N \in \mathcal{M}})$ satisfies the Coxeter relations with respect to I, $\mathcal{M}$, $(r_i)_{i \in I}$ and the numbers $(m_{i,j}^N)_{i,j \in I, N \in \mathcal{M}}$. Hence there is a functor $\rho : \mathcal{G} \to \mathcal{W}$ such that ρ is the identity on the objects and $\rho(t_i^N) = s_i^N$ for all $i \in I$ and $N \in \mathcal{M}$.*

Proof. Let $N \in \mathcal{M}$ and $i, j \in I$. We have $m_{i,i}^N = 1$ by (R2) and if $m_{i,j}^N < \infty$ we have $(r_i r_j)^{m_{i,j}^N}(N) = N$ by (R4). Let $m_{i,j}^N$ be finite and let

$$\omega := (s_i s_j)^{m_{i,j}^N} \mathrm{id}_N \in \mathrm{Hom}_{\mathcal{W}}(N, N).$$

It remains to show that $\omega = \mathrm{id}_N$. In case $i = j$ this follows from Equations (2.2). So let $i \neq j$. Then, by Lemmata 4.13 and 4.16, we have

$$\omega(e_i) = e_i \quad \text{and} \quad \omega(e_j) = e_j. \tag{4.17}$$

Let $h \in I \setminus \{i, j\}$, $k \in \{i, j\}$, $X \in \mathcal{M}$ and $x \in (e_h + \mathbb{Z}\{e_i, e_j\}) \cap \boldsymbol{\Delta}_+^X$. Equation (4.1), Relation (4.2) and the fact that $x \in e_h + \mathbb{Z}\{e_i, e_j\}$ yield that

$$s_k^X(x) \in e_h + \mathbb{Z}\{e_i, e_j\}$$

and by Lemma 4.2 and the fact that $x \in \boldsymbol{\Delta}_+^X$ and $x \neq e_k$ we have that

$$s_k^X(x) \in \boldsymbol{\Delta}_+^{r_i(N)}.$$

Hence we have

$$s_k^X((e_h + \mathbb{Z}\{e_i, e_j\}) \cap \boldsymbol{\Delta}_+^X) \subset (e_h + \mathbb{Z}\{e_i, e_j\}) \cap \boldsymbol{\Delta}_+^{r_k(X)}$$

and it follows inductively that

$$\omega(e_h) = (s_i s_j)^{m_{i,j}^N} \mathrm{id}_N(e_h) \in (e_h + \mathbb{Z}\{e_i, e_j\}) \cap \boldsymbol{\Delta}_+^N$$

and in particular

$$\omega(e_h) \in e_h + \mathbb{N}_0\{e_i, e_j\}. \tag{4.18}$$

Let $\tilde{\boldsymbol{\Delta}}^N := \boldsymbol{\Delta}^N \cap \mathbb{Z}^{\{h,i,j\}}$ and $\tilde{\boldsymbol{\Delta}}_+^N := \boldsymbol{\Delta}_+^N \cap \mathbb{N}_0^{\{h,i,j\}}$. By definition of ω, Lemma 4.16 and (R3') yield that $\omega(\tilde{\boldsymbol{\Delta}}^N) = \tilde{\boldsymbol{\Delta}}^N$. By Equations (4.17) and (4.18) we have that

$\omega(\tilde{\mathbf{\Delta}}_+^N) \subset \tilde{\mathbf{\Delta}}_+^N$ and, by (R1') and linearity, it follows that $\omega(\tilde{\mathbf{\Delta}}_+^N) = \tilde{\mathbf{\Delta}}_+^N$. Thus there are nonnegative integers $\lambda_h, \lambda_i, \lambda_j \in \mathbb{N}_0$ such that

$$\omega(\lambda_h e_h + \lambda_i e_i + \lambda_j e_j) = e_h.$$

On the other hand, Equations (4.17) and (4.18) yield that

$$\omega(\lambda_h e_h + \lambda_i e_i + \lambda_j e_j) \in \lambda_h e_h + \lambda_i e_i + \lambda_j e_j + \mathbb{N}_0\{e_i, e_j\}.$$

Hence $\lambda_h = 1$ and $\lambda_i = \lambda_j = 0$, i.e. $\omega(e_h) = e_h$. Since $h \in I \setminus \{i, j\}$ was chosen arbitrarily, this proves that $\omega = \mathrm{id}_N$.

□

Lemma 4.18. *Let $i, j \in I$, $i \neq j$, $N \in \mathcal{M}$ and let $d := m_{i,j}^N$. Further, let $i_{2k} := j$ and $i_{2k+1} := i$ for all $k \in \mathbb{Z}$.*

(1) *The roots*

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \in \mathbf{\Delta}_+^N, \quad 0 \leq n < d$$

are positive and pairwise distinct.

(2) *For all $0 < n < d - 1 (:= \infty$ if d is not finite) there are positive integers $k_n, l_n \in \mathbb{N}$ such that*

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) = k_n e_{i_0} + l_n e_{i_1}.$$

(3) *If d is finite, then*

$$e_{i_0} = \mathrm{id}_N s_{i_1} \cdots s_{i_{d-1}}(e_{i_d}).$$

Proof. Lemmata 4.8 and 4.16 yield that the roots

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \in \mathbf{\Delta}_+^N \cap \mathbb{N}_0\{e_{i_0}, e_{i_1}\}, \quad 0 \leq n < d$$

are positive and pairwise distinct and that for all $0 \leq n < d - 1$

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \neq e_{i_0}.$$

In particular we have for all $1 \leq n < d - 1$ that

$$\mathrm{id}_N s_{i_1} \cdots s_{i_n}(e_{i_{n+1}}) \notin \{e_{i_0}, e_{i_1}\}.$$

This proves part (1) and (2). Part (3) follows from part (1) of Corollary 4.10 and Lemma 4.16. □

Corollary 4.19. *Let $i, j, k \in I$ and $N \in \mathcal{M}$. Then $s_i^N(e_k) = e_j$ implies $i \neq j = k$ and $m_{i,j}^N = 2$.*

Proof. By Equation (4.1) we have $s_i^N(e_i) = -e_i$, hence $i \neq k$. It follows by Relation (4.2) that

$$e_j = s_i^N(e_k) \in e_k + \mathbb{N}_0 e_i.$$

This implies $j = k$ and thus $i \neq j$ and $m_{i,j}^N \geq 2$. Since $s_i^N(e_j) = e_j$, part (1) of Lemma 4.18 yields that $m_{i,j}^N \leq 2$. Hence $m_{i,j}^N = 2$. □

Proposition 4.20. *There is a functor* $F : \mathcal{W} \to \mathbb{Z}^\times$ *(where we consider* $\mathbb{Z}^\times$ *as a groupoid with exactly one object) such that*

$$F(s_i^N) = -1$$

for all $i \in I$ *and* $N \in \mathcal{M}$.

Proof. Let

$$F(s_{i_1} \cdots s_{i_n} \mathrm{id}_N) := (-1)^n$$

for all $i_1, \dots, i_n \in I$, $n \in \mathbb{N}_0$ and $N \in \mathcal{M}$. Remark 2.7 yields that, in order to see that this is well-defined, it suffices to show that for all $i_n, \dots, i_1 \in I$, $n \in \mathbb{N}_0$ and $N \in \mathcal{M}$ we have that

$$s_{i_1} \cdots s_{i_n} \mathrm{id}_N = \mathrm{id}_N \quad \text{implies} \quad n \equiv 0 \mod 2.$$

But this is easy, since equation $s_{i_1} \cdots s_{i_n} \mathrm{id}_N = \mathrm{id}_N$ and (R1') imply that

$$|s_{i_1} \cdots s_{i_n} \mathrm{id}_N(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^N| = |\boldsymbol{\Delta}_+^N \cap \boldsymbol{\Delta}_-^N| = 0,$$

and on the other hand, Corollary 4.4 (1) yields that

$$|s_{i_1} \cdots s_{i_n} \mathrm{id}_N(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^N| \in n - 2\mathbb{N}_0,$$

hence $n \equiv 0 \mod 2$. Thus F is well-defined and obviosly preserves identities and composition of morphisms. □

Corollary 4.21. *There is a functor* $\mathrm{sgn} : \mathcal{G} \to \mathbb{Z}^\times$ *such that*

$$\mathrm{sgn}(t_i^N) = -1$$

for all $i \in I$ *and* $N \in \mathcal{M}$.

Proof. By Proposition 4.17 there is a functor $\rho : \mathcal{G} \to \mathcal{W}$ such that ρ is the identity on the objects and $\rho(t_i^N) = s_i^N$ for all $i \in I$ and $N \in \mathcal{M}$. By Proposition 4.20 there is a functor $F : \mathcal{W} \to \mathbb{Z}^\times$ such that $F(s_i^N) = -1$ for all $i \in I$ and $N \in \mathcal{M}$. Let $\mathrm{sgn} := F \circ \rho$. Then $\mathrm{sgn}(s_i^N) = -1$ for all $i \in I$ and $N \in \mathcal{M}$. □

Lemma 4.22. *Let* $i_1, \dots, i_n \in I$, $n \in \mathbb{N}$ *and* $N \in \mathcal{M}$. *If*

$$l(t_{i_1} \cdots t_{i_n} \mathrm{id}_N) < n,$$

then there exist $j_2, \dots, j_{n-1} \in I$ *such that*

$$t_{i_2} \cdots t_{i_n} \mathrm{id}_N = t_{i_1} t_{j_2} \cdots t_{j_{n-1}} \mathrm{id}_N.$$

Proof. By assumption there are $j_2, \dots, j_r \in I$, $r \in \mathbb{N}$, $r \le n$ such that

$$t_{i_1} \cdots t_{i_n} \mathrm{id}_N = t_{j_2} \cdots t_{j_r} \mathrm{id}_N. \tag{4.19}$$

Applying sgn to both sides yields that $n \equiv r - 1 \mod 2$. This and relation $r \le n$ imply that $r \le n - 1$ and that there is an integer $k \in \mathbb{N}_0$ such that $r + 2k = n - 1$. By Remark 3.5 we have that

$$t_{j_2} \cdots t_{j_r} \mathrm{id}_N = (t_{j_2} t_{j_2})^k t_{j_2} \cdots t_{j_r} \mathrm{id}_N.$$

Hence, without loss of generality, we may assume that $r = n-1$. Thus Equation (4.19) and Remark 3.5 yield that

$$t_{i_2}\cdots t_{i_n}\mathrm{id}_N = t_{i_1}t_{j_2}\cdots t_{j_{n-1}}\mathrm{id}_N.$$

□

Lemma 4.23. *Let $i_1,\dots,i_n \in I$, $n \geq 2$, and $N \in \mathcal{M}$, $X := r_{i_2}...r_{i_{n-1}}(N)$ such that*

$$l(\mathrm{id}_{r_{i_1}(X)}t_{i_2}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)}) = n-1. \tag{4.20}$$

and

$$\mathrm{id}_{r_{i_1}(X)}s_{i_2}\cdots s_{i_{n-1}}\mathrm{id}_N(e_{i_n}) = e_{i_1}. \tag{4.21}$$

Let $d := m^N_{i_1,i_2}$. Then

a) *Relation $d \leq n-1$ holds and there are $j_{d+1},\dots,j_{n-1} \in I$ such that*

$$\mathrm{id}_X t_{i_1}\cdots t_{i_{n-1}}\mathrm{id}_N = \mathrm{id}_X t_{i_{1\,(\mathrm{mod}\,2)}}\cdots t_{i_{d\,(\mathrm{mod}\,2)}}t_{j_{d+1}}\cdots t_{j_{n-1}}\mathrm{id}_N \tag{4.22}$$

where we set $2k+1\,(\mathrm{mod}\,2) := 1$ and $2k\,(\mathrm{mod}\,2) := 2$ for all $k \in \mathbb{Z}$.

b) *For all $j_{d+1},\dots,j_{n-1} \in I$ such that Equation (4.22) holds, we have that*

$$s_{j_{d+1}}\cdots s_{j_{n-1}}\mathrm{id}_N(e_{i_n}) = e_{i_{d+1\,(\mathrm{mod}\,2)}}.$$

Proof. If $d = 1$, then we have $i_1 = i_2$ by the definition of $m^N_{i_1,i_2}$. Thus a) and b) hold trivially. Now, assume that $d \geq 2$, i.e. $i_1 \neq i_2$. Then, by Equation (4.21), it follows that that $n \geq 3$.

We will now show part a). In case $d = 2$ relation $d \leq n-1$ holds, since $n \geq 3$, and Equation (4.22) holds trivially for $j_k := i_k$ for all $3 \leq k \leq n-1$. Thus a) holds for $d = 2$. Now, assume that $d \geq 3$. In order to prove a) it suffices to show that for all $2 \leq p \leq \min(d,n)$ relation $p < n$ holds and there are $j_{p+1},\dots,j_{n-1} \in I$ such that

$$\mathrm{id}_X t_{i_1}\cdots t_{i_{n-1}}\mathrm{id}_N = \mathrm{id}_X t_{i_{1\,(\mathrm{mod}\,2)}}\cdots t_{i_{p\,(\mathrm{mod}\,2)}}t_{j_{p+1}}\cdots t_{j_{n-1}}\mathrm{id}_N. \tag{4.23}$$

We prove this by induction on p. The base step $p = 2$ holds trivially, since relation $n \geq 3$ holds by assumption. Now, assume that $2 \leq p \leq \min(d,n)-1$, $p < n$ and that there are $j_{p+1},\dots,j_{n-1} \in I$ such that Equation (4.23) holds. By Remark 3.5 and Proposition 4.17 it follows that

$$\mathrm{id}_{r_{i_1}(X)}s_{i_1}\cdots s_{i_{n-1}}\mathrm{id}_N = \mathrm{id}_{r_{i_1}(X)}s_{i_{2\,(\mathrm{mod}\,2)}}\cdots s_{i_{p\,(\mathrm{mod}\,2)}}s_{j_{p+1}}\cdots s_{j_{n-1}}\mathrm{id}_N.$$

and thus Equation (4.21) yields that

$$\mathrm{id}_{r_{i_1}(X)}s_{i_{2\,(\mathrm{mod}\,2)}}\cdots s_{i_{p\,(\mathrm{mod}\,2)}}s_{j_{p+1}}\cdots s_{j_{n-1}}\mathrm{id}_N(e_{i_n}) = e_{i_1}.$$

By Remark 2.7 and Proposition 4.17 we have thus

$$s_{j_{p+1}}\cdots s_{j_{n-1}}\mathrm{id}_N(e_{i_n}) = s_{i_{p\,(\mathrm{mod}\,2)}}\cdots s_{i_{2\,(\mathrm{mod}\,2)}}\mathrm{id}_{r_{i_1}(X)}(e_{i_1}). \tag{4.24}$$

Now, assume that $p+1 \geq n$. Then induction hypothesis $p < n$ implies $p = n-1$. Thus Equation (4.24) yields that

$$e_{i_n} = s_{i_{n-1\,(\mathrm{mod}\,2)}} \cdots s_{i_{2\,(\mathrm{mod}\,2)}} \mathrm{id}_{r_{i_1}(X)}(e_{i_1}). \tag{4.25}$$

By Lemma 4.16 (applied to $\{i_1, i_2\}$) we have that

$$s_{i_{n-1\,(\mathrm{mod}\,2)}} \cdots s_{i_{2\,(\mathrm{mod}\,2)}} \mathrm{id}_{r_{i_1}(X)}(e_{i_1}) \in \mathbb{Z}\{e_{i_1}, e_{i_2}\}.$$

Hence it follows that $i_n \in \{i_1, i_2\}$. Recall that $p \leq \min(d,n) - 1$, $n \geq 3$ and $p = n-1$. Thus we have

$$1 \leq n-2 < d-1 (:= \infty \text{ if } d \text{ is not finite}).$$

Hence Proposition 4.15 and Lemma 4.18 (2) imply that

$$\underbrace{s_{i_{n-1\,(\mathrm{mod}\,2)}} \cdots s_{i_{2\,(\mathrm{mod}\,2)}}}_{n-2 \text{ factors}} \mathrm{id}_{r_{i_1}(X)}(e_{i_1}) \notin \{e_{i_1}, e_{i_2}\}.$$

This is a contradiction to Equation (4.25) and the fact that $i_n \in \{i_1, i_2\}$. Thus it follows that $p+1 < n$. We are now going to apply induction hypothesis (ii). Recall that $p \leq \min(d,n) - 1$. Hence we have $1 \leq p-1 < d-1$. Thus, by Proposition 4.15 and Lemma 4.18 (2), there are positive integers $h_1, h_2 \in \mathbb{N}$ such that

$$s_{i_{p\,(\mathrm{mod}\,2)}} \cdots s_{i_{2\,(\mathrm{mod}\,2)}} \mathrm{id}_{r_{i_1}(X)}(e_{i_1}) = h_1 e_{i_1} + h_2 e_{i_2}.$$

Hence Equation (4.24) implies

$$s_{j_{p+1}} \cdots s_{j_{n-1}} \mathrm{id}_N(e_{i_n}) = h_1 e_{i_1} + h_2 e_{i_2}. \tag{4.26}$$

Now, by Remark 3.5 and Equation (4.23) we have that

$$\mathrm{id}_{r_{i_1}(X)} t_{i_2} \cdots t_{i_n} \mathrm{id}_{r_{i_n}(N)} = \mathrm{id}_{r_{i_1}(X)} t_{i_{2\,(\mathrm{mod}\,2)}} \cdots t_{i_{p\,(\mathrm{mod}\,2)}} t_{j_{p+1}} \cdots t_{j_{n-1}} t_{i_n} \mathrm{id}_{r_{i_n}(N)}.$$

and hence Equation (4.20) implies that

$$l(\mathrm{id}_{r_{i_1}(X)} t_{i_{2\,(\mathrm{mod}\,2)}} \cdots t_{i_{p\,(\mathrm{mod}\,2)}} t_{j_{p+1}} \cdots t_{j_{n-1}} t_{i_n} \mathrm{id}_{r_{i_n}(N)}) = n-1$$

and by Relation (3.2) it follows that

$$l(t_{i_{p\,(\mathrm{mod}\,2)}} t_{j_{p+1}} \cdots t_{j_{n-1}} \mathrm{id}_N) = n-p. \tag{4.27}$$

Recall that by assumption $n \geq 3$ and $2 \leq p \leq \min(d,n) - 1$. Thus we have $1 \leq n-p \leq d-1$. By Equations (4.26), (4.27) and induction hypothesis (ii) for $n-p$ (with $j = i_n$ and $k = i_{p+1\,(\mathrm{mod}\,2)}$) there are $k_{p+2}, \ldots, k_{n-1} \in I$ such that

$$t_{j_{p+1}} \cdots t_{j_{n-1}} \mathrm{id}_N = t_{i_{p+1\,(\mathrm{mod}\,2)}} t_{k_{p+2}} \cdots t_{k_{n-1}} \mathrm{id}_N.$$

Thus induction hypothesis (4.23) yields that

$$\mathrm{id}_X t_{i_1} \cdots t_{i_{n-1}} \mathrm{id}_N = \mathrm{id}_X t_{i_{1\,(\mathrm{mod}\,2)}} \cdots t_{i_{p+1\,(\mathrm{mod}\,2)}} t_{k_{p+2}} \cdots t_{k_{n-1}} \mathrm{id}_N.$$

This completes the inductive step for p. In particular we have $\min(d,n) < n$, i.e. $d \leq n-1$, and thus we may set $p = d$, which yields that Equation (4.22) holds for some $j_{d+1}, \ldots, j_{n-1} \in I$. This completes the proof of a).

We will now prove part b). Let $j_{d+1},\dots,j_{n-1}\in I$ such that

$$\mathrm{id}_X t_{i_1}\cdots t_{i_{n-1}}\mathrm{id}_N=\mathrm{id}_X t_{i_{1\,(\mathrm{mod}\,2)}}\cdots t_{i_{d\,(\mathrm{mod}\,2)}}t_{j_{d+1}}\cdots t_{j_{n-1}}\mathrm{id}_N.$$

By Remark (3.5) it follows that

$$t_{i_{d\,(\mathrm{mod}\,2)}}\cdots t_{i_{2\,(\mathrm{mod}\,2)}}t_{i_2}\cdots t_{i_{n-1}}\mathrm{id}_N=t_{j_{d+1}}\cdots t_{j_{n-1}}\mathrm{id}_N$$

and thus, by Proposition 4.17, we have that

$$s_{i_{d\,(\mathrm{mod}\,2)}}\cdots s_{i_{2\,(\mathrm{mod}\,2)}}s_{i_2}\cdots s_{i_{n-1}}\mathrm{id}_N=s_{j_{d+1}}\cdots s_{j_{n-1}}\mathrm{id}_N.$$

By Equation (4.21) it follows that

$$s_{i_{d\,(\mathrm{mod}\,2)}}\cdots s_{i_{2\,(\mathrm{mod}\,2)}}\mathrm{id}_{r_{i_1}(X)}(e_{i_1})=s_{j_{d+1}}\cdots s_{j_{n-1}}\mathrm{id}_N(e_{i_n})$$

and by Proposition 4.15 and Lemma 4.18 (3) we have that

$$s_{i_{d\,(\mathrm{mod}\,2)}}\cdots s_{i_{2\,(\mathrm{mod}\,2)}}\mathrm{id}_{r_{i_1}(X)}(e_{i_1})=e_{i_{d+1\,(\mathrm{mod}\,2)}}.$$

Thus we have

$$s_{j_{d+1}}\cdots s_{j_{n-1}}\mathrm{id}_N(e_{i_n})=e_{i_{d+1\,(\mathrm{mod}\,2)}}.$$

This completes the proof of part b). □

Lemma 4.24. *Let $i_1,\dots,i_n\in I$, $n\in\mathbb{N}$ and $N\in\mathcal{M}$.*

(1) *If $s_{i_1}\cdots s_{i_{n-1}}\mathrm{id}_N(e_{i_n})\in\boldsymbol{\Delta}_-^X$, where $X:=r_{i_1}\cdots r_{i_{n-1}}(N)$, then*

$$l(t_{i_1}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)})<n.$$

(2) *If $l(t_{i_1}\cdots t_{i_n}\mathrm{id}_N)=n$, and there are $k,j\in I$ with $k\neq i_1$, $h_1,h_2\in\mathbb{N}$ such that*

$$s_{i_2}\cdots s_{i_n}\mathrm{id}_N(e_j)=h_1e_{i_1}+h_2e_k,$$

then there exist $j_2,\dots,j_{n-1}\in I$ with

$$t_{i_2}\cdots t_{i_n}\mathrm{id}_N=t_kt_{j_2}\cdots t_{j_{n-1}}\mathrm{id}_N.$$

(3) *One has*

$$l(t_{i_1}\cdots t_{i_n}\mathrm{id}_N)=|s_{i_1}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^{r_{i_1}\cdots r_{i_n}(N)}|.$$

Proof. We prove (1), (2) and (3) simultaneously by induction on n. Let $n=1$. Part (1) holds trivially, since $e_{i_1}\notin\boldsymbol{\Delta}_-^N$. Part (2) holds trivially, since for all $k\in I\setminus\{i_1\}$, $j\in I$, $h_1,h_2\in\mathbb{N}$ we have $e_j\neq h_1e_{i_1}+h_2e_k$. We eventually show that both sides of the equation in part (3) are equal to 1. Suppose $t_{i_1}\mathrm{id}_N=\mathrm{id}_N$. Then applying the functor sgn to both sides yields a contradiction. Thus $t_{i_1}\mathrm{id}_N\neq\mathrm{id}_N$ and $l(t_{i_1}\mathrm{id}_N)=1$. By Lemma 4.2 we have that

$$|s_{i_1}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^{r_{i_1}(N)}|=1,$$

thus (3) holds for $n=1$. Now, assume that $n\geq 2$ and that (1), (2) and (3) hold for $1,\dots,n-1$.

First we show that part (1) holds for n. Let $\mathrm{id}_X s_{i_1}\cdots s_{i_{n-1}}\mathrm{id}_N(e_{i_n}) \in \boldsymbol{\Delta}^X_-$, where $X := r_{i_1}\cdots r_{i_{n-1}}(N)$. We have to show that

$$l(\mathrm{id}_X t_{i_1}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)}) < n.$$

If $i_1 = i_2$ or $l(\mathrm{id}_{r_{i_1}(X)} t_{i_2}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)}) < n-1$, then Relation (3.2) yields that

$$l(\mathrm{id}_X t_{i_1}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)}) < n-1$$

and we are done. So, we may assume that $i_1 \neq i_2$ and

$$l(\mathrm{id}_{r_{i_1}(X)} t_{i_2}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)}) = n-1. \tag{4.28}$$

By induction hypothesis (1) for $n-1$ and (R1') it follows that

$$\mathrm{id}_{r_{i_1}(X)} s_{i_2}\cdots s_{i_{n-1}}\mathrm{id}_N(e_{i_n}) \in \boldsymbol{\Delta}^{r_{i_1}(X)}_+.$$

By assumption we have

$$\mathrm{id}_X s_{i_1}\cdots s_{i_{n-1}}\mathrm{id}_N(e_{i_n}) \in \boldsymbol{\Delta}^X_-,$$

hence Lemma 4.2 yields that

$$\mathrm{id}_{r_{i_1}(X)} s_{i_2}\cdots s_{i_{n-1}}\mathrm{id}_N(e_{i_n}) = e_{i_1}. \tag{4.29}$$

Let $d := m^N_{i_1,i_2}$. By assumption we have $i_1 \neq i_2$, thus $d \geq 2$. Equations (4.28), (4.29) and Lemma (4.23) yield that:

a) Relation $d \leq n-1$ holds and there are $j_{d+1},\dots,j_{n-1} \in I$ such that

$$\mathrm{id}_X t_{i_1}\cdots t_{i_{n-1}}\mathrm{id}_N = \mathrm{id}_X t_{i_{1\,(\mathrm{mod}\,2)}}\cdots t_{i_{d\,(\mathrm{mod}\,2)}} t_{j_{d+1}}\cdots t_{j_{n-1}}\mathrm{id}_N \tag{4.30}$$

where we set $2k+1\,(\mathrm{mod}\,2) := 1$ and $2k\,(\mathrm{mod}\,2) := 2$ for all $k \in \mathbb{Z}$.

b) Equation $s_{j_{d+1}}\cdots s_{j_{n-1}}\mathrm{id}_N(e_{i_n}) = e_{i_{d+1\,(\mathrm{mod}\,2)}}$ holds.

Since $d \leq n-1 < \infty$, we have, by the definition of d and Proposition (4.15), that $\mathrm{id}_X(t_{i_1}t_{i_2})^d = \mathrm{id}_X$. By Remark 3.5 it follows that

$$\mathrm{id}_X t_{i_{1\,(\mathrm{mod}\,2)}}\cdots t_{i_{d\,(\mathrm{mod}\,2)}} = \mathrm{id}_X t_{i_{2\,(\mathrm{mod}\,2)}}\cdots t_{i_{d+1\,(\mathrm{mod}\,2)}}.$$

Thus it follows from Equation (4.30) that

$$\mathrm{id}_X t_{i_1}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)} = \mathrm{id}_X t_{i_{2\,(\mathrm{mod}\,2)}}\cdots t_{i_{d+1\,(\mathrm{mod}\,2)}} t_{j_{d+1}}\cdots t_{j_{n-1}} t_{i_n}\mathrm{id}_{r_{i_n}(N)}. \tag{4.31}$$

By b) and Equation (4.1) we obtain

$$\underbrace{s_{i_{d+1\,(\mathrm{mod}\,2)}} s_{j_{d+1}}\cdots s_{j_{n-1}}}_{n-d\text{ factors}}\mathrm{id}_N(e_{i_n}) \in \boldsymbol{\Delta}_-^{r_{i_{d+1\,(\mathrm{mod}\,2)}} r_{j_{d+1}}\cdots r_{j_{n-1}}(N)}.$$

Since $d \geq 2$, we have that $n-d+1 \leq n-1$. Thus, by induction hypothesis (1), it follows that

$$l(t_{i_{d+1\,(\mathrm{mod}\,2)}} t_{j_{d+1}}\cdots t_{j_{n-1}} t_{i_n}) < n-d+1.$$

Hence, by Equation (4.31) and Relation (3.2), it follows that

$$l(\mathrm{id}_X t_{i_1}\cdots t_{i_n}\mathrm{id}_{r_{i_n}(N)}) \leq (d-2+1)+(n-d) = n-1.$$

Thus (1) holds for n.

Next we demonstrate that part (3) holds for n. We have to show that

$$l(t_{i_1}\cdots t_{i_n}\mathrm{id}_N)=|s_{i_1}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^X| \tag{4.32}$$

where $X:=r_{i_1}\cdots r_{i_n}(N)$. Suppose that $l(t_{i_1}\cdots t_{i_n}\mathrm{id}_N)<n$. Then there are $j_1,\dots,j_k\in I$, $0\le k\le n-1$ such that

$$\mathrm{id}_X t_{i_1}\cdots t_{i_n}\mathrm{id}_N=\mathrm{id}_X t_{j_1}\cdots t_{j_k}\mathrm{id}_N.$$

By Proposition (4.17) it follows that

$$|s_{i_1}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^X|=|s_{j_1}\cdots s_{j_k}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^X|.$$

Thus, if $k=0$, then Equation (4.32) holds since both of its sides are zero. If $1\le k\le n-1$ then induction hypothesis (3) yields that

$$l(t_{j_1}\cdots t_{j_k}\mathrm{id}_N)=|s_{j_1}\cdots s_{j_k}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^X|$$

and thus Equation (4.32) holds. Hence we may assume that

$$l(t_{i_1}\cdots t_{i_n}\mathrm{id}_N)=n. \tag{4.33}$$

By Relation (3.2) it follows that

$$l(t_{i_2}\cdots t_{i_n}\mathrm{id}_N)=n-1.$$

and hence induction hypothesis (3) yields

$$|s_{i_2}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^{r_{i_1}(X)}|=n-1. \tag{4.34}$$

Now, Equations (4.33) and (3.3) imply that

$$l(\mathrm{id}_N t_{i_n}\cdots t_{i_1}\mathrm{id}_X)=n.$$

We have already proved (1) for n, hence it follows that

$$\mathrm{id}_N s_{i_n}\cdots s_{i_2}(e_{i_1})\notin\boldsymbol{\Delta}_-^N$$

and thus, by (R1'), we have that

$$e_{i_1}\in s_{i_2}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N).$$

By Corollary 4.3 it follows that

$$|s_{i_1}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^X|=|s_{i_2}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^{r_{i_1}(X)}|+1$$

and hence Equation (4.34) implies that

$$|s_{i_1}\cdots s_{i_n}\mathrm{id}_N(\boldsymbol{\Delta}_+^N)\cap\boldsymbol{\Delta}_-^X|=n.$$

Combining the last equation with Equation (4.33) yields that (3) holds for n.

Eventually we show that part (2) holds for n. Let $\tilde{\omega}:=t_{i_2}\cdots t_{i_n}\mathrm{id}_N$, $\omega:=s_{i_2}\cdots s_{i_n}\mathrm{id}_N$ and $X:=r_{i_2}\cdots r_{i_n}(N)$. Suppose that $l(t_{i_1}\tilde{\omega})=n$ and that there are $k\in I\setminus\{i_1\}$, $j\in I$, $h_1,h_2\in\mathbb{N}$ such that

$$\omega(e_j)=h_1e_{i_1}+h_2e_k. \tag{4.35}$$

We have to show that there are $j_2, \dots, j_{n-1} \in I$ with

$$\tilde{\omega} = t_k t_{j_2} \cdots t_{j_{n-1}} \mathrm{id}_N.$$

Note that by Lemma 4.22 it suffices to show that $l(t_k\tilde{\omega}) < n$. We have already proved part (3) for n and by assumption we have $l(t_{i_1}\tilde{\omega}) = n$, thus it follows that

$$|s_{i_1}\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^{r_{i_1}(X)}| = n. \tag{4.36}$$

By Corollary 4.4 (1) we have that $|\omega(\mathbf{\Delta}_+^N) \cap \mathbf{\Delta}_-^X| \le n-1$ and thus, by Corollary 4.3 and Equation (4.36), it follows that there is a positive root $\beta \in \mathbf{\Delta}_+^N$ such that $\omega(\beta) = e_{i_1}$. By (R2) and (R3') there is a root $\beta' \in \mathbf{\Delta}^N$ such that $\omega(\beta') = e_{i_k}$. Thus Equation (4.35) yields that

$$\omega(e_j) = \omega(h_1\beta + h_2\beta'),$$

hence

$$e_j = h_1\beta + h_2\beta'. \tag{4.37}$$

Suppose that $\beta' \in \mathbf{\Delta}_+^N$. Since $h_1, h_2 \ge 1$ and $\beta, \beta' \in \mathbb{N}_0^I \setminus \{0\}$, it follows that $h_1\beta + h_2\beta'$ is a linear combination of the basis $(e_i)_{i\in I}$ such that at least two coefficients are greater than or equal to 1 or at least one coefficient is greater than or equal to 2. But this is a contradiction to Equation (4.37). Therefore we have that $\beta' \notin \mathbf{\Delta}_+^N$ and thus, by the definition of β' and (R1'), it follows that

$$\omega^{-1}(e_{i_k}) = \beta' \in \mathbf{\Delta}_-^N.$$

We have already proved (1) for n, hence it follows that

$$l(\tilde{\omega}^{-1}t_k) < n.$$

and, by Equation (3.3), we obtain

$$l(t_k\tilde{\omega}) < n.$$

By Lemma 4.22 it follows that there are $j_2, \dots, j_{n-1} \in I$ such that

$$\tilde{\omega} = t_k t_{j_2} \cdots t_{j_{n-1}} \mathrm{id}_N.$$

Thus (2) holds for n. This completes the inductive step. □

Theorem 4.25. *The pair $(\mathcal{W}(\mathcal{C}), (s_i^N)_{i\in I, N\in\mathcal{M}})$ is a Coxeter groupoid with respect to I, $\mathcal{M}$, $(r_i)_{i\in I}$ and the numbers $(m_{i,j}^N)_{i,j\in I, N\in\mathcal{M}}$.*

Proof. By Proposition 4.17 there is a functor $F : \mathcal{G} \to \mathcal{W}$ such that F is the identity on the objects and $F(t_i^N) = s_i^N$ for all $i \in I$ and $N \in \mathcal{M}$. We show that F is fully faithful. The functor F is obviously a full functor, since the morphisms of the Coxeter groupoid $\mathcal{G}$ are generated by the family of morphisms $(t_i^N)_{i\in I, N\in\mathcal{M}}$. We know that $\mathcal{G}$ is a groupoid, so in order to show that F is faithful, it suffices to show that for all objects $N \in \mathcal{M}$ and morphisms $\omega \in \mathrm{Hom}_{\mathcal{G}}(N,N)$ equation $F(\omega) = \mathrm{id}_N$ implies $\omega = \mathrm{id}_N$. So, let $N \in \mathcal{M}$ and $\omega \in \mathrm{Hom}_{\mathcal{G}}(N,N)$ such that $F(\omega) = \mathrm{id}_N$. By Lemma 4.24 (3) it follows that $l(\omega) = 0$ and hence $\omega = \mathrm{id}_N$.

Thus the functor F is fully faithful. We know that F is the identity on the objects, hence F is an equivalence of categories and therefore the Weyl groupoid $\mathcal{W}$ is Coxeter. □

Remark 4.26. Theorem 4.25 allows us to identify $\mathcal{G}$ with $\mathcal{W}$.

Corollary 4.27. *Given $N, X \in \mathcal{M}$, $\omega \in \mathrm{Hom}_{\mathcal{W}}(N, X)$ one has*

$$l(\omega) = |\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X|.$$

Proof. This follows from Lemma 4.24 (3) and Remark 4.26. □

Corollary 4.28. *Let $N, X \in \mathcal{M}$, $\omega \in \mathrm{Hom}_{\mathcal{W}}(N, X)$ and $i \in I$. Then*

$$l(s_i\omega), l(\omega s_i) \in \{l(\omega) \pm 1\}$$

and

$$l(s_i\omega) = l(\omega) + 1 \quad \textit{if and only if} \quad \omega^{-1}(e_i) \in \boldsymbol{\Delta}_+^N$$
$$l(\omega s_i) = l(\omega) + 1 \quad \textit{if and only if} \quad \omega(e_i) \in \boldsymbol{\Delta}_+^X.$$

Proof. We know by Corollary 4.3 that

$$|s_i\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^{r_i(X)}| = |\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| \pm 1$$

and that

$$|s_i\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^{r_i(X)}| = |\omega(\boldsymbol{\Delta}_+^N) \cap \boldsymbol{\Delta}_-^X| + 1 \quad \text{if and only if} \quad \omega^{-1}(e_i) \in \boldsymbol{\Delta}_+^N.$$

By Corollary 4.27 this may be rewritten as

$$l(s_i\omega) = l(\omega) \pm 1$$

and

$$l(s_i\omega) = l(\omega) + 1 \quad \text{if and only if} \quad \omega^{-1}(e_i) \in \boldsymbol{\Delta}_+^N.$$

We have thus proved the first part of the claim. By Equation (3.3) it follows that

$$l(\omega s_i) = l(s_i\omega^{-1}) = l(\omega^{-1}) \pm 1 = l(\omega) \pm 1$$

and

$$l(\omega s_i) = l(\omega) + 1 \quad \text{if and only if} \quad \omega(e_i) \in \boldsymbol{\Delta}_+^X.$$

□

Corollary 4.29. *Let $N, X \in \mathcal{M}$, $\omega \in \mathrm{Hom}_{\mathcal{W}}(N, X)$ and $i, j \in I$. Then the following assertions are equivalent.*

a) $\omega(e_j) = e_i$
b) $l(s_i\omega) = l(\omega s_j) = l(\omega) + 1$ *and* $l(s_i\omega s_j) = l(\omega)$.

Proof. Assume that $\omega(e_j) = e_i$. Then Corollary 4.28 yields that

$$l(s_i\omega) = l(\omega s_j) = l(\omega) + 1.$$

By assumption and Equation (4.1) we also have that

$$s_i\omega(e_j) = s_i(e_i) = -e_i \in \mathbf{\Delta}_-^X,$$

thus Corollary 4.28 yields that

$$l(s_i\omega s_j) = l(s_i\omega) - 1 = l(\omega).$$

This completes one direction of the proof. For the converse, assume that

$$l(s_i\omega) = l(\omega s_j) = l(\omega) + 1$$

and

$$l(s_i\omega s_j) = l(\omega).$$

Then one has

$$l(s_i\omega s_j) = l(\omega) = l(s_i\omega) - 1$$

and thus Corollary 4.28 yields

$$s_i\omega(e_j) \in \mathbf{\Delta}_-^X. \tag{4.38}$$

By assumption we have that $l(\omega s_j) = l(\omega) + 1$, hence Corollary 4.28 yields that

$$\omega(e_j) \in \mathbf{\Delta}_+^X.$$

Thus, by Equation (4.38) and Lemma 4.2, we have that

$$\omega(e_j) = e_i.$$

This completes the proof. □

4.4. **Finite Root Systems.**

Corollary 4.30. *Let $N \in \mathcal{M}$. Then the following assertions are equivalent.*

(1) *The set $\mathbf{\Delta}^N$ is finite.*
(2) *There is a morphism $\omega \in \mathrm{Hom}(N, \mathcal{W})$ with maximal length.*
(3) *There is a morphism $\omega' \in \mathrm{Hom}(\mathcal{W}, N)$ with maximal length.*

If the above holds, then the morphisms ω, ω' are uniquely determined and equation

$$l(\omega) = |\mathbf{\Delta}_+^N| = l(\omega')$$

holds. In particular we have $\omega' = \omega^{-1}$.

Proof. Suppose that (1) holds. We show that there is a unique morphism with domain N and maximal length, and that this morphism has length $|\mathbf{\Delta}_+^N|$. We begin with uniqueness. Suppose that $\omega \in \mathrm{Hom}(N, \mathcal{W})$ has maximal length. Let X be the codomain of ω. Then, by Corollary 4.28, we have $\omega^{-1}(e_i) \in \mathbf{\Delta}_-^N$ for all $i \in I$. Hence $\omega^{-1}(\mathbf{\Delta}_+^X) \subset \mathbf{\Delta}_-^N$. By (R1') it follows that $\omega^{-1}(\mathbf{\Delta}_+^X) = \mathbf{\Delta}_-^N$ and thus

$\omega(\mathbf{\Delta}^N_\pm) = \omega(\mathbf{\Delta}^X_\mp)$. Now, suppose that $\omega_1, \omega_2 \in \mathrm{Hom}(N, \mathcal{W})$ both have maximal length. Let X_i be the codomain of ω_i for all $i \in \{1, 2\}$. Then

$$\omega_2\omega_1^{-1}(\mathbf{\Delta}^{X_1}_+) = \omega_2(\mathbf{\Delta}^N_-) = \mathbf{\Delta}^{X_2}_+$$

and thus, by Corollary 4.27, it follows that

$$l(\omega_2\omega_1^{-1}) = |\omega_2\omega_1^{-1}(\mathbf{\Delta}^{X_1}_+) \cap \mathbf{\Delta}^{X_2}_-| = 0.$$

Hence $\omega_2\omega_1^{-1} = \mathrm{id}_{X_1}$ and thus $\omega_1 = \omega_2$. This proves uniqueness, next we prove existence. We know by Corollary 4.27 that each morphism $\omega \in \mathrm{Hom}(N, \mathcal{W})$ has length $l(\omega) \leq |\mathbf{\Delta}^N_+|$. Hence any morphism with domain N and length $|\mathbf{\Delta}^N_+|$ has maximal length. We show the existence of such a morphism by induction. Let $\omega \in \mathrm{Hom}(N, \mathcal{W})$ such that $l(\omega) < |\mathbf{\Delta}^N_+|$. Let X be the codomain of ω. Assume that $\omega^{-1}(e_i) \in \mathbf{\Delta}^N_-$ for all $i \in I$. Then $\omega^{-1}(\mathbf{\Delta}^X_+) \subset \mathbf{\Delta}^N_-$ and thus, by (R1'), $\omega(\mathbf{\Delta}^N_\pm) = \mathbf{\Delta}^X_\mp$. By Corollary 4.27 it follows that $l(\omega) = |\mathbf{\Delta}^N_+|$, a contradiction to the assumption $l(\omega) < |\mathbf{\Delta}^N_+|$. Thus there is an element $i \in I$ such that $\omega^{-1}(e_i) \in \mathbf{\Delta}^N_+$. Then Corollary 4.28 yields that $l(s_i\omega) = l(\omega)+1$. Thus, by induction, we obtain a morphism $\omega \in \mathrm{Hom}(N, W)$ with length $|\mathbf{\Delta}^N_+|$. This completes this part of the proof.
Now, suppose that (2) holds, i.e. there is a morphism $\omega \in \mathrm{Hom}(N, \mathcal{W})$ with maximal length. We show that the set $\mathbf{\Delta}^N$ must be finite. Let X be the codomain of ω. Corollary 4.28 yields that $\omega^{-1}(e_i) \in \mathbf{\Delta}^N_-$ for all $i \in I$. Hence we have $\omega^{-1}(\mathbf{\Delta}^X_+) \subset \mathbf{\Delta}^N_-$ and Corollary 4.27 yields that

$$|\mathbf{\Delta}^N_-| = |\mathbf{\Delta}^N_- \cap \omega^{-1}(\mathbf{\Delta}^X_+)| = l(\omega^{-1}) < \infty.$$

This proves that $\mathbf{\Delta}^N$ is finite. The rest of the claim follows easily, since the map

$$\mathrm{Hom}(N, \mathcal{W}) \to \mathrm{Hom}(\mathcal{W}, N), \omega \mapsto \omega^{-1}$$

is bijective and preserves the length of morphisms. □

Proposition 4.31. *The pair* $\mathcal{R}^{re} = \mathcal{R}^{re}(\mathcal{C}, \mathbf{\Delta}^{N\,\mathrm{re}})$ *is a root system with* $\mathcal{W}(\mathcal{R}^{re}) = \mathcal{W}(\mathcal{R})$*.*

Proof. We have to check whether axioms (R1) - (R4) are satisfied. (R1) holds, since for all $x \in \mathbf{\Delta}^{N\,\mathrm{re}}$ there are $\omega \in \mathrm{Hom}(N, \mathcal{W})$, $i \in I$ with $x = \omega(e_i)$ and hence $-x = \omega s_i(e_i) \in \mathbf{\Delta}^{N\,\mathrm{re}}$. (R2) holds, since $\mathbf{\Delta}^{N\,\mathrm{re}}$ is a subset of $\mathbf{\Delta}^N$. (R3) is trivial. (R4) holds, since by Lemmata 4.10 and 4.16 one has

$$|\mathbb{N}_0 e_i, e_j \cap \mathbf{\Delta}^N_+| = |\mathbb{N}_0 e_i, e_j \cap \mathbf{\Delta}^{N\,\mathrm{re}}|$$

for all $N \in \mathcal{M}$ and $i, j \in I$. □

Proposition 4.32. *Suppose that* $\mathcal{W}$ *is connected. Then the following assertions are equivalent.*

(1) $\mathbf{\Delta}^N_+$ *is finite for some* $N \in \mathcal{M}$*.*
(2) $\mathbf{\Delta}^{N\,\mathrm{re}}_+$ *is finite for some* $N \in \mathcal{M}$*.*

(3) $\mathrm{Hom}(N, \mathcal{W})$ *is finite for some* $N \in \mathcal{M}$.
(4) $\mathrm{Hom}(\mathcal{W}, N)$ *is finite for some* $N \in \mathcal{M}$.
(5) *The groupoid* $\mathcal{W}$ *is finite.*

Moreover, assertions (1)-(4) hold for some $N \in \mathcal{M}$ *if and only if they hold for all* $N \in \mathcal{M}$.

Proof. It is trivial, that (1)-(4) hold for all $N \in \mathcal{M}$ if and only if they hold for at least one $N \in \mathcal{M}$. Assertions (3) and (4) are equivalent, since for all $N \in \mathcal{M}$ the map

$$\mathrm{Hom}(N, \mathcal{W}) \to \mathrm{Hom}(\mathcal{W}, N), \omega \mapsto \omega^{-1}$$

is bijective. Assertions (4) and (5) are equivalent, since $\mathcal{W}$ is connected. Now, assume that (1) holds, i.e. there is a $N \in \mathcal{M}$ such that $d := |\boldsymbol{\Delta}_+^N| < \infty$. Then, by Corollary 4.30, all mophisms $\omega \in \mathrm{Hom}(N, \mathcal{W})$ have length less than or equal to d, hence $\mathrm{Hom}(N, \mathcal{W})$ is finite. Now, assume that (5) holds. Then there is an upper bound for the length of each morphism of $\mathcal{W}$, hence, by Corollary 4.30, it follows that $\boldsymbol{\Delta}_+^N$ is finite for all $N \in \mathcal{M}$. We have thus proved that assertions (1),(3),(4) and (5) are equivalent. Thus it follows, by Proposition 4.31, that (2) and (5) are equvialent too. This completes the proof. $\square$

5. A Comparison of Terminology

In this chapter we will briefly discuss the terms and definitions given in [HY08] and show that they are compatible with the notions introduced in [CH09] and [HS09].

5.1. **Generalized Root Systems.**

Proposition 5.1. *Given an arbitrary set I, there is a group $F_2(I)$ and a map $\alpha: I \to F_2(I)$ such that:*

(1) *For all $i \in I$ one has $\alpha(i)^2 = e$.*
(2) *Let F_2' be a group and $\alpha' : I \to F_2'$ a map, such that for all $i \in I$ equation $\alpha'(i)^2 = e$ holds. Then there is a unique group homorphism $\varphi : F_2(i) \to F_2'$ with $\varphi \circ \alpha = \alpha'$.*

Proposition 5.2. *Let I be a set. Then for each $x \in F_2(I)$ there are unique $i_1, \dots, i_n \in I$, $n \in \mathbb{N}_0$ with $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$ such that*

$$x = \alpha(i_1) \cdots \alpha(i_n).$$

This allows us to identify elements of I with their images under the map α.

Definition 5.3. Let I, $\mathcal{M}$ be sets and $\triangleright$ a left group action of $F_2(I)$ on $\mathcal{M}$. For all $i, j \in I$, $N \in \mathcal{M}$ let

$$\Theta(i,j,N) := \{(ij)^n \triangleright N | n \in \mathbb{Z}\},$$
$$\theta(i,j,N) := |\Theta(i,j,N)| \in \mathbb{N} \cup \{\infty\}.$$

Proposition 5.4. *Let I, $\mathcal{M}$ be sets and $\triangleright$ a left group action of $F_2(I)$ on $\mathcal{M}$. Then for all $i, j \in I$, $N \in \mathcal{M}$ we have that*

$$\Theta(i,j,N) = \Theta(j,i,N),$$
$$\Theta(i,j,i \triangleright N) = \Theta(i,j,j \triangleright N) = \{i \triangleright X | X \in \Theta(i,j,N)\}.$$

If I' is a subset of I, then we may regard $F_2(I')$ as a subgroup of $F_2(I)$. Thus an action $\triangleright$ of $F_2(I)$ on $\mathcal{M}$ induces an action $\triangleright'$ of $F_2(I')$ on $\mathcal{M}$. The group action $\triangleright'$ is called the restriction of $\triangleright$ to the subgroup $F_2(I')$.

Proposition 5.5. *Let I, $\mathcal{M}$ be sets and $\triangleright$ a left group action of $F_2(I)$ on $\mathcal{M}$. Let $i, j \in I$, $N \in \mathcal{M}$ and let $\triangleright'$ be the restriction of the group action $\triangleright$ to the subgroup $F_2(\{i,j\})$. Then for each X that lies in the orbit of N under the group action $\triangleright'$ we have that*

$$\theta(i,j,N) = \theta(i,j,X).$$

Proposition 5.6. *Let I, $\mathcal{M}$ be sets and $\triangleright$ a left group action of $F_2(I)$ on $\mathcal{M}$. Let $i, j \in I$ and $N \in cM$. If $(ij)^k \triangleright N \neq N$ for all $k \in \mathbb{N}$, then $\theta(i,j,N) = \infty$. Otherwise, if $(ij)^k \triangleright N = N$ for some $k \in \mathbb{N}$, then*

$$\theta(i,j,N) = \min\{k \in \mathbb{N} | (ij)^k \triangleright N = N\}.$$

and thus

$$\{k \in \mathbb{Z} | (ij)^k \triangleright N = N\} = \mathbb{Z}\theta(i, j, N).$$

Definition 5.7. A generalized root system is a quadrupel

$$((R^N, \pi_N, S_N)_{N \in \mathcal{M}}, \mathcal{M}, I, \triangleright),$$

such that:

(1) I and $\mathcal{M}$ are nonempty sets and $\triangleright$ is a transitive left group action of $F_2(I)$ on $\mathcal{M}$.
(2) Let $V_0 := \mathbb{R}^{(I)}$. For all $N \in \mathcal{M}$ the set R^N is a subset of V_0, $\pi_N = (e_i^N)_{i \in I}$ is a basis of V_0 and $S_N = (\sigma_i^N)_{i \in I}$ is a family of $\mathbb{R}$-linear automorphisms $\sigma_i^N \in \mathrm{GL}(V_0)$.
(3) (a) $\sigma_i^{i \triangleright N} \sigma_i^N = \mathrm{id}_{V_0}$ for all $i \in I$, $N \in \mathcal{M}$,
 (b) $\sigma_i^N(e_i^N) = -e_i^{i \triangleright N}$ for all $i \in I$, $N \in \mathcal{M}$,
 (c) $\sigma_i^N(e_j^N) \in e_j^{i \triangleright N} + \mathbb{N}_0 e_i^{i \triangleright N}$ for all $i, j \in I$, $i \neq j$, $N \in \mathcal{M}$,
(4) $R^N = R_+^N \cup -R_+^N$ where $R_+^N := \mathbb{N}_0 \pi_N \cap R^N$ for all $N \in \mathcal{M}$,
(5) $\mathbb{R} e_{i,N} \cap R^N = \{e_{i,N}, -e_{i,N}\}$ for all $i \in I$, $N \in \mathcal{M}$,
(6) $\sigma_i^N(R^N) = R^{i \triangleright N}$ for all $i \in I$, $N \in \mathcal{M}$,
(7) $\theta(i, j, N)$ is finite and divides m_{ij}^N for all $i, j \in I$, $i \neq j$, $N \in \mathcal{M}$ such that the number $m_{ij}^N := |\mathbb{N}_0\{e_i^N, e_j^N\} \cap R_+^N|$ is finite.

Remark 5.8. Note that, by axiom (5) and the definition of $\theta(i, j, N)$, we have that

$$\theta(i, i, N) = 1 = |\mathbb{N}_0\{e_i^N, e_j^N\} \cap R_+^N|$$

for all $i \in I$, $N \in \mathcal{M}$. Hence, by Proposition 5.6, axiom (7) may be replaced by

(7') $(ij)^{m_{ij}^N} \triangleright N = N$ for all $i, j \in I$, $i \neq j$, $N \in \mathcal{M}$ such that the number $m_{ij}^N := |\mathbb{N}_0\{e_i^N, e_j^N\} \cap R_+^N|$ is finite.

Remark 5.9. There are canonical bijections between the set of left group actions $\triangleright$ of $F_2(I)$ on $\mathcal{M}$, the set of group homorphisms φ from $F_2(I)$ to the symmetric group $S_{\mathcal{M}}$ and, by definition of $F_2(I)$, the set of families $(r_i)_{i \in I}$ of bijections $r_i \in S_{\mathcal{M}}$ with $r_i^2 = \mathrm{id}_{\mathcal{M}}$.

Remark 5.10. There is a quadruple

$$\mathcal{C} = \mathcal{C}(I, \mathcal{M}, (r_i)_{i \in I}, (A^N)_{N \in \mathcal{M}})$$

that satisfies all axioms of a Cartan scheme except that I might be infinite, such that

$$\sigma_i^N(e_j^N) = e_j^{r_i(N)} - a_{ij}^N e_i^{r_i(N)} \qquad \text{for all } i, j \in I, N \in \mathcal{M}.$$

Proof. Define $r_i(N) = i \triangleright N$ for all $i \in I$ and $N \in \mathcal{M}$. Then (C1) holds. By axiom (3) (b) and (c) we know that there are matrices $A^N = (a_{ij}^N)_{i,j \in I}$, $N \in \mathcal{M}$

with integer coefficients satisfying (M1) and $a_{ij}^N \leq 0$ for all $i \neq j$ such that

$$\sigma_i^N(e_j^N) = e_j^{r_i(N)} - a_{ij}^N e_i^{r_i(N)}$$

holds for all $i, j \in I, N \in \mathcal{M}$. It follows by axiom (3) (a) that

$$\begin{aligned} 0 &= e_j^N - \sigma_i^{r_i(N)} \sigma_i^N(e_j^N) \\ &= e_j^N - \sigma_i^{r_i(N)}(e_j^{r_i(N)} - a_{ij}^N e_i^{r_i(N)}) \\ &= (a_{ij}^{r_i(N)} - a_{ij}^N) e_i^N, \end{aligned}$$

hence (C2) holds. Now, assume that $i \neq j$, $a_{ij}^N = 0$. Then

$$\begin{aligned} \sigma_i^N \sigma_j^{r_j(N)}(e_i^{r_j(N)}) &= \sigma_i^N(e_i^N - a_{ji}^{r_j(N)} e_j^N) \\ &= -e_i^{r_i(N)} - a_{ji}^{r_j(N)} e_j^{r_i(N)} \end{aligned}$$

We have assumed that $i \neq j$, hence, by axiom (4), this root is negative. But $a_{ji}^{r_j(N)} \leq 0$, thus $a_{ji}^{r_j(N)} = 0$. We already know that $a_{ji}^{r_j(N)} = a_{ji}^N$, hence it follows that (M2) holds. □

5.2. **Coxeter groupoids.**

Definition 5.11. A semigroup is a set M together with a map

$$\mu : M \times M \to M, (x, y) \mapsto xy$$

such that for all $x, y, z \in M$ equation $(xy)z = x(yz)$ holds. A semigroup morphism is a map between semigroups that preserves multiplication.

Proposition 5.12. *Let M be an arbitrary set. Then there is a semigroup $L(M)$ and a map $\alpha : M \to L(M)$ with the following universal property. Given a semigroup L' and a map $\alpha' : M \to L'$, there is a unique semigroup morphism $\varphi : L(M) \to L'$ such that equation $\varphi \circ \alpha = \alpha'$ holds. We call $L(M)$ the free semigroup generated by the set M.*

Proposition 5.13. *Let M be a set. Then for each $x \in L(M)$ there are unique $m_1, \ldots, m_n \in M$, $n \in \mathbb{N}$ such that*

$$x = \alpha(m_1) \cdots \alpha(m_n).$$

This allows us to identify elements of M with their images under the map α.

Proposition 5.14. *Let M be a set and $Q \subset L(M) \times L(M)$. Then there is a semigroup C and a semigroup morphism $\alpha : L(M) \to C$ such that:*

1. *For each pair $(x, y) \in Q$ equation $\alpha(x) = \alpha(y)$ holds.*
2. *Let C' be a semigroup and $\alpha' : L(M) \to C'$ be a semigroup morphism such that for each pair $(x, y) \in Q$ equation $\alpha'(x) = \alpha'(y)$ holds. Then there is a unique semigroup morphism $\varphi : C \to C'$ such that equation $\varphi \circ \alpha = \alpha'$ holds.*

We call C the semigroup generated by the set M under relation Q.

Definition 5.15. Let I and $\mathcal{M}$ be nonempty sets and $\triangleright$ a transitive left group action of $F_2(I)$ on $\mathcal{M}$. For all $N \in \mathcal{M}$ and $i, j \in I$ let

$$m_{i,j}^N = m_{j,i}^N \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\})$$

be a multiple of $\theta(i, j, N)$. Let $\widetilde{\mathcal{W}}$ be the semigroup generated by 0, $(e_N)_{N\in\mathcal{M}}$, $(s_i^N)_{i\in I, N\in\mathcal{M}}$ under the following relations.

$$\begin{aligned} 0 &= e_N 0 = 0 e_N = s_i^N 0 = 0 s_i^N = 00 \\ e_N^2 &= e_N \\ e_N e_{N'} &= 0 \text{ for } N \neq N' \\ e_{i\triangleright N} s_i^N &= s_i^N e_N = s_i^N \\ s_i^{i\triangleright N} s_i^N &= e_N \\ \underbrace{\dots s_i^{ji\triangleright N} s_j^{i\triangleright N} s_i^N}_{m_{i,j}^N factors} &= \underbrace{\dots s_j^{ij\triangleright N} s_i^{j\triangleright N} s_j^N}_{m_{i,j}^N factors} \text{ if } m_{i,j}^N \text{ is finite.} \end{aligned}$$

Then the quadrupel

$$(\widetilde{\mathcal{W}}, I, \mathcal{M}, \triangleright, (m_{i,j}^N)_{i,j\in I, N\in\mathcal{M}})$$

is called a Coxeter groupoid.

Remark 5.16. Each nonzero element x of $\widetilde{\mathcal{W}}$ can be written as a product

$$x = s_{i_n} \cdots s_{i_1}^N := s_{i_n}^{i_{n-1}\cdots i_1 \triangleright N} \cdots s_{i_2}^{i_1\triangleright n} s_{i_1}^N$$

for some $N \in \mathcal{M}$, $i_1, \dots, i_n \in I$, $n \in \mathbb{N}$.

Proposition 5.17. *There is a semigroup morphism* $\widetilde{\operatorname{sgn}} : \widetilde{\mathcal{W}} \to \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}^{(\mathcal{M})})$ *such that for all* $N, N' \in \mathcal{M}$ *and* $i \in I$

$$\begin{aligned} \widetilde{\operatorname{sgn}}(0)(N) &= 0 \\ \widetilde{\operatorname{sgn}}(e_N)(N') &= \delta_{N,N'} N' \\ \widetilde{\operatorname{sgn}}(s_i^N)(N') &= -\delta_{N,N'} i \triangleright N. \end{aligned}$$

Corollary 5.18. *Let* $i_1, \dots, i_n, j_1, \dots, j_m \in I$, $m, n \in \mathbb{N}$, $N, N' \in \mathcal{M}$ *such that*

$$s_{i_1} \cdots s_{i_n}^N = s_{j_1} \cdots s_{j_m}^{N'}.$$

Then $N = N'$ *and* $i_1 \cdots i_n \triangleright N = j_1 \cdots j_m \triangleright N'$.

Proposition 5.19. $\widetilde{\mathcal{W}}$ *may be considered as a Coxeter groupoid in the sense of Definition 3.1.*

Proof. For all $i \in I$ and $N \in \mathcal{M}$ let $m_{ii}^N := 1$. Let $\mathcal{W}$ be the category with objects $\operatorname{Ob}(\mathcal{W}) = \mathcal{M}$ and morphisms $\operatorname{Hom}(\mathcal{W}) = \widetilde{\mathcal{W}} \setminus \{0\}$, such that for all $N \in \mathcal{M}$,

$i_1, \dots i_n \in I$

$$\begin{aligned} \mathrm{id}_N &= e_N \\ \mathrm{dom}(s_{i_1} \cdots s_{i_n}^N) &= N \\ \mathrm{cod}(s_{i_1} \cdots s_{i_n}^N) &= i_1 \cdots i_n \triangleright N. \end{aligned}$$

By Corollary 5.18 this is a well-defined groupoid. $\mathcal{W}$ satisfies the Coxeter relations with respect to I, $\mathcal{M}$, $(s_i^N)_{i \in I, N \in \mathcal{M}}$ and the numbers $(m_{i,j}^N)_{i,j \in I, N \in \mathcal{M}}$. One can show that $\mathcal{W}$ satisfies the universal property of Definition 3.1 as well.

□

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