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SCALING LIMITS OF RANDOM GRAPHS FROM SUBCRITICAL CLASSES

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We study the uniform random graph C_n with n vertices drawn from a subcritical class of connected graphs. Our main result is that the rescaled graph $\mathsf{C}_n/\sqrt{n}$ converges to the Brownian continuum random tree $\mathcal{T}_{\mathsf{e}}$ multiplied by a constant scaling factor that depends on the class under consideration. In addition, we provide sub-Gaussian tail bounds for the diameter $\mathrm{D}(\mathsf{C}_n)$ and height $\mathrm{H}(\mathsf{C}_n^\bullet)$ of the rooted random graph $\mathsf{C}_n^\bullet$. We give analytic expressions for the scaling factor in several cases, including for example the class of outerplanar graphs. Our methods also enable us to study first passage percolation on C_n, where we also show the convergence to $\mathcal{T}_{\mathsf{e}}$ under an appropriate rescaling.

1. Introduction. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. We can associate in a natural way a metric space $(V(G), d_G)$ to G, where $d_G(u, v)$ is the number of edges on a shortest path that contains u and v in G. In this work, we study the case where G is a random graph, and we consider several properties of the associated metric space as the number of vertices of G becomes large.

In the series of seminal papers [3–5], Aldous studied the fundamental case of G being a critical Galton–Watson random tree with n vertices, where the offspring distribution has finite nonzero variance. Among other results, he showed that the asymptotic properties of the associated metric space admit an *universal* description: they can be depicted, up to an appropriate rescaling, in terms of "continuous trees" whose archetype is the so-called *Brownian Continuum Random Tree* (CRT for short). Since Aldous's pioneering work, the CRT has been identified as the limiting object of many different classes of discrete structures, in particular trees; see, for example, Haas and Miermont [26] and references therein, and for planar maps, see, for example, Albenque and Marckert [2], Bettinelli [9], Caraceni [13], Curien, Haas and Kortchemski [15] and Janson and Stefansson [29].

Although the aforementioned papers identify the CRT as the universal limiting object in various settings, much less is known about the scaling limit of random

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graphs from complex graph classes. In this paper, we study in a unified way the asymptotic distribution of distances in random graphs from so-called *subcritical classes*, where informally a class is called subcritical if for a typical graph with n vertices the largest block (i.e., inclusion maximal 2-connected subgraph) has $O(\log n)$ vertices. Random graphs from such classes have been the object of intense research in the last years; see, for example [7, 18, 19, 41], especially because of their close connection to the class of planar graphs. Prominent examples of classes that are subcritical are outerplanar and series-parallel graphs. However, with the notable exception of [19], most research on such random graphs has focused on *additive* parameters, like the number of vertices of a given degree; the fine study of *global* properties, like the distribution of the distances, poses a significant challenge.

In the present paper we study the random graph C_n drawn uniformly from the set of connected graphs with n vertices of a subcritical class $\mathcal{C}$. Our first main result is Theorem 5.1, which shows that, up to an appropriate rescaling, the associated metric space converges in distribution to a multiple of the CRT. Postponing the introduction of the appropriate notation to later sections (see the outline), our main result asserts that there is a constant $s = s(\mathcal{C}) > 0$ such that

$$\big(V(\mathsf{C}_n), sn^{-1/2}d_{\mathsf{C}_n}\big) \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}},$$

where $\mathcal{T}_{\mathsf{e}}$ is the CRT and convergence is with respect to the Gromov–Hausdorff metric. In particular, this establishes that the CRT is the universal scaling limit for random graphs from subcritical clases, and it proves (in a strong form) a conjecture by Drmota and Noy [19]. The proof of Theorem 5.1 (see Section 5) gives a natural combinatorial characterization of the "scaling" constant s. Our methods are based on the algebraic concept of $\mathcal{R}$-enriched trees; more specifically, we use a size-biased enriched tree in order to study a coupling of C_n with an appropriate conditioned critical Galton–Watson tree T_n on the same vertex set. Our main step establishes that with a probability that converges to 1 as $n \to \infty$ for any two vertices u, v in C_n the distance $d_{\mathsf{C}_n}(u, v)$ is concentrated around $d_{\mathsf{T}_n}(u, v)$ multiplied by a constant factor $\kappa \geq 1$ that depends *only* on $\mathcal{C}$, and which is, very roughly speaking, the average length of a shortest path between two random distinct vertices in a random block of C_n. Thus, the constant s turns out to be the product of two quantities: the constant involved in the scaling limit of T_n, and the reciprocal of κ. In Section 8, we exploit this characterization of s and compute its value for several important classes, including outerplanar graphs.

As a consequence of our main result, we obtain the following statements; see Corollary 5.2. The *diameter* $\mathrm{D}(G)$ of a graph G is defined as the maximum distance of any pair of vertices, that is, as $\max_{u,v \in V(G)} d_G(u, v)$. The *height* $\mathrm{H}(G^\bullet)$ of a pointed graph $G^\bullet = (G, r)$, which is G rooted at a vertex r, is $\max_{v \in V(G)} d_G(r, v)$. Then the limit distribution for the diameter of C_n and the height of $\mathsf{C}_n^\bullet$ satisfy for $x > 0$, as $n \to \infty$

$$\mathbb{P}(\mathrm{D}(\mathsf{C}_n) > s^{-1}n^{1/2}x) \to \sum_{k=1}^{\infty}(k^2-1)\left(\frac{2}{3}k^4x^4 - 4k^2x^2 + 2\right)e^{-k^2x^2/2},$$

$$\mathbb{P}(\mathrm{H}(\mathsf{C}_n^\bullet) > s^{-1}n^{1/2}x) \to 2\sum_{k=1}^{\infty}(4k^2x^2-1)e^{-2k^2x^2}.$$

Apart from the convergence in distribution, we also show sharp tail bounds for the diameter and the height; see Theorem 6.1. In particular, we show that there are constants $C, c > 0$ such that *for all* n and $x \geq 0$

$$\mathbb{P}(\mathrm{D}(\mathsf{C}_n) \geq x) \leq C\exp(-cx^2/n) \quad \text{and} \quad \mathbb{P}(\mathrm{H}(\mathsf{C}_n^\bullet) \geq x) \leq C\exp(-cx^2/n).$$

A similar result was shown for critical Galton–Watson random trees by Addario-Berry, Devroye and Janson [1], and our proof of these bounds builds on the methods in that paper. From this, we deduce that all moments of the rescaled height and diameter converge as well. In particular, we obtain the universal and remarkable asymptotic behaviour

$$\mathbb{E}[\mathrm{D}(\mathsf{C}_n)] \sim \frac{2^{3/2}}{3s}\sqrt{\pi n} \sim \frac{4}{3}\mathbb{E}[\mathrm{H}(\mathsf{C}_n^\bullet)].$$

This improves the previously best known bounds $c_1\sqrt{n} \leq \mathbb{E}[\mathrm{D}(\mathsf{C}_n)] \leq c_2\sqrt{n\log n}$ given in [19]. The higher moments can also be determined and are depicted in Sections 5 and 2.4.

In addition to the previous results, we demonstrate that our proof strategy is powerful enough to enable us to study the far more general setting of *first passage percolation*: suppose that the edges of C_n are equipped with independent random "lengths," drawn from a distribution that has exponential moments, and let the distance of two vertices u, v be the minimum sum of those lengths along a path that contains both u and v. Our last main result shows that again, up to an appropriate rescaling, the associated metric space converges to a multiple of the CRT; see Section 7 for the details.

Outline. Section 2 fixes some basic notation and summarizes several results related to Galton–Watson random trees and the Continuum Random Tree (CRT). In particular, Section 2.4 states the distribution and expressions for arbitrarily high moments of the height and diameter of the CRT—to our knowledge, these results are scattered across several papers and we provide a concise presentation. Section 3 is devoted to the definition of combinatorial species, $\mathcal{R}$-enriched trees and subcritical graph classes. In this part of the paper, we collect some general and relevant properties of the these objects—many of them were already known in special cases, and we put them in a broader context. In Section 4, we describe a construction of a powerful object called the *size-biased $\mathcal{R}$-enriched tree* that is novel in this context and will allow us to study systematically the distribution of distances in random graphs from subcritical graph classes. Subsequently, in Section 5 we show our main result: the convergence of the rescaled random graphs toward a multiple of the CRT. Section 6 complements this result by proving sub-Gaussian tail-bounds for the height and diameter. Section 7 is devoted to several extensions of our results, in particular first passage percolation. The paper closes with many examples, including among others the prominent class of outerplanar graphs. We leave determining the explicit scaling constant for the class of series-parallel graphs as an open problem.

2. Galton–Watson trees and the CRT. We briefly summarize required notions and results related to the Brownian Continuum Random Tree (CRT) and refer the reader to [5, 35] for a thorough treatment.

2.1. *Graphs and* (*plane*) *trees*. All graphs considered in this paper are undirected and may not contain multiple edges or loops. That is, a graph G consists of a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges that are two-element subsets of $V(G)$. If $|V(G)| \in \mathbb{N}$ we say that $|G| := |V(G)|$ is the *size* of G. Following Diestel [17], we recall and fix basic definitions and notation. Two vertices $v, w \in V(G)$ are said to be *adjacent* if $\{v, w\} \in E(G)$. We will often write vw instead of $\{v, w\}$. A *path* P is a graph such that

$$V(P) = \{v_0, \dots, v_\ell\}, \qquad E(P) = \{v_0v_1, \dots, v_{\ell-1}v_\ell\}$$

with the v_i being distinct. The number $\ell = \ell(P)$ of edges of a path is its *length*. We say P *connects* its end-vertices v_0 and v_ℓ and we often write $P = v_0v_1 \cdots v_\ell$. If P has length at least two we call the graph $C_\ell = P + v_0v_\ell$ obtained by adding the edge v_0v_ℓ a *cycle*. The *complete graph* with n vertices in which each pair of distinct vertices is adjacent is denoted by K_n.

We say the graph G is *connected* if any two vertices $u, v \in V(G)$ are connected by a path in G. The length of a shortest path connecting the vertices u and v is called the *distance* of u and v and it is denoted by $d_G(u, v)$. Clearly, d_G is a metric on the vertex set $V(G)$. A graph G together with a distinguished vertex $v \in V(G)$ is called a *rooted* graph with root-vertex v. The *height* $h(w)$ of a vertex $w \in V(G)$ is its distance from the root. The *height* $\mathrm{H}(G)$ is the maximum height of the vertices in G. A *tree* T is a nonempty connected graph without cycles. Any two vertices of a tree are connected by a unique path. If T is rooted, then the vertices $w' \in V(T)$ that are adjacent to a vertex w and have height $h(w) + 1$ form the *offspring* set of the vertex w. Its cardinality is the *outdegree* $d^+(w)$ of w.

The *Ulam–Harris tree* is an infinite rooted tree with vertex set $\bigcup_{n \geq 0} \mathbb{N}^n$ consisting of finite sequences of natural numbers. The empty string $\varnothing$ is the root and the offspring of any vertex v is given by the concatenations $v1, v2, v3, \dots$. In particular, the labelling of the vertices induces a linear order on each offspring set. A *plane tree* is defined as a subtree of the Ulam–Harris tree that contains the root such that the offspring set of each vertex v is of the form $\{v1, v2, \ldots, vk\}$ for some integer $k \geq 0$ depending on v.

2.2. *Galton–Watson trees*. Throughout this section, we fix an integer-valued random variable $\xi \geq 0$. By abuse of language, we will often not distinguish between ξ and its distribution. A ξ-*Galton–Watson* tree T is the family tree of a Galton–Watson branching process with offspring distribution ξ, interpreted as a (possibly infinite) plane tree. It is well known that if $\mathbb{P}(\xi = 1) < 1$, then T is almost surely finite if and only if $\mathbb{E}[\xi] \leq 1$. If $\mathbb{E}[\xi] = 1$, we call T *critical*. Let $\mathsf{supp}(\xi) = \{k | \mathbb{P}(\xi = k) > 0\}$ denote the *support* of ξ and define the *span* $d = \mathsf{span}(\xi)$ as the greatest common divisor of $\{k - \ell | k, \ell \in \mathsf{supp}(\xi)\}$. If T is finite, then

$$|\mathsf{T}| = 1 + \sum_{v \in V(\mathsf{T})} d^+_{\mathsf{T}}(v) \equiv 1 \bmod d.$$

Conversely, if ξ is not almost surely positive, then $\mathbb{P}(|\mathsf{T}| = n) > 0$ for all large enough $n \in \mathbb{N}$ satisfying $n \equiv 1 \bmod d$. We need the following standard result for the probability that a critical Galton–Watson tree has size n.

LEMMA 2.1 ([31], page 105). *Suppose that ξ has expected value one and finite nonzero variance σ^2. Let $(\xi_i)_{i \in \mathbb{N}}$ be an infinite family of i.i.d. copies of ξ. Then, for any n with $n \equiv 1 \bmod d$, where $d = \mathsf{span}(\xi)$,*

$$\mathbb{P}\big(|\mathsf{T}| = n\big) = n^{-1}\mathbb{P}\left(\sum_{i=1}^{n} \xi_i = n - 1\right) \sim \frac{d}{\sqrt{2\pi\sigma^2}} n^{-3/2} \qquad \text{as } n \to \infty.$$

We also state some results given in [1, 28] that will be useful in our arguments. Suppose that ξ satisfies $\mathbb{E}[\xi] = 1$ and has finite nonzero variance. Let $n \equiv 1 \bmod d$ be sufficiently large such that $\mathbb{P}(\mathsf{T} = n) > 0$ and let T_n denote the Galton–Watson tree conditioned on having size n. By [1], Theorem 1.2 and [1], page 6, there are constants $C_1, c_1, C_2, c_2 > 0$ such that the height $\mathrm{H}(\mathsf{T}_n)$ satisfies the following tail bounds for all n and $h \geq 0$:

$$\mathbb{P}\big(\mathrm{H}(\mathsf{T}_n) \leq h\big) \leq C_1 \exp\big(-c_1(n-2)/h^2\big), \tag{1}$$

$$\mathbb{P}\big(\mathrm{H}(\mathsf{T}_n) \geq h\big) \leq C_2 \exp\big(-c_2 h^2/n\big). \tag{2}$$

2.3. *Convergence of rescaled conditioned Galton–Watson trees. Throughout this and the following subsections in Section 2 we write T for a critical ξ-Galton–Watson tree having finite nonzero variance σ^2. Moreover, n will always denote an integer satisfying $n \equiv 1 \bmod \mathsf{span}(\xi)$ and is assumed to be large enough such that the conditioned tree T_n having exactly n vertices is well defined.*

Given a plane tree T of size n we consider its canonical *depth-first search* walk $(v_i)_{0 \leq i \leq 2(n-1)}$ that starts at the root and always traverses the leftmost unused edge

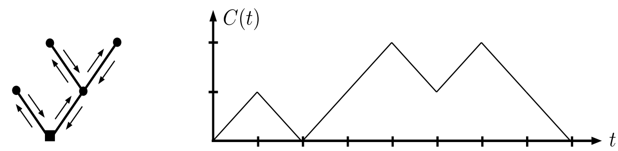

FIG. 1. *The contour function of a plane tree.*

first. That is, v_0 is the root of T and given $v_0, \dots, v_i$ walk if possible to the leftmost unvisited son of v_i. If v_i has no sons or all sons have already been visited, then try to walk to the parent of v_i. If this is not possible either, being only the case when v_i is the root of T and all other vertices have already been visited, then terminate the walk. The corresponding heights $c(i) := d(\text{root of } T, v_i)$ define the search-depth function c of the tree T. The *contour function* $C : [0, 2(n-1)] \to \mathbb{R}_+$ is defined by $C(i) = c(i)$ for all integers $0 \le i \le 2(n-1)$ with linear interpolation between these values; see Figure 1 for an example. It can be shown that after a suitable rescaling the contour process of T_n converges to a normalized Brownian excursion.

THEOREM 2.2 ([36], Theorem 6.1). *Let T_n be a critical ξ-Galton–Watson tree conditional on having n vertices, where ξ has finite nonzero variance σ^2. Let C_n denote the contour function of T_n. Then*

$$\left(\frac{\sigma}{2\sqrt{n}} C_n\big(t2(n-1)\big)\right)_{0 \le t \le 1} \xrightarrow{(d)} \mathsf{e}$$

in $\mathcal{C}([0,1], \mathbb{R}_+)$, where $\mathsf{e} = (\mathsf{e}_t)_{0 \le t \le 1}$ denotes the Brownian excursion of duration one.

This result is due to Aldous [5], Theorem 23, who stated it for aperiodic offspring distributions. See also [21, 32] for further extensions. Theorem 2.2 can be formulated as a convergence of random trees with respect to the Gromov–Hausdorff metric. We introduce the required notions following Le Gall and Miermont [37]. A continuous function $g : [0,1] \to [0, \infty)$ with $g(0) = g(1) = 0$ induces a pseudo-metric on the interval $[0, 1]$ by

$$d(u, v) = g(u) + g(v) - 2 \inf_{u \le s \le v} g(s)$$

for $0 \le u \le v \le 1$. This defines a metric on the quotient $T_g = [0, 1]/\sim$, where $u \sim v$ if and only if $d(u, v) = 0$; we denote the corresponding metric space by (T_g, d_g) and call the equivalence class $r_0(T_g)$ of the origin its *root*.

Given a metric space (E, d) the set of compact subsets is again a metric space with respect to the *Hausdorff metric*

$$\delta_{\mathrm{H}}\big(K, K'\big) = \inf\big\{\varepsilon > 0 | K \subset U_\varepsilon\big(K'\big), K' \subset U_\varepsilon(K)\big\},$$

where $U_\varepsilon(K) = \{x \in E | d(x, K) \le \varepsilon\}$. The sets $\mathbb{K}$ ($\mathbb{K}^\bullet$) of isometry classes of (pointed) compact metric spaces, where a pointed space is a triple (E, d, r) where

(E, d) is a metric space and $r \in E$ is a distinguished element, are Polish spaces with respect to the *(pointed) Gromov–Hausdorff metric*

$$d_{\mathrm{GH}}\big((E_1, d_1), (E_2, d_2)\big) = \inf_{\varphi_1, \varphi_2} \delta_{\mathrm{H}}\big(\varphi_1(E_1), \varphi_2(E_2)\big),$$

$$d_{\mathrm{GH}}\big((E_1, d_1, r_1), (E_2, d_2, r_2)\big) = \inf_{\varphi_1, \varphi_2} \max\big\{\delta_{\mathrm{H}}\big(\varphi_1(E_1), \varphi_2(E_2)\big), d_E\big(\varphi_1(r_1), \varphi_2(r_2)\big)\big\},$$

where the infimum is in both cases taken over all isometric embeddings $\varphi_1 : E_1 \to E$ and $\varphi_2 : E_2 \to E$ into a common metric space (E, d_E) ([37], Theorem 3.5). We will make use of the following characterisation of the Gromov–Hausdorff metric. Given two compact metric spaces (E_1, d_1) and (E_2, d_2) a *correspondence* between them is a subset $R \subset E_1 \times E_2$ such that any point $x \in E_1$ corresponds to at least one point $y \in E_2$ and vice versa. The *distortion* of R is

$$\mathsf{dis}(R) = \sup\big\{\big|d_1(x_1, x_2) - d_2(y_1, y_2)\big| \big| (x_1, y_1), (x_2, y_2) \in R\big\}.$$

PROPOSITION 2.3 ([37], Proposition 3.6). *Given two pointed compact metric spaces (E_1, d_1, r_1) and (E_2, d_2, r_2) we have that*

$$2d_{\mathrm{GH}}\big((E_1, d_1, r_1), (E_2, d_2, r_2)\big) = \inf_R \mathsf{dis}(R),$$

where R ranges over all correspondences between E_1 and E_2 such that r_1, r_2 correspond to each other.

An important consequence is that the mapping

$$\big(\big\{g \in \mathcal{C}\big([0, 1], \mathbb{R}_+\big) | g(0) = g(1) = 0\big\}, \|\cdot\|_\infty\big) \to \big(\mathbb{K}^\bullet, d_{GH}\big), \qquad g \mapsto T_g$$

is Lipschitz-continuous ([37], Corollary 3.7).

DEFINITION 2.4. The random metric space $(\mathcal{T}_{\mathsf{e}}, d_{\mathsf{e}}, r_0(\mathcal{T}_{\mathsf{e}}))$ coded by the Brownian excursion of duration one e is called the Brownian continuum random tree (CRT).

We may view T_n as a random pointed metric space $(V(\mathsf{T}_n), d_{\mathsf{T}_n}, \varnothing) \in \mathbb{K}$. This space is close to the real tree encoded by its contour function, hence Theorem 2.2 implies convergence with respect to the Gromov–Hausdorff metric; see [36], page 740.

THEOREM 2.5. *Let T_n be a critical ξ-Galton–Watson tree conditional on having n vertices, where ξ has finite nonzero variance σ^2. As n tends to infinity, T_n with edges rescaled to length $\frac{\sigma}{2\sqrt{n}}$ converges in distribution to the CRT, that is,*

$$\left(V(\mathsf{T}_n), \frac{\sigma}{2\sqrt{n}} d_{\mathsf{T}_n}, \varnothing\right) \xrightarrow{(d)} \big(\mathcal{T}_{\mathsf{e}}, d_{\mathsf{e}}, r_0(\mathcal{T}_{\mathsf{e}})\big) \tag{3}$$

in the metric space $(\mathbb{K}^\bullet, d_{\mathrm{GH}})$.

This invariance principle is due to Aldous [5] and there exist various extensions, see for example [21, 22, 26]. Adopting common terminology, instead of (3) we will often write in the sequel

$$\frac{\sigma}{2\sqrt{n}}\mathsf{T}_n \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}}.$$

2.4. *Height and diameter of the CRT.* The height $\mathrm{H}(\mathsf{T}_n)$ and diameter $\mathrm{D}(\mathsf{T}_n)$ of T_n may be recovered from its contour function. From the results in Section 2.3, it follows that

$$\frac{\sigma}{2\sqrt{n}}\mathrm{H}(\mathsf{T}_n) \xrightarrow{(d)} \mathrm{H}(\mathcal{T}_{\mathsf{e}}) \overset{(d)}{=} \sup_{0\le t\le 1} \mathsf{e}(t), \tag{4}$$

$$\frac{\sigma}{2\sqrt{n}}\mathrm{D}(\mathsf{T}_n) \xrightarrow{(d)} \mathrm{D}(\mathcal{T}_{\mathsf{e}}) \overset{(d)}{=} \sup_{0\le t_1\le t_2\le 1} \Big(\mathsf{e}(t_1) + \mathsf{e}(t_2) - 2 \inf_{t_1\le t\le t_2} \mathsf{e}(t)\Big). \tag{5}$$

Since $\mathrm{D}(\mathsf{T}_n) \le 2\mathrm{H}(\mathsf{T}_n)$ the tail bound (1) implies that all moments in (4) and (5) converge. It is well known that $\mathrm{H}(\mathcal{T}_{\mathsf{e}})/\sqrt{2}$ follows a *Theta distribution*, that is,

$$\mathbb{P}\big(\mathrm{H}(\mathcal{T}_{\mathsf{e}}) > x\big) = 2\sum_{k=1}^{\infty}(4k^2x^2 - 1)\exp(-2k^2x^2) \tag{6}$$

for all $x > 0$. The moments of the height are given by

$$\mathbb{E}[\mathrm{H}(\mathcal{T}_{\mathsf{e}})] = \sqrt{\pi/2} \quad \text{and} \quad \mathbb{E}[\mathrm{H}(\mathcal{T}_{\mathsf{e}})^k] = 2^{-k/2}k(k-1)\Gamma(k/2)\zeta(k), \quad k \ge 2. \tag{7}$$

This follows from standard results on the Brownian excursion; see, for example, [10, 14], or by calculating directly the limit distribution of extremal parameters of a class of trees that converges to the CRT (see, e.g., [42]). The distribution of the diameter is given by

$$\mathbb{P}\big(\mathrm{D}(\mathcal{T}_{\mathsf{e}}) > x\big) = \sum_{k=1}^{\infty}(k^2 - 1)\Big(\frac{2}{3}k^4x^4 - 4k^2x^2 + 2\Big)\exp(-k^2x^2/2). \tag{8}$$

This expression may be obtained (by tedious calculations) from results of Szekeres [43], who proved the existence of a limit distribution for the diameter of rescaled random unordered labelled trees. It is also proved directly in the continuous setting by Wang [44]. The moments of this distribution were calculated for example in Broutin and Flajolet [12] and are given by

$$\mathbb{E}[\mathrm{D}(\mathcal{T}_{\mathsf{e}})] = \frac{4}{3}\sqrt{\pi/2}, \qquad \mathbb{E}[\mathrm{D}(\mathcal{T}_{\mathsf{e}})^2] = \frac{2}{3}\Big(1 + \frac{\pi^2}{3}\Big), \qquad \mathbb{E}[\mathrm{D}(\mathcal{T}_{\mathsf{e}})^3] = 2\sqrt{2\pi}, \tag{9}$$

$$\mathbb{E}[\mathrm{D}(\mathcal{T}_{\mathsf{e}})^k] = \frac{2^{k/2}}{3}k(k-1)(k-3)\Gamma(k/2)\big(\zeta(k-2) - \zeta(k)\big), \qquad k \ge 4. \tag{10}$$

3. Combinatorial species and subcritical graph classes. We recall parts of the theory of combinatorial species and Boltzmann samplers to the extend required in this paper. A reader who is already familiar with the framework of subcritical graph classes may skip some parts of this section. However, we stress the importance of the representation of connected graphs as enriched trees in Section 3.5 and the coupling of random graphs with a Galton–Watson tree in Section 3.7. Moreover, several intermediate lemmas that we state and prove here were already shown in previous papers, albeit under stronger assumptions.

3.1. *Combinatorial species*. The framework of combinatorial species allows for a unified treatment of a wide range of combinatorial objects. We give only a concise introduction and refer to Joyal [30] and Bergeron, Labelle and Leroux [6] for a detailed discussion. The essentially equivalent language of *combinatorial classes* was developed by Flajolet and Sedgewick [24].

A *combinatorial species* may be defined as a functor $\mathcal{F}$ that maps any finite set U of *labels* to a finite set $\mathcal{F}[U]$ of $\mathcal{F}$*-objects* and any bijection $\sigma : U \to V$ of finite sets to its (bijective) *transport function* $\mathcal{F}[\sigma] : \mathcal{F}[U] \to \mathcal{F}[V]$ *along* σ, such that composition of maps and the identity are preserved. We say that a species $\mathcal{G}$ is a *subspecies* of $\mathcal{F}$ and write $\mathcal{G} \subset \mathcal{F}$ if $\mathcal{G}[U] \subset \mathcal{F}[U]$ for all finite sets U and $\mathcal{G}[\sigma] = \mathcal{F}[\sigma]|_U$ for all bijections $\sigma : U \to V$. Given two species $\mathcal{F}$ and $\mathcal{G}$, an *isomorphism* $\alpha : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$ from $\mathcal{F}$ to $\mathcal{G}$ is a family of bijections $\alpha = (\alpha_U : \mathcal{F}[U] \to \mathcal{G}[U])_U$ where U ranges over all finite sets, such that $\mathcal{G}[\sigma]\alpha_U = \alpha_V \mathcal{F}[\sigma]$ for all bijective maps $\sigma : U \to V$. The species $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* if there exists and isomorphism from one to the other. This is denoted by $\mathcal{F} \simeq \mathcal{G}$ or, by abuse of notation, just $\mathcal{F} = \mathcal{G}$. An element $\gamma_U \in \mathcal{F}[U]$ has size $|\gamma_U| := |U|$ and two $\mathcal{F}$-objects γ_U and γ_V are termed isomorphic if there is a bijection $\sigma : U \to V$ such that $\mathcal{F}[\sigma](\gamma_U) = \gamma_V$. We will often just write $\sigma.\gamma_U = \gamma_V$ instead, if there is no risk of confusion. An isomorphism class of $\mathcal{F}$-structures is called an *unlabeled* $\mathcal{F}$-object.

We will mostly be interested in subspecies of the species of finite simple graphs and use basic species such as the species of linear orders SEQ or the SET-species defined by $\text{SET}[U] = \{U\}$ for all U. Moreover, let 0 denote the empty species, 1 the species with a single object of size 0 and $\mathcal{X}$ the species with a single object of size 1.

Given $n \in \mathbb{N}_0$ we set $[n] := \{1, \dots, n\}$ and, where there is no danger of confusion, use the notation $\mathcal{F}_n$ for the set $\mathcal{F}[n] = \mathcal{F}[\{1, \dots, n\}]$. By abuse of notation, we will often let $\mathcal{F}$ also refer to the set $\bigcup_n \mathcal{F}_n$. The *exponential generating series* of a combinatorial species $\mathcal{F}$ is defined by $F(x) = \sum_{n \ge 0} |\mathcal{F}_n| x^n / n!$. In general, this is a formal power series that may have radius of convergence zero. If the series $F(x)$ has positive radius of convergence, we say that $\mathcal{F}$ is an *analytic species* and $F(x)$ is its *exponential generating function*. For any power series $f(x)$, we let $[x^n] f(x)$ denote the coefficient of x^n.

3.2. *Operations on species*. The framework of combinatorial species offers a large variety of constructions that create new species from others. In the following, let $\mathcal{F}$, $(\mathcal{F}_i)_{i\in\mathbb{N}}$ and $\mathcal{G}$ denote species and U an arbitrary finite set. The *sum* $\mathcal{F}+\mathcal{G}$ is defined by the disjoint union

$$(\mathcal{F}+\mathcal{G})[U]=\mathcal{F}[U]\sqcup\mathcal{G}[U].$$

More generally, the infinite sum $(\sum_i \mathcal{F}_i)$ may be defined by $(\sum_i \mathcal{F}_i)[U] = \bigsqcup_i \mathcal{F}_i[U]$ if the right-hand side is finite for all finite sets U. The *product* $\mathcal{F}\cdot\mathcal{G}$ is defined by the disjoint union

$$(\mathcal{F}\cdot\mathcal{G})[U]=\bigsqcup_{\substack{(U_1,U_2)\\ U_1\cap U_2=\varnothing, U_1\cup U_2=U}} \mathcal{F}[U_1]\times\mathcal{G}[U_2]$$

with componentwise transport. Thus, n-sized objects of the product are pairs of $\mathcal{F}$-objects and $\mathcal{G}$-objects whose sizes add up to n. If the species $\mathcal{G}$ has no objects of size zero, we can form the *substitution* $\mathcal{F}\circ\mathcal{G}$ by

$$(\mathcal{F}\circ\mathcal{G})[U]=\bigsqcup_{\pi\text{ partition of }U}\mathcal{F}[\pi]\times\prod_{Q\in\pi}\mathcal{G}[Q].$$

An object of the substitution may be interpreted as an $\mathcal{F}$-object whose labels are substituted by $\mathcal{G}$-objects. The transport along a bijection σ is defined by applying the induced map $\overline{\sigma}:\pi\to\overline{\pi}=\{\sigma(Q)|Q\in\pi\}$ of partitions to the $\mathcal{F}$-object and the restricted maps $\sigma|_Q$ with $Q\in\pi$ to their corresponding $\mathcal{G}$-objects. We will often write $\mathcal{F}(\mathcal{G})$ instead of $\mathcal{F}\circ\mathcal{G}$. The *rooted* or *pointed* $\mathcal{F}$-species is given by

$$\mathcal{F}^\bullet[U]=\mathcal{F}[U]\times U$$

with componentwise transport. That is, a pointed object is formed by distinguishing a label, named the *root* of the object, and any transport function is required to preserve roots. The *derived* species $\mathcal{F}'$ is defined by

$$\mathcal{F}'[U]=\mathcal{F}\big[U\cup\{*_U\}\big]$$

with $*_U$ referring to an arbitrary fixed element not contained in the set U (e.g., we could take $*_U=U$). The transport along a bijective map $\sigma:U\to V$ is done by applying the canonically extended bijection $\sigma':U\sqcup\{*_U\}\to V\sqcup\{*_V\}$ with $\sigma'(*_U)=*_V$ to the object. Derivation and pointing are related by an isomorphism $\mathcal{F}^\bullet\simeq\mathcal{X}\cdot\mathcal{F}'$.

Note that $\mathcal{F}'^\bullet$ and $\mathcal{F}^{\bullet\prime}$ are in general different species. In $\mathcal{F}^{\bullet\prime}$ objects, the root and $*$-label may coincide, since

$$\mathcal{F}^{\bullet\prime}[U]=\mathcal{F}^\bullet\big[U\cup\{*_U\}\big]$$

implies that a $\mathcal{F}^{\bullet\prime}$-object over U is a $\mathcal{F}$-object over $U\cup\{*_U\}$ together with a distinguished element from $U\cup\{*_U\}$. On the other hand, $\mathcal{F}'^\bullet$-objects are always rooted at non-$*$-labels, since

$$\mathcal{F}'^\bullet[U]=\mathcal{F}'[U]\times U$$

TABLE 1
Relation between combinatorial constructions and generating series

$\sum_i \mathcal{F}_i$	$\sum_i F_i(x)$
$\mathcal{F} \cdot \mathcal{G}$	$F(x)G(x)$
$\mathcal{F} \circ \mathcal{G}$	$F(G(x))$
$\mathcal{F}^\bullet$	$x\frac{d}{dx}F(x)$
$\mathcal{F}'$	$\frac{d}{dx}F(x)$

SET	$\exp(x)$
SEQ	$1/(1-x)$
0	0
1	1
$\mathcal{X}$	x

implies that a $\mathcal{F}'^\bullet$-object over U is a $\mathcal{F}$-object over $U \cup \{*_U\}$ together with a distinguished element from U.

Explicit formulas for the exponential generating series of these constructions are summarized in Table 1. The notation is quite suggestive: up to (canonical) isomorphism, each operation considered in this section is associative. The sum and product are commutative operations and satisfy a distributive law, that is,

$$\mathcal{F} \cdot (\mathcal{G}_1 + \mathcal{G}_2) \simeq \mathcal{F} \cdot \mathcal{G}_1 + \mathcal{F} \cdot \mathcal{G}_2 \tag{11}$$

for any two species $\mathcal{G}_1$ and $\mathcal{G}_2$. The operation of deriving a species is additive and satisfies a product rule and a chain rule, analogous to the derivative in calculus:

$$(\mathcal{F} \cdot \mathcal{G})' \simeq \mathcal{F}' \cdot \mathcal{G} + \mathcal{F} \cdot \mathcal{G}' \quad \text{and} \quad \mathcal{F}(\mathcal{G})' \simeq \mathcal{F}'(\mathcal{G}) \cdot \mathcal{G}'. \tag{12}$$

Recall that for the chain rule to apply we have to require $\mathcal{G}[\varnothing] = \varnothing$, since otherwise $\mathcal{F}(\mathcal{G})$ is not defined. A thorough discussion of these facts is beyond the scope of this introduction. We refer the inclined reader to Joyal [30] and Bergeron, Labelle and Leroux [6].

3.3. *Combinatorial specifications*. In this section, we briefly recall Joyal's implicit species theorem that allows us to define combinatorial species up to unique isomorphism and construct recursive samplers that draw objects of a species randomly (see Section 3.6 below). In order to state the theorem, we need to introduce the concept of *multisort species*. As it is sufficient for our applications, we restrict ourselves to the 2-sort case.

A 2-*sort species* $\mathcal{H}$ is a functor that maps any pair $U = (U_1, U_2)$ of finite sets to a finite set $\mathcal{H}[U] = \mathcal{H}[U_1, U_2]$ and any pair $\sigma = (\sigma_1, \sigma_2)$ of bijections $\sigma_i : U_i \to V_i$ to a bijection $\mathcal{H}[\sigma] : \mathcal{H}[U] \to \mathcal{H}[V]$ in such a way, that identity maps and composition of maps are preserved. The operations of sum, product and composition extend naturally to the multisort-context. Let $\mathcal{H}$ and $\mathcal{K}$ be 2-sort species and $U = (U_1, U_2)$ a pair of finite sets. The *sum* is defined by

$$(\mathcal{H} + \mathcal{K})[U] = \mathcal{H}[U] \sqcup \mathcal{K}[U].$$

We write $U = V + W$ if $U_i = V_i \cup W_i$ and $V_i \cap W_i = \varnothing$ for all i. The *product* is defined by

$$(\mathcal{H} \cdot \mathcal{K})[U] = \bigsqcup_{V+W=U} \mathcal{H}[V] \times \mathcal{K}[W].$$

The *partial derivatives* are given by

$$\partial_1 \mathcal{H}[U] = H\big[U_1 \cup \{*_{U_1}\}, U_2\big] \quad \text{and} \quad \partial_2 \mathcal{H}[U] = H\big[U_1, U_2 \cup \{*_{U_2}\}\big].$$

In order state Joyal's implicit species theorem, we also require the substitution operation for multisort species; this will allow us to define species "recursively" up to (canonical) isomorphism. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be (1-sort) species and M a finite set. A structure of the *composition* $\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2)$ over the set M is a quadrupel $(\pi, \chi, \alpha, \beta)$ such that:

1. π is partition of the set M.
2. $\chi : \pi \to \{1, 2\}$ is a function assigning to each class a sort.
3. α a function that assigns to each class $Q \in \pi$ a $\mathcal{F}_{\chi(Q)}$-object

$$\alpha(Q) \in \mathcal{F}_{\chi(Q)}[Q].$$

4. β a $\mathcal{H}$-structure over the pair $(\chi^{-1}(1), \chi^{-1}(2))$.

This construction is *functorial*: any pair of isomorphisms α_1, α_2 with

$$\alpha_i : \mathcal{F}_i \xrightarrow{\sim} \mathcal{G}_i$$

induces an isomorphism

$$\mathcal{H}[\alpha_1, \alpha_2] : \mathcal{H}(\mathcal{F}_1, \mathcal{F}_2) \xrightarrow{\sim} \mathcal{H}(\mathcal{G}_1, \mathcal{G}_2).$$

Let $\mathcal{H}$ be a 2-sort species and recall that $\mathcal{X}$ denotes the species with a unique object of size one. A solution of the system $\mathcal{Y} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ is pair $(\mathcal{A}, \alpha)$ of a species $\mathcal{A}$ with $\mathcal{A}[0] = 0$ and an isomorphism $\alpha : \mathcal{A} \xrightarrow{\sim} \mathcal{H}(\mathcal{X}, \mathcal{A})$. An isomorphism of two solutions $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ is an isomorphism of species $u : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha} & \mathcal{H}(\mathcal{X}, \mathcal{A}) \\ \big\downarrow u & & \big\downarrow \mathcal{H}(\mathrm{id}, u) \\ \mathcal{B} & \xrightarrow{\beta} & \mathcal{H}(\mathcal{X}, \mathcal{B}) \end{array}$$

We may now state Joyal's implicit species theorem.

THEOREM 3.1 ([30], Théorème 6). *Let $\mathcal{H}$ be a 2-sort species satisfying $\mathcal{H}(0, 0) = 0$. If $(\partial_2 \mathcal{H})(0, 0) = 0$, then the system $\mathcal{Y} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ has up to isomorphism only one solution. Moreover, between any two given solutions there is exactly one isomorphism.*

We say that an isomorphism $\mathcal{F} \simeq \mathcal{H}(\mathcal{X}, \mathcal{F})$ is a *combinatorial specification* for the species $\mathcal{F}$ if the 2-sort species $\mathcal{H}$ satisfies the requirements of Theorem 3.1.

3.4. *Block-stable graph classes.* Any graph may be decomposed into its *connected components*, that is, its maximal connected subgraphs. These connected components allow a *block-decomposition* which we recall in the following. Let C be a connected graph. If removing a vertex v (and deleting all adjacent edges) disconnects the graph, we say that v is a *cutvertex* of C. The graph C is *2-connected*, if it has size at least three and no cutvertices.

A *block* of an arbitrary graph G is a maximal connected subgraph $B \subset G$ that does not have a cutvertex (of itself). It is well known (see, e.g., [17]) that any block is either 2-connected or an edge or a single isolated point. Moreover, the intersection of two blocks is either empty or a cutvertex of a connected component of G. If G is connected, then the bipartite graph whose vertices are the blocks and the cutvertices of G and whose edges are pairs $\{v, B\}$ with $v \in B$ is a tree and called the *block-tree* of G.

Let $\mathcal{G}$ denote a subspecies of the species of graphs, $\mathcal{C} \subset \mathcal{G}$ the subspecies of connected graphs in $\mathcal{G}$ and $\mathcal{B} \subset \mathcal{C}$ the subspecies of all graphs in $\mathcal{C}$ that are 2-connected or consist of only two vertices joined by an edge. We say that $\mathcal{G}$ or $\mathcal{C}$ is a *block-stable* class of graphs, if $\mathcal{B} \neq 0$ and $G \in \mathcal{G}$ if and only if every block of G belongs to $\mathcal{B}$ or is a single isolated vertex. Block-stable classes satisfy the following combinatorial specifications that can be found, for example, in [6, 27, 30]:

$$\mathcal{G} \simeq \mathrm{SET} \circ \mathcal{C} \quad \text{and} \quad \mathcal{C}^\bullet \simeq \mathcal{X} \cdot (\mathrm{SET} \circ \mathcal{B}' \circ \mathcal{C}^\bullet). \tag{13}$$

The first correspondence expresses the fact that we may form any graph on a given vertex set U by partitioning U and constructing a connected graph on each partition class. The specification for rooted connected graphs, illustrated in Figure 2, is based on the construction of the block-tree. The idea is to interpret $\mathcal{B}' \circ \mathcal{C}^\bullet$-objects as graphs by connecting the roots of the $\mathcal{C}^\bullet$ objects on the partition classes and the $*$-vertex with edges according to the $\mathcal{B}'$-object on the partition. An object of $\mathrm{SET} \circ (B' \circ \mathcal{C}^\bullet)$ can then be interpreted as a graph by identifying the $*$-vertices of the $B' \circ \mathcal{C}^\bullet$ objects. This construction is compatible with graph isomorphisms,

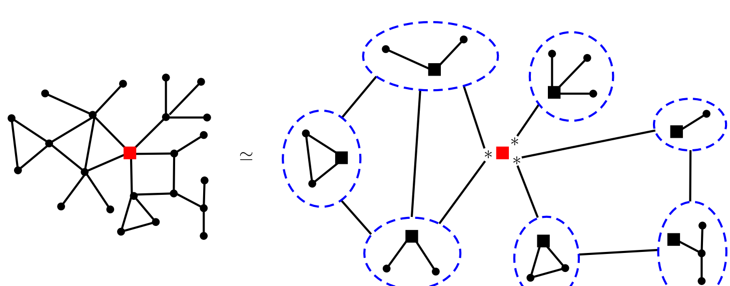

FIG. 2. *Decomposition of a rooted graph from $\mathcal{C}^\bullet$ into a $\mathcal{X} \cdot (\mathrm{SET} \circ \mathcal{B}' \circ \mathcal{C}^\bullet)$ structure. Labels are omitted and the roots are marked with squares.*

hence $\mathcal{C}' \simeq \mathrm{SET} \circ \mathcal{B}' \circ \mathcal{C}^\bullet$ and the second specification in (13) follows. By the rules for computing the generating series of species, we obtain the equations

$$G(x) = \exp\bigl(C(x)\bigr) \quad \text{and} \quad C^\bullet(x) = x \exp\bigl(B'\bigl(C^\bullet(x)\bigr)\bigr). \tag{14}$$

The following lemma was given in Panagiotou and Steger [41] and Drmota et al. [18] under some additional assumptions.

LEMMA 3.2. *Let $\mathcal{C}$ be a block-stable class of connected graphs, $\mathcal{B} \neq 0$ its subclass of all graphs that are 2-connected or a single edge. Then $C(z)$ has radius of convergence $\rho < \infty$ and the sums $y := C^\bullet(\rho)$ and $\lambda := B'(y)$ are finite and satisfy*

$$y = \rho \exp(\lambda). \tag{15}$$

PROOF. It suffices to consider the case $\rho > 0$. By assumption, we have $\mathcal{B} \neq 0$ and hence there is a $k \in \mathbb{N}$ such that $[z^k]B'(z) \neq 0$. Thus, by (14) we have, say, $C^\bullet(z) = czC^\bullet(z)^{2k} + R(z)$ for some constant $c > 0$ and $R(z)$ a power series in z with nonnegative coefficients. This implies $\lim_{x\uparrow\rho} C^\bullet(x) < \infty$ and thus ρ and $C^\bullet(\rho)$ are both finite. The coefficients of all power series involved in (14) are nonnegative, and so it follows that $y = \rho \exp(\lambda)$ and thus $\lambda < \infty$. $\square$

We will only be interested in the case where $\mathcal{C}$ is analytic. The following observation (made, e.g., also in [19]) shows that this is equivalent to requiring that $\mathcal{B}$ is analytic. We include a short proof for completeness.

PROPOSITION 3.3. *Let $\mathcal{C}$ be a block-stable class of connected graphs, $\mathcal{B} \neq 0$ its subclass of all graphs that are 2-connected or a single edge. Then $\mathcal{C}$ is analytic if and only if $\mathcal{B}$ is analytic.*

PROOF. By nonnegativity of coefficients, we see easily that $\rho > 0$ implies that $\mathcal{B}$ is analytic. Conversely, suppose that $B(z)$ has positive radius of convergence $R > 0$. By the inverse function theorem, the block-stability equation $f(z) = z \exp(B'(f(z)))$ has an analytic solution whose expansion at the point 0 agrees with the series $C^\bullet(z)$ by Lagrange's inversion formula. Hence, $\mathcal{C}$ is an analytic class. $\square$

3.5. *$\mathcal{R}$-Enriched trees.* The class $\mathcal{A}$ of rooted trees[2] is known to satisfy the decomposition

$$\mathcal{A} \simeq \mathcal{X} \cdot \mathrm{SET}(\mathcal{A}).$$

This is easy to see: in order to form a rooted tree on a given set of vertices, we choose a root vertex v, partition the remaining the vertices, endow each partition

[2] *Arborescence* is the French word for rooted tree, hence the notation $\mathcal{A}$.

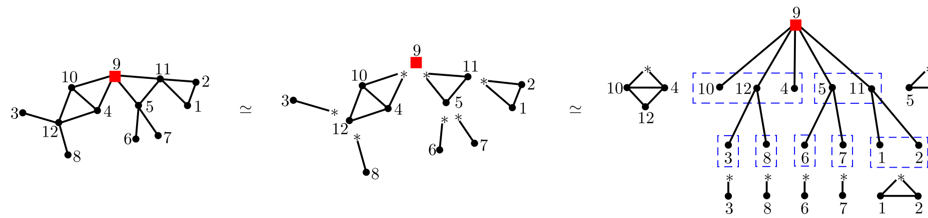

FIG. 3. *Correspondence of the classes* $\mathcal{C}^\bullet$ *and* $\mathrm{SET}(\mathcal{B}')$*-enriched trees.*

class with a structure of a rooted tree and connect the vertex v with their roots. More generally, given a species $\mathcal{R}$ the class $\mathcal{A}_\mathcal{R}$ of $\mathcal{R}$-enriched trees is defined by the combinatorial specification

$$\mathcal{A}_\mathcal{R} \simeq \mathcal{X} \cdot \mathcal{R}(\mathcal{A}_\mathcal{R}).$$

In other words, an $\mathcal{R}$-enriched tree is a rooted tree such that the offspring set of any vertex is endowed with an $\mathcal{R}$-structure. Natural examples are labeled ordered trees, which are SEQ-enriched trees, and plane trees, which are unlabeled ordered trees. Ordered and unordered tree families defined by restrictions on the allowed outdegree of internal vertices also fit in this framework. $\mathcal{R}$-enriched trees were introduced by Labelle [33] in order to provide a combinatorial proof of Lagrange Inversion. They have applications in various fields of mathematics; see, for example, [16, 34, 40].

The combinatorial specification (13) together with Theorem 3.1 allows us to identify a block-stable graph class $\mathcal{C}^\bullet$ with the class $\mathcal{R}$-enriched trees where $\mathcal{R} = \mathrm{SET}(\mathcal{B}')$, that is, rooted trees from $\mathcal{A}$ where the offspring set of each vertex is partitioned into nonempty sets and each of these sets carries a $\mathcal{B}'$-structure. Compare with Figure 3.

COROLLARY 3.4. *Let* $\mathcal{C}$ *be a block-stable class of connected graphs,* $\mathcal{B} \neq 0$ *its subclass of all graphs that are* 2*-connected or a single edge. Then there is a unique isomorphism between* $\mathcal{C}^\bullet$ *and the class* $\mathcal{A}_{\mathrm{SET}\circ\mathcal{B}'}$ *of pairs* (T, α) *with* $T \in \mathcal{A}$ *and* α *a function that assigns to each* $v \in V(T)$ *a* (*possibly empty*) *set* $\alpha(v) \in (\mathrm{SET} \circ \mathcal{B}')[M_v]$ *of derived blocks whose vertex sets partition the offspring set* M_v *of* v.

PROOF. By the isomorphism given in (13), the classes $\mathcal{A}_{\mathrm{SET}\circ\mathcal{B}'}$ and $\mathcal{C}^\bullet$ are both solutions of the system $\mathcal{Y} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ with $\mathcal{H}(\mathcal{X}, \mathcal{Y}) = \mathcal{X} \cdot \mathrm{SET} \circ \mathcal{B}' \circ \mathcal{Y}$. Joyal's implicit species Theorem 3.1 yields that there is a unique isomorphism between any two solutions. □

3.6. *Boltzmann samplers*. Boltzmann samplers provide a method of generating efficiently random discrete combinatorial objects. They were introduced in

TABLE 2
Rules for the construction of Boltzmann samplers

$\mathcal{F} = \mathcal{A} + \mathcal{B}$	**if** $\mathsf{Bern}(A(x)/(A(x)+B(x)))$ **then** **return** $\Gamma A(x)$ **else return** $\Gamma B(x)$
$\mathcal{F} = \mathcal{A} \cdot \mathcal{B}$	**return** $(\Gamma A(x), \Gamma B(x))$ relabeled uniformly at random
$\mathcal{F} = \mathcal{A} \circ \mathcal{B}$	$\gamma \leftarrow \Gamma A(y)$ with $y = B(x)$ **for** $i = 1$ **to** $\|\gamma\|$ $\gamma_i \leftarrow \Gamma B(x)$ **return** $(\gamma, (\gamma_i)_i)$ relabeled uniformly at random
$\mathcal{F} = \mathrm{SET}$	$m \leftarrow \mathsf{Pois}(x)$ **return** the unique structure of size m

Duchon, Flajolet, Louchard and Schaeffer [20] and were developed further in Flajolet, Fusy and Pivoteau [23]. Following these sources, we will briefly recall the theory of Boltzmann samplers to the extend required for the applications in this paper. Let $\mathcal{F} \neq 0$ be an analytic species of structures and F its exponential generating function. Given a parameter $x > 0$ such that $0 < F(x) < \infty$, a Boltzmann sampler $\Gamma F(x)$ is a random generator that draws an object $\gamma \in \mathcal{F}$ with probability

$$\mathbb{P}(\Gamma F(x) = \gamma) = \frac{x^{|\gamma|}}{F(x)|\gamma|!}.$$

In particular, if we condition on a fixed output size n, we get the uniform distribution on $\mathcal{F}_n$. We describe Boltzmann samplers using an informal pseudo-code notation. Given a specification of the species of structures $\mathcal{F}$ in terms of other species using the operations of sums, products and composition, we obtain a Boltzmann sampler for $\mathcal{F}$ in terms of samplers for the other species involved. The rules for the construction of Boltzmann samplers are summarized in Table 2. We let $\mathsf{Bern}(p)$ and $\mathsf{Pois}(\lambda)$ denote Bernoulli and Poisson distributed generators.

Note that if $\mathcal{F} = \mathcal{A}\mu\mathcal{B}$ with $\mathcal{A}, \mathcal{B} \neq 0$, $\mu \in \{+, \cdot, \circ\}$ and $0 < \mathcal{F}(x) < \infty$, then the samplers of $\mathcal{A}$ and $\mathcal{B}$ are almost surely called with valid parameters, since the coefficients of all power-series involved are nonnegative.

Given a combinatorial specification $\mathcal{Y} \simeq \mathcal{H}(\mathcal{X}, \mathcal{Y})$ satisfying the conditions of Theorem 3.1 we may apply the rules above to construct a recursive Boltzmann sampler that is guaranteed to terminate almost surely. In our setting, this allows us to construct a Boltzmann sampler for block-stable graph classes. More specifically, let $\mathcal{C}$ be a block-stable class of connected graphs such that the radius of convergence ρ of the generating series $C(z)$ is positive. The rooted class $\mathcal{C}^\bullet$ has a combinatorial specification given in (13) in terms of the subclass $\mathcal{B}$ of edges and 2-connected graphs. By Lemma 3.2, we know that $y = C^\bullet(\rho)$ and $\lambda = B'(y)$ are finite.

Since ρ is an admissible parameter for the Boltzmann-distribution of $\mathcal{C}^\bullet$, we may apply the rules in Table 2 in order to obtain an explicit sampler $\Gamma C^\bullet(\rho)$. By the rule concerning products of species, we have to start with independent calls to the samplers $\Gamma X(\rho)$ and $\Gamma(\mathrm{SET} \circ B' \circ C^\bullet)(\rho)$, and relabel uniformly at random afterward. The sampler $\Gamma X(\rho)$ generates (deterministically) a single root-vertex. The rule for compositions states that a Boltzmann sampler for $(\mathrm{SET} \circ B') \circ C^\bullet$ is obtained by starting with $\Gamma(\mathrm{SET} \circ B')(y)$, and making independent calls to $\Gamma C^\bullet(\rho)$ for each atom (i.e., non-$*$-vertex) of the result. The graph is then constructed from these objects according to the isomorphism illustrated in Figure 2. Putting everything together, we obtain the following recursive procedure.

COROLLARY 3.5. *Let $\mathcal{C}$ be a block-stable class of connected graphs, $\mathcal{B} \neq 0$ its subclass of all graphs that are 2-connected or a single edge. The following recursive procedure terminates almost surely and samples according to the Boltzmann distribution for $\mathcal{C}^\bullet$ with parameter ρ.*

$\Gamma C^\bullet(\rho)$: $\gamma \leftarrow$ a single root vertex
  $M \leftarrow \Gamma(\mathrm{SET} \circ B')(y)$
  for each derived block B' in M
    merge the $*$-vertex of B' with γ
    for each non $*$-vertex v of B'
      $C_v \leftarrow \Gamma C^\bullet(\rho)$
      merge v with the root of C_v
  return the resulting graph, relabeled uniformly at random

We may interpret this sampler as a deterministic recursive procedure whose input data is a list of $\mathrm{SET} \circ \mathcal{B}'$-objects $(M_1, M_2, \dots)$ (which we choose independently at random according to a Boltzmann distribution) from which it always reads the next so far unused entry in the second line of the pseudo code.

This procedure was used before in the study of certain block-stable graph classes; see, for example, [41]. Using the rules for the composition and the SET-species, we also obtain an explicit description of a Boltzmann sampler for the species $\mathrm{SET} \circ \mathcal{B}'$:

$\Gamma(\mathrm{SET} \circ B')(y)$: $m \leftarrow \mathsf{Pois}(\lambda)$
  for $k = 1 \cdots m$
    $B'_k \leftarrow \Gamma B'(y)$
  return $\{B'_1, \dots, B'_m\}$, relabeled u.a.r.

3.7. *Subcritical graph classes.* Let $\mathcal{C}$ be a block-stable class of connected graphs and $\mathcal{B}$ its subclass of all graphs that are 2-connected or a single edge. Assume that $\mathcal{B}$ is nonempty and analytic, hence $\mathcal{C}$ is analytic as well by Proposition 3.3. Denote by ρ and R the radii of convergence of the corresponding exponential generating series $C(z)$ and $B(z)$. By Lemma 3.2, we know that ρ, $y = C^\bullet(\rho)$ and $\lambda = B'(y)$ are finite. The following proposition provides a coupling of a Boltzmann-distributed random graph drawn from the class $\mathcal{C}$ with a Galton–Watson tree. This will play a central role in the proof of the main theorem.

PROPOSITION 3.6. *Let* (T, α) *denote the enriched tree corresponding to the Boltzmann Sampler* $\Gamma C^\bullet(\rho)$ *given in Corollary* 3.5. *Then the rooted labeled unordered tree* T *is distributed like the outcome of the following process*:

1. *Draw a Galton–Watson tree with offspring distribution* ξ *given by the probability generating function* $\varphi(z) = \exp(B'(yz) - \lambda)$.
2. *Distribute labels uniformly at random.*
3. *Discard the ordering on the offspring sets.*

PROOF. The sampler $\Gamma C^\bullet(\rho)$ given in Corollary 3.5 starts with a single root-vertex and a set M of $\mathcal{B}'$-objects drawn according to $\Gamma(\mathrm{SET} \circ B')(y)$. Each non-$*$-vertex of the blocks in M corresponds to an offspring vertex of the root in the tree T. Thus, the root receives total offspring with size distributed according to $|\Gamma(\mathrm{SET} \circ B')(y)|$, which by definition of the Boltzmann distribution has probability generating function $\exp(B'(yz) - \lambda)$. For any offspring vertex, the sampler proceeds with a recursive call to $\Gamma C^\bullet(\rho)$. After this recursive procedure terminates, the vertices of the resulting graph are relabeled uniformly at random. Thus, T is distributed like a Galton–Watson tree with offspring distribution given by the pgf $\varphi(z)$, except that we neglect all orderings on the offspring sets and relabel the vertices uniformly at random after constructing the tree. □

Let ξ denote the offspring distribution given in Proposition 3.6. As discussed above, the rules governing Boltzmann samplers guarantee that the sampler $\Gamma C^\bullet(\rho)$ terminates almost surely. Hence, we have

$$1 \geq \mathbb{E}[\xi] = \varphi_\ell'(1) = yB''(y) = B'^\bullet(y)$$

and in particular $y \leq R$, where R is the radius of convergence of B. We define sub-criticality depending on whether this inequality is strict.

DEFINITION 3.7. A block-stable class of connected graphs $\mathcal{C}$ is termed subcritical if $y < R$.

Prominent examples of subcritical graph classes are trees, outer-planar graphs and series-parallel graphs; the class of planar graphs does not fall into this framework [7, 18], that is, it satisfies $y = R$. The following lemma was proved in [41], Lemma 2.8, by analytic methods.

LEMMA 3.8. *If* $B'^\bullet(R) \geq 1$, *then* $B'^\bullet(y) = 1$. *If* $B'^\bullet(R) \leq 1$, *then* $y = R$. *In particular*, $\mathcal{C}$ *is subcritical if and only if* $B'^\bullet(R) > 1$.

Thus, if $B'^\bullet(R) \geq 1$, then the offspring distribution ξ has expected value 1 and variance

$$\sigma^2 = 1 + B'''(y)y^2 = \mathbb{E}[|\Gamma B'^\bullet(y)|]$$

with $\Gamma B'^{\bullet}(y)$ denoting a Boltzmann sampler for the class $\mathcal{B}'^{\bullet}$ with parameter y. By Proposition 3.6, the size of the outcome of the sampler $\Gamma C^{\bullet}(\rho)$ is distributed like the size of a ξ-Galton–Watson tree. Hence, we may apply Lemma 2.1 to obtain the following result, which was shown in [18] under stronger assumptions.

COROLLARY 3.9. *Let $\mathcal{C}$ be an analytic block-stable class of graphs, and let ξ be the distribution from Proposition 3.6. Suppose that $B'^{\bullet}(R) \geq 1$ and $B'''(y) < \infty$, that is, ξ has finite variance. Let $d = \mathsf{span}(\xi)$. Then, as $n \equiv 1 \bmod d$ tends to infinity,*

$$\mathbb{P}(|\Gamma C^{\bullet}(\rho)| = n) \sim \frac{d}{\sqrt{2\pi \mathbb{E}[|\Gamma B'^{\bullet}(y)|]}} n^{-3/2}$$

and

$$|\mathcal{C}_n| \sim \frac{yd}{\sqrt{2\pi \mathbb{E}[|\Gamma B'^{\bullet}(y)|]}} n^{-5/2} \rho^{-n} n!.$$

3.8. *Deviation inequalities*. We will make use of the following moderate deviation inequality for one-dimensional random walks found in most textbooks on the subject.

LEMMA 3.10. *Let $(X_i)_{i \in \mathbb{N}}$ be independent copies of a real-valued random variable X with $\mathbb{E}[X] = 0$. Let $S_n = X_1 + \cdots + X_n$. Suppose there is a $\delta > 0$ such that $\mathbb{E}[e^{\theta X}] < \infty$ for $|\theta| < \delta$. Then there is a $c > 0$ such that for every $1/2 < p < 1$ there is a number N such that for all $n \geq N$ and $0 < \varepsilon < 1$*

$$\mathbb{P}(|S_n/n^p| \geq \varepsilon) \leq 2 \exp(-c\varepsilon^2 n^{2p-1}).$$

4. A size-biased random $\mathcal{R}$-enriched tree.

4.1. *A size-biased random $\mathcal{R}$-enriched tree*. An important ingredient in our forthcoming arguments will be an accurate description of the distribution of the blocks on sufficiently long paths in random graphs from an analytic block-stable class of connected graphs $\mathcal{C}$. In order to study this distribution, we will make use of a special case of a *size-biased* random $\mathcal{R}$-enriched tree. The use of size-biased structures to study distances for large random trees is a fruitful approach used in classic and recent literature (see, e.g., [1, 38]), and applying it to $\mathcal{R}$-enriched trees allows for a particular short and elegant proof of our main result.

In order to motivate our construction and to create a direct analogy to previous work, we begin with a concise description of the *size biased ξ-Galton–Watson tree* $\mathsf{T}^{(\ell)}$ from [1], where $\xi \geq 0$ is an integer valued random variable with $\mathbb{E}[\xi] = 1$. The size-biased tree $\mathsf{T}^{(\ell)}$ is a random plane tree that has a distinguished vertex having height ℓ. For $\ell = 0$, it is distributed like the ξ-Galton–Watson tree T and the distinguished vertex coincides with the root. For $\ell \geq 1$ it is constructed as follows.

There are two types of vertices called *normal* and *mutant*. We start with a mutant root. Normal vertices have offspring according to an independent copy of ξ and each of their offspring is declared normal as well. Mutant vertices have offspring according to an independent copy of the *size-biased offspring distribution* $\hat{\xi}$, given by

$$\mathbb{P}(\hat{\xi} = m) = m\mathbb{P}(\xi = m).$$

For each mutant vertex, one of its children is selected uniformly at random and declared its heir. The remaining offspring is declared normal. The heir is declared mutant if it has height less than ℓ, and normal otherwise. The distinguished vertex is given by the unique heir with height ℓ.

The construction guarantees that for any plane tree T and any vertex u with height ℓ in T

$$\mathbb{P}(\mathsf{T}^{(\ell)} = (T, u)) = \mathbb{P}(\mathsf{T} = T); \tag{16}$$

see equation (26) in [1]. The importance of (16) is that it may be used to study properties of the ξ-Galton–Watson tree T by making use of the fact that the outdegrees of the ℓ mutant vertices of $\mathsf{T}^{(\ell)}$ are independent and identically distributed.

Our *size-biased $\mathcal{R}$-enriched tree* is constructed in a similar though more involved fashion. Let $\mathcal{C}$ be an analytic block-stable class of connected graphs and $\mathcal{B} \neq 0$ its subclass of graphs that are 2-connected or a single edge. Recall that by Corollary 3.4 the class $\mathcal{C}^\bullet$ may be identified with the class of $\mathcal{R}$-enriched trees with $\mathcal{R} := \mathrm{SET} \circ \mathcal{B}'$, that is, pairs (T, α) with T being a rooted labeled unordered tree and α a function that assigns to each $v \in V(T)$ a (possibly empty) set $\alpha(v)$ of derived blocks whose vertex sets partition the offspring set of v.

In our construction, we make use of Boltzmann samplers $\Gamma R(y)$ and $\Gamma R^\bullet(y)$ for the classes $\mathcal{R}$ and $\mathcal{R}^\bullet$, and of the Boltzmann sampler $\Gamma A_R(\rho)$ for $\mathcal{A}_{\mathcal{R}}$. With this notation at hand, we are going to construct the size-biased $\mathcal{R}$-enriched tree $\mathsf{A}_{\mathcal{R}}^{(\ell)}$ as a random $\mathcal{R}$-enriched tree (T, α) together with a distinguished vertex v having height $h_{\mathsf{T}}(v) = \ell$.

For $\ell = 0$, it is given by $\Gamma A_R(\rho)$ and the distinguished vertex coincides with the root. For $\ell \geq 1$, we consider again two kinds of vertices termed normal and mutant. We begin with a single mutant root. The offspring of any normal vertex is given by an independent copy of $\Gamma R(y)$ and each vertex of that copy is declared normal as well. Mutant vertices have an independent copy of $\Gamma R^\bullet(y)$ as offspring, and all vertices except for the root in the $\mathcal{R}^\bullet$-object are declared normal. The root in the $\mathcal{R}^\bullet$-object is declared mutant unless it is in the ℓth copy of $\Gamma R^\bullet(y)$. The final result is then obtained by letting the root of the ℓth copy of $\Gamma R^\bullet(y)$ be the distinguished vertex and additionally distributing the labels of the resulting pointed enriched tree uniformly at random. The construction of the size-biased $\mathcal{R}$-enriched tree is illustrated in Figure 4.

Note the analogies in the construction to the size-biased ξ-Galton–Watson tree: there, normal vertices have offspring according to ξ, and similarly in $\mathsf{A}_{\mathcal{R}}^{(\ell)}$ before

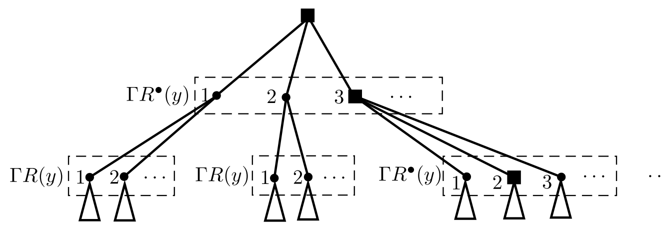

FIG. 4. *Illustration of the size-biased $\mathcal{R}$-enriched tree.*

the relabeling, the $\mathcal{R}$-structures of normal vertices are independent $\Gamma R(\rho)$ structures. Moreover, in $\mathsf{T}^{(\ell)}$ mutant vertices have an offspring distributed like $\hat{\xi}$, while in $\mathsf{A}_{\mathcal{R}}^{(\ell)}$ the offspring is given by an independent copy of $\Gamma R^\bullet(y)$; since $\mathcal{R}^\bullet = \mathcal{X} \cdot \mathcal{R}'$ the analogy with the probability generating function $\mathbb{E}[z^{\hat{\xi}}] = z \frac{\mathrm{d}}{\mathrm{d}z} \mathbb{E}[z^{\xi}]$ of the size-biased offspring distribution $\hat{\xi}$ is established.

The main aim of this section is to establish a statement analogous to (16).

LEMMA 4.1. *With the notation in this section, let ρ denote the radius of convergence of the exponential generating series $C(z)$ and set $y = C^\bullet(\rho)$. Let $A = (T, \alpha) \in \mathcal{A}_{\mathcal{R}}$ be an $\mathcal{R}$-enriched tree and let u be a vertex in A having height $h_T(u) = \ell$. Then*

$$\mathbb{P}(\mathsf{A}_{\mathcal{R}}^{(\ell)} = (A, u)) = (\rho R'(y))^{-\ell} \mathbb{P}(\Gamma A_R(\rho) = A). \tag{17}$$

In order to facilitate the proof of (17), we first consider a decomposition of enriched trees along paths that start at the root. More specifically, consider the species $\mathcal{A}_{\mathcal{R}}^\bullet$ of pointed enriched trees, that is of enriched trees $A = (T, \alpha)$ together with a distinguished vertex u of T. In order to avoid confusion, we call u the *outer root*, and the root of T the *inner root*. The directed path in T from the inner root to the outer root is termed the *spine*. The species $\mathcal{A}_{\mathcal{R}}^\bullet$ admits the following classical decomposition due to Labelle [33], Theorem A. First, we split the species into summands

$$\mathcal{A}_{\mathcal{R}}^\bullet \simeq \sum_{\ell \geq 0} \mathcal{A}_{\mathcal{R}}^{(\ell)}$$

with $\mathcal{A}_{\mathcal{R}}^{(\ell)}$ denoting the subspecies of all pointed $\mathcal{R}$-enriched trees whose spine has length ℓ. In the case $\ell = 0$, the inner and outer root coincide, yielding $\mathcal{A}_{\mathcal{R}}^{(0)} \simeq \mathcal{A}_{\mathcal{R}}$. For $\ell \geq 1$, we are going to argue that there is an isomorphism

$$\mathcal{A}_{\mathcal{R}}^{(\ell)} \simeq \mathcal{X} \cdot \mathcal{R}'(\mathcal{A}_{\mathcal{R}}) \cdot \mathcal{A}_{\mathcal{R}}^{(\ell-1)}, \tag{18}$$

as illustrated in Figure 5.

Indeed, suppose that we are given an arbitrary $\mathcal{A}_{\mathcal{R}}^{(\ell)}$-object. The maximal (enriched) subtree rooted at the successor v of the inner root along the spine is an

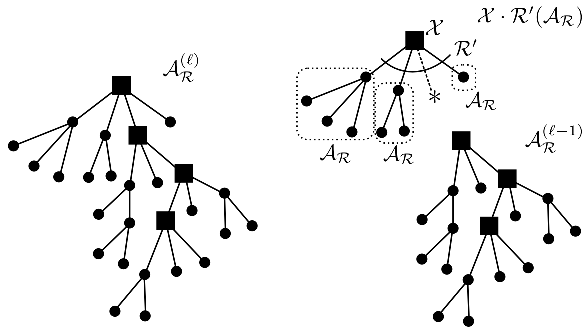

FIG. 5. *The decomposition of $\mathcal{A}_{\mathcal{R}}^{(\ell)}$, with the squares marking the vertices on the spine.*

$\mathcal{A}_{\mathcal{R}}^{(\ell-1)}$-object, as the length of its spine is decreased by 1. If we cut this tree away and replace v with a $*$-vertex, we are left with the inner root, accounting for the factor $\mathcal{X}$ in (18), together with an $\mathcal{R}'$-object whose non-$*$-labels are the roots of $\mathcal{A}_{\mathcal{R}}$-objects, accounting for the factor $\mathcal{R}'(\mathcal{A}_{\mathcal{R}})$.

By iterating (18), we arrive at

$$\mathcal{A}_{\mathcal{R}}^{(\ell)} \simeq (\mathcal{X} \cdot \mathcal{R}'(\mathcal{A}_{\mathcal{R}}))^{\ell} \cdot \mathcal{A}_{\mathcal{R}}. \tag{19}$$

PROOF OF LEMMA 4.1. Recall that $\rho > 0$ denotes the radius of convergence of the exponential generating series $C^{\bullet}(z) = A_R(z)$ and that $y = A_R(\rho) < \infty$ by Lemma 3.2. We are going to study a Boltzmann sampler $\Gamma A_R^{(\ell)}(\rho)$ for the class $\mathcal{A}_{\mathcal{R}}^{(\ell)}$. Of course, we first have to check whether ρ is an admissible parameter for the Boltzmann distribution, that is, if $A_R^{(\ell)}(\rho) < \infty$. This is easily confirmed, as the isomorphism in (19) yields

$$A_R^{(\ell)}(\rho) = (\rho R'(A_R(\rho)))^{\ell} A_R(\rho) = (\rho R'(y))^{\ell} y = (\rho B''(y) e^{B'(y)})^{\ell} y.$$

By Lemmas 3.2 and 3.8, we have that $B'(y), B''(y) < \infty$, i.e. $A_R^{(\ell)}(\rho)$ is finite. We infer that for any pointed enriched tree (A, u) from $\mathcal{A}_{\mathcal{R}}^{(\ell)}$ with k vertices

$$\mathbb{P}(\Gamma A_R^{(\ell)}(\rho) = (A, u)) = \rho^k / A_R^{(\ell)}(\rho) = \rho^k (\rho R'(y))^{-\ell} / y.$$

Moreover, letting $\Gamma A_R(\rho)$ denote a Boltzmann sampler for $\mathcal{A}_{\mathcal{R}}$, we have that

$$\mathbb{P}(\Gamma A_R(\rho) = A) = \rho^k / y$$

and hence

$$\mathbb{P}(\Gamma A_R^{(\ell)}(\rho) = (A, u)) = (\rho R'(y))^{-\ell} \mathbb{P}(\Gamma A_R(\rho) = A). \tag{20}$$

Thus, in order to prove (17), it suffices to establish that the size-biased random tree $\mathsf{A}_R^{(\ell)}$ follows a Boltzmann distribution with parameter ρ. To see that this is indeed the case, we apply the rules in Table 2 to the isomorphism in (18) in order to obtain the following sampling procedure for $\Gamma A_R^{(\ell)}(\rho)$:

1. If $\ell = 0$, then return $\Gamma A_R(\rho)$. Otherwise, proceed with the following steps.
2. Make independent calls to Boltzmann samplers $\Gamma X(\rho)$, $\Gamma R'(A_R)(\rho)$, and $\Gamma A_R^{(\ell-1)}(\rho)$.
3. Apply the isomorphism in (18) to the result in order to obtain an $\mathcal{A}_{\mathcal{R}}^{(\ell)}$-structure.
4. Relabel uniformly at random.

The rule for the composition in Table 2 yields the following description for $\Gamma R'(A_R)(\rho)$:

(a) Call $\Gamma R'(y)$ with $y = A_R(\rho)$. Let R' denote the result.
(b) For each non-$*$-label v of R' call $\Gamma A_R(\rho)$ and let A_v denote the result.
(c) Relabel $(R', (A_v)_v)$ uniformly at random.

Since everything is labeled uniformly at random in step 4, we may skip step (c) each time the sampler $\Gamma R'(A_R)(\rho)$ is called. In the same way, every time the sampler $\Gamma A_R(\rho) = \Gamma C^\bullet(\rho)$ described in Section 3.6 is called, we may also skip the relabelling at the end.

After completing step 3, there are two kinds of vertices in the resulting enriched tree: normal and special. The offspring of a normal vertex is distributed according to an independent copy of $\Gamma A_R(\rho)$, and each of its children is also normal. The offspring of a special vertex is given by taking an independent copy of $\Gamma R'(y)$ and replacing the $*$-vertex by regular vertex, so that it receives a label afterwards in step 4. (Recall that in the definition of the derivative operator for species we stated that $*$-vertices do not receive labels.) Moreover, the (former) $*$-vertex is also special, except if it is in the ℓth independent copy of $\Gamma R'(y)$. Note that since $\mathcal{R}^\bullet \simeq \mathcal{R}' \cdot \mathcal{X}$, the product rule for the construction of Boltzmann samplers yields that the sampler $\Gamma R^\bullet(y)$ is given by taking $\Gamma R'(y)$, replacing the $*$-vertex by a regular vertex, and relabeling everything uniformly at random. Thus, up to relabeling, the offspring of special vertices is distributed according to an independent copy of $\Gamma R^\bullet(y)$. We have thus identified the size-biased enriched tree $\mathsf{A}_R^{(\ell)}$ as a Boltzmann sampler $\Gamma A_R^{(\ell)}(\rho)$ for the class $\mathcal{A}_{\mathcal{R}}^{(\ell)}$. □

4.2. *Uniform random graphs via size-biased $\mathcal{R}$-enriched trees.* In this section, we show a lemma that will enable us to study random graphs from block-stable classes through the size-biased $\mathcal{R}$-enriched tree. We begin with a simple observation. Note that the $\mathcal{R}$-objects along the spine of $\mathsf{A}_{\mathcal{R}}^{(\ell)}$ are drawn according to ℓ independent copies of $\Gamma R^\bullet(y)$. In the setting of a block-stable class of connected graphs $\mathcal{C}$, we have that $\mathcal{R} = \mathrm{SET} \circ \mathcal{B}'$, where $\mathcal{B} \neq 0$ denotes the subclass of blocks of $\mathcal{C}$. Using (12), we obtain

$$\mathcal{R}^\bullet \simeq (\mathrm{SET} \circ \mathcal{B}') \cdot \mathcal{B}'^\bullet$$

and the sampler $\Gamma R^\bullet(y)$ is given by independent calls of $\Gamma(\mathrm{SET} \circ B')(y)$ and $\Gamma B'^\bullet(y)$. Hence, up to relabelling of vertices, the blocks along the spine are drawn according to ℓ independent copies of $\Gamma B'^{\bullet}(y)$—this observation will be used in the proof of our main result later.

We are going to apply the following general lemma in Section 5 in order to show that the blocks along sufficiently long paths in random graphs behave asymptotically like the spine of $\mathsf{A}_{\mathcal{R}}^{(\ell)}$ for a corresponding integer ℓ.

LEMMA 4.2. *Let $\mathcal{E}$ be a property of pointed $\mathcal{R}$-enriched trees (i.e., a subset of $\mathcal{A}_{\mathcal{R}}^{\bullet}$) and let $n \in \mathbb{N}$ be such that $\mathcal{A}_{\mathcal{R}}[n]$ is nonempty. Consider the function*

$$f : \mathcal{A}_{\mathcal{R}}[n] \to \mathbb{R}, \qquad A \mapsto \sum_{v \in [n]} \mathbb{1}_{(A,v) \in \mathcal{E}}$$

counting the number of "admissible" outer roots with respect to $\mathcal{E}$. Let $\mathsf{A}_n \in \mathcal{A}_{\mathcal{R}}[n]$ be drawn uniformly at random. Then $\mathbb{E}[f(\mathsf{A}_n)]$ is given by

$$\mathbb{P}(|\Gamma A_R(\rho)| = n)^{-1} \sum_{\ell=0}^{n-1} (\rho R'(y))^{\ell} \mathbb{P}(\mathsf{A}_{\mathcal{R}}^{(\ell)} \text{ has size } n \text{ and satisfies } \mathcal{E}).$$

PROOF. First, observe that

$$\sum_{v=1}^{n} \mathbb{P}((\mathsf{A}_n, v) \in \mathcal{E}) = \sum_{\ell=0}^{n-1} \sum_{(A,u) \in \mathcal{E} \cap \mathcal{A}_{\mathcal{R}}^{(\ell)}[n]} \mathbb{P}(\mathsf{A}_n = A).$$

By (17), we have for all $(A, u) \in \mathcal{E} \cap \mathcal{A}_{\mathcal{R}}^{(\ell)}[n]$ that

$$\begin{aligned}\mathbb{P}(\Gamma A_R(\rho) = A \mid |\Gamma A_R(\rho)| = n) \\ = (\rho R'(y))^{\ell} \mathbb{P}(\mathsf{A}_{\mathcal{R}}^{(\ell)} = (A, u)) \mathbb{P}(|\Gamma A_R(\rho)| = n)^{-1}.\end{aligned}$$

This proves the claim. □

5. Convergence toward the CRT. Let $\mathcal{C}$ be an analytic block-stable class of connected graphs and $\mathcal{B} \neq 0$ its subclass of all graphs that are 2-connected or a single edge. We let $\rho > 0$ denote the radius of convergence of the exponential generating series $C(z)$ and set $y = C^{\bullet}(\rho)$. As before, we identify $\mathcal{C}^{\bullet}$ with the class $\mathcal{A}_{\mathcal{R}}$ of $\mathcal{R}$-enriched trees with $\mathcal{R} = \mathrm{SET} \circ \mathcal{B}'$. By Proposition 3.6, we know that if we draw an $\mathcal{R}$-enriched tree (T, α) according to the Boltzmann distribution with parameter ρ, then T is distributed like a ξ-Galton–Watson tree with $\xi := |\Gamma(\mathrm{SET} \circ \mathcal{B}')(y)|$, relabelling uniformly at random and discarding the ordering on the offspring sets.

Throughout this section, let $n \equiv 1 \bmod \mathsf{span}(\xi)$ denote a large enough integer such that the probability of a ξ-GWT having size n is positive. Let $\mathsf{C}_n \in \mathcal{C}_n$ be drawn uniformly at random and generate $\mathsf{C}_n^{\bullet} \in \mathcal{C}_n^{\bullet}$ by uniformly choosing a root from $[n]$. We let (T_n, α_n) be the corresponding enriched tree.

For any pointed derived block $B \in \mathcal{B}'^{\bullet}$, we let $\mathsf{sp}(B) := d_B(*, \text{root})$ denote the length of a shortest path connecting the $*$-vertex with the root. In this section, we prove our main result.

THEOREM 5.1. *Let $\mathcal{C}$ be a subcritical class of connected graphs. Then*

$$\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{C}_n^{\bullet} \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}} \quad \text{and} \quad \frac{\sigma}{2\kappa\sqrt{n}}\mathsf{C}_n \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}}$$

with respect to the (pointed) Gromov–Hausdorff metric. The constants are given by $\sigma^2 = \mathbb{E}[|\mathsf{B}|]$ and $\kappa = \mathbb{E}[|\mathsf{sp}(\mathsf{B})|]$ with $\mathsf{B} \in \mathcal{B}'^{\bullet}$ a random block drawn according to the Boltzmann distribution with parameter $y = \mathcal{C}^{\bullet}(\rho)$, and in particular $\sigma^2 = 1 + B'''(y)y^2$.

As a consequence, we obtain the limit distributions for the height and diameter of $\mathsf{C}_n^{\bullet}$.

COROLLARY 5.2. *Let $\mathcal{C}$ be a subcritical class of connected graphs. Then the rescaled height $\frac{\sigma}{2\kappa\sqrt{n}}\mathrm{H}(\mathsf{C}_n^{\bullet})$ and diameter $\frac{\sigma}{2\kappa\sqrt{n}}\mathrm{D}(\mathsf{C}_n)$ converge in distribution to $\mathrm{H}(\mathcal{T}_{\mathsf{e}})$ and $\mathrm{D}(\mathcal{T}_{\mathsf{e}})$, i.e. for all $x > 0$, as n tends to infinity*

$$\mathbb{P}\left(\mathrm{H}(\mathsf{C}_n^{\bullet}) > \frac{2\kappa\sqrt{n}}{\sigma}x\right) \to 2\sum_{k=1}^{\infty}(4k^2x^2 - 1)\exp(-2k^2x^2),$$

$$\mathbb{P}\left(\mathrm{D}(\mathsf{C}_n) > \frac{2\kappa\sqrt{n}}{\sigma}x\right) \to \sum_{k=1}^{\infty}(k^2 - 1)\left(\frac{2}{3}k^4x^4 - 4k^2x^2 + 2\right)\exp(-k^2x^2/2).$$

Moreover, all moments converge as well. In particular,

$$\mathbb{E}[\mathrm{D}(\mathsf{C}_n)] \sim \frac{2^{5/2}\kappa}{3\sigma}\sqrt{\pi n} \sim \frac{4}{3}\mathbb{E}[\mathrm{H}(\mathsf{C}_n^{\bullet})].$$

Expressions for arbitrarily high moments are given in (7) and (9).

PROOF. The limiting distributions are given in (6) and (8). In order to show convergence of the moments, we argue that the rescaled height and diameter are bounded in the space L^p for all $1 < p < \infty$. This follows for example from the sub-Gaussian tail-bounds of Theorem 6.1 given in Section 6 below (note that the proof of Theorem 6.1 does not depend on the results in this section). $\square$

In the following, we are going to prove Theorem 5.1. The idea is to show that the pointed Gromov–Hausdorff distance of $\mathsf{C}_n^{\bullet}$ and $\kappa\mathsf{T}_n$ is small with a probability that tends to 1 as n becomes large and use the convergence of T_n toward a multiple of the CRT $\mathcal{T}_{\mathsf{e}}$.

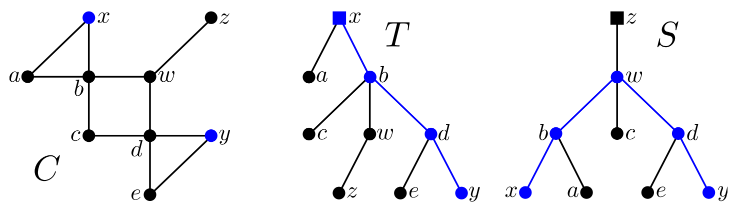

FIG. 6. *The trees T and S correspond to the rooted graphs (C, x) and (C, z).*

DEFINITION 5.3. Let $C \in \mathcal{C}$. For any $x, y \in V(C)$ set $\bar{d}_C(x, y) := d_T(x, y)$ with (T, α) the enriched tree corresponding to (C, x), that is, C rooted at the vertex x.

Less formally speaking, $\bar{d}_C(x, y)$ denotes the minimum number of blocks required to cover the edges of a shortest path linking x and y. As the example illustrated in Figure 6 shows, the distance between x and y in the tree corresponding to a root $z \neq x, y$ might differ from $\bar{d}_C(x, y)$. The following lemma ensures that this difference is bounded.

LEMMA 5.4. *Let $C \in \mathcal{C}$ be a connected graph and x, y, z vertices of C. Let S be the tree corresponding to the graph C rooted at z. Then*

$$\bar{d}_C(x, y) \le d_S(x, y) \le \bar{d}_C(x, y) + 1.$$

Moreover, $\bar{d}_C$ is a metric on the vertex set $V(C)$.

PROOF. Suppose that $x \neq y$ are two distinct vertices of the graph C. Let T denote the tree corresponding to the rooted graph (C, x) and let P denote the unique path in T that joins x and y. Then $\bar{d}_C(x, y)$ equals the length $\ell(P)$ of P. Similarly, let S denote the tree corresponding to the rooted graph (C, z) and let Q denote the unique path in T that joins x and y. Again, the length $\ell(Q)$ equals the distance $d_S(x, y)$.

The path P consists of precisely the vertices of C whose removal would separate x from y. Any such vertex must clearly also lie on the path Q. Conversely, if Q contains a vertex u that does not lie on P, then the removal of u would not separate x from y. This may only happen if u is equal to the lowest common ancestor w of x and y in the tree S. See Figure 6 for an example of this case. Hence, the lengths of the paths P and Q satisfy

$$\ell(P) \le \ell(Q) \le \ell(P) + 1,$$

and equality holds if and only if w lies on P. Thus,

$$d_S(x, y) = \bar{d}_C(x, y) + \mathbb{1}_{\{w \notin P\}}.$$

The case $z = y$ yields that $\bar{d}_C$ is symmetric. The triangle inequality follows from this fact and

$$\bar{d}_C(x, y) \le d_S(x, y) \le d_S(x, z) + d_S(z, y) = \bar{d}_C(z, x) + \bar{d}_C(z, y).$$

Clearly, $\bar{d}_C$ is also reflexive and hence a metric. $\square$

In the following lemma, which is the most important ingredient in the proof of Theorem 5.1, we apply the results on pointed enriched trees of Section 4.

LEMMA 5.5. *Let $\mathcal{C}$ be a subcritical class of connected graphs and set $\kappa = \mathbb{E}[\mathsf{sp}(\Gamma B'^{\bullet}(y))]$. Then for all $s > 1$ and $0 < \varepsilon < 1/2$ with $2\varepsilon s > 1$ we have with a probability that tends to 1 as n becomes large that all $x, y \in V(\mathsf{C}_n)$ with $\bar{d}_{\mathsf{C}_n}(x, y) \ge \log^s(n)$ satisfy*

$$\left|d_{\mathsf{C}_n}(x, y) - \kappa \bar{d}_{\mathsf{C}_n}(x, y)\right| \le \bar{d}_{\mathsf{C}_n}(x, y)^{1/2+\varepsilon}.$$

PROOF. We denote $L_n = \log^s(n)$ and $t_\ell = \ell^{1/2+\varepsilon}$. Let $\mathcal{E} \subset \mathcal{A}_{\mathcal{R}}^{\bullet} \simeq \mathcal{C}^{\bullet\bullet}$ with $\mathcal{R} = \mathrm{SET} \circ \mathcal{B}'$ denote the set of all bipointed graphs or pointed enriched trees $((C, x), y) \simeq ((T, \alpha), y)$, where we call x the *inner root* and y the *outer root*, such that

$$d_T(x, y) \ge L_{|T|} \quad \text{and} \quad \left|d_C(x, y) - \kappa d_T(x, y)\right| > t_{d_T(x,y)}.$$

We will bound the probability that there exist vertices x and y with $((\mathsf{C}_n, x), y) \in \mathcal{E}$. First, observe that

$$\sum_{x,y\in[n]} \mathbb{P}\big(((\mathsf{C}_n, x), y) \in \mathcal{E}\big) = \sum_{((C,x),y)\in\mathcal{E}} \mathbb{P}(\mathsf{C}_n = C) = n \sum_{y=1}^{n} \mathbb{P}\big((\mathsf{C}_n^{\bullet}, y) \in \mathcal{E}\big).$$

By assumption, we may apply Corollary 3.9 to obtain $\mathbb{P}(|\Gamma C^{\bullet}(\rho)| = n) = \Theta(n^{-3/2})$. Moreover, Lemma 3.8 asserts that $B'^{\bullet}(y) = 1$ and thus, with Lemma 3.2

$$\rho R'(y) = \rho B''(y) e^{B'(y)} = y B''(y) = 1.$$

Hence, by applying Lemma 4.2 we obtain that

$$\begin{aligned} &\mathbb{P}\big(((\mathsf{C}_n, x), y) \in \mathcal{E} \text{ for some } x, y\big) \\ &\quad \le O\big(n^{5/2}\big) \sum_{\ell=L_n}^{n-1} \mathbb{P}\big(\mathsf{A}_{\mathcal{R}}^{(\ell)} \text{ has size } n \text{ and satisfies } \mathcal{E}\big). \end{aligned}$$

The height of the outer root in $\mathsf{A}_{\mathcal{R}}^{(\ell)}$ is distributed like the sum of ℓ independent random variables, each distributed like the distance of the $*$-vertex and the root in the corresponding derived block of $\Gamma(\mathrm{SET} \circ B')^{\bullet}(y)$. Since $(\mathrm{SET} \circ \mathcal{B}')^{\bullet} \simeq (\mathrm{SET} \circ \mathcal{B}') \cdot \mathcal{B}'^{\bullet}$, these variables are actually $\mathsf{sp}(\Gamma B'^{\bullet}(y))$-distributed. Hence,

$$\mathbb{P}\big(\mathsf{A}_{\mathcal{R}}^{(\ell)} \in \mathcal{E}, \big|\mathsf{A}_{\mathcal{R}}^{(\ell)}\big| = n\big) \le \mathbb{P}\big(\left|\eta_1 + \cdots + \eta_\ell - \ell \mathbb{E}[\eta_1]\right| > t_\ell\big)$$

with $(\eta_i)_i$ i.i.d. copies of $\eta := \mathsf{sp}(\Gamma B'^{\bullet}(y))$. Clearly, we have that $\eta \le |\Gamma B'^{\bullet}(y)|$. Since $\mathcal{C}$ is subcritical it follows that there is a constant $\delta > 0$ such that $\mathbb{E}[e^{\eta\theta}] < \infty$ for all θ with $|\theta| \le \delta$. Hence, we may apply the standard moderate deviation inequality for one-dimensional random walk stated in Lemma 3.10 to obtain for some constant $c > 0$

$$\mathbb{P}\big(((\mathsf{C}_n, x), y) \in \mathcal{E} \text{ for some } x, y\big) \le O\big(n^{7/2}\big)\exp\big(-c(\log n)^{2s\varepsilon}\big) = o(1). \qquad \square$$

It remains to clarify what happens if $\bar{d}_{\mathsf{C}_n}$ is small. We prove the following statement for random graphs from block-stable classes that are not necessarily subcritical.

PROPOSITION 5.6. *Let $\mathcal{C}$ be a block-stable class of connected graphs. Suppose that $B'^{\bullet}(y) = 1$ and the offspring distribution ξ has finite second moment, that is, $B'''(y) < \infty$. Let $\mathsf{lb}(\mathsf{C}_n)$ denote the size of the largest block in C_n. Then*

1. *For any $x, y \in \mathsf{C}_n$ we have $d_{\mathsf{C}_n}(x, y) \le \bar{d}_{\mathsf{C}_n}(x, y)\mathsf{lb}(\mathsf{C}_n)$.*
2. *If the offspring distribution ξ is bounded, then so is $\mathsf{lb}(\mathsf{C}_n)$. Otherwise, for any sequence K_n we have $\mathbb{P}(\mathsf{lb}(\mathsf{C}_n) \ge K_n) = O(n)\mathbb{P}(\xi \ge K_n)$.*

PROOF. We have that $d_{\mathsf{C}_n} \le \bar{d}_{\mathsf{C}_n}(\mathsf{lb}(\mathsf{C}_n) - 1)$ and $\mathsf{lb}(\mathsf{C}_n) = \mathsf{lb}(\mathsf{C}_n^{\bullet}) \le \Delta(\mathsf{T}_n) + 1$ with $\Delta(\mathsf{T}_n)$ denoting the largest out-degree. Recall that $\Delta(\mathsf{T}_n)$ is distributed like the maximum degree of a ξ-Galton–Watson tree conditioned to have n vertices. By assumption, the offspring distribution ξ has expected value $\mathbb{E}[\xi] = B'^{\bullet}(y) = 1$ and finite variance.

If ξ is bounded, then so is the largest outdegree of T_n. Otherwise, as argued in the proof of [28], equation (19.20), for any sequence K_n

$$\mathbb{P}\big(\Delta(\mathsf{T}_n) \ge K_n\big) \le \big(1 + o(1)\big)n\mathbb{P}(\xi \ge K_n). \tag{21}$$

Applying (21) yields $\mathbb{P}(\mathsf{lb}(\mathsf{C}_n) \ge K_n) \le (1 + o(1))n\mathbb{P}(\xi \ge K_n)$ for any sequence K_n. $\square$

Note that if $\mathcal{C}$ is subcritical then this implies that $\mathsf{lb}(\mathsf{C}_n) = O(\log n)$ with a probability that tends to 1: the definition of the Boltzmann model and the fact that y is smaller than the radius of convergence of $B(z)$ guarantee that there is a constant $\beta < 1$ such that

$$\mathbb{P}(\xi = k) = \mathbb{P}\big(\big|\Gamma\big(\mathrm{SET} \circ B'\big)(y)\big| = k\big) = O\big(\beta^k\big).$$

Combined with the bounds of Lemma 5.5, this yields the following concentration result.

COROLLARY 5.7. *Let $\mathcal{C}$ be a subcritical class of connected graphs. Then for all $s > 1$ and $0 < \varepsilon < 1/2$ with $2\varepsilon s > 1$ we have with a probability that tends to 1 as n becomes large that for all vertices $x, y \in V(\mathsf{C}_n)$*

$$\big|d_{\mathsf{C}_n}(x, y) - \kappa\bar{d}_{\mathsf{C}_n}(x, y)\big| \le \bar{d}_{\mathsf{C}_n}(x, y)^{1/2+\varepsilon} + O\big(\log^{s+1}(n)\big).$$

We may now prove the main theorem.

PROOF OF THEOREM 5.1. Lemma 5.4 implies that $\bar{d}_{\mathsf{C}_n} \le d_{\mathsf{T}_n} \le \bar{d}_{\mathsf{C}_n} + 1$. By Corollary 5.7, Proposition 2.3 and considering the distortion of the identity map as correspondence between the vertices of T_n and $\mathsf{C}_n^\bullet$, it follows that with a probability that tends to 1 as n becomes large

$$d_{\mathrm{GH}}\big(\mathsf{C}_n^\bullet/(\kappa\sqrt{n}), \mathsf{T}_n/\sqrt{n}\big) \le \mathrm{D}(\mathsf{T}_n)^{3/4}/\sqrt{n} + o(1).$$

Using the tail bounds (1) for the diameter $\mathrm{D}(\mathsf{T}_n)$, we obtain that

$$d_{\mathrm{GH}}\big(\mathsf{C}_n^\bullet/(\kappa\sqrt{n}), \mathsf{T}_n/\sqrt{n}\big)$$

converges in probability to zero. Recall that the variance of the offspring distribution ξ is given by $\sigma^2 = \mathbb{E}[|\Gamma B'^\bullet(y)|]$. By Theorem 2.5, we have that $\frac{\sigma}{2\sqrt{n}}\mathsf{T}_n \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}}$, and thus $\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{C}_n^\bullet \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}}$. □

6. Sub-Gaussian tail bounds for the height and diameter. In this section, we prove sub-Gaussian tail bounds for the height and diameter of the random graphs $\mathsf{C}_n^\bullet$ and C_n. Our proof builds on results obtained in [1]. Recall that (T_n, α_n) denotes the enriched tree corresponding to the graph $\mathsf{C}_n^\bullet$ and that T_n has a natural coupling with a ξ-Galton–Watson conditioned on having size n, see Proposition 3.6. With (slight) abuse of notation, we also write T_n for the conditioned ξ-Galton–Watson tree within this section. We prove the following statement for random graphs from block-stable classes that are not necessarily subcritical.

THEOREM 6.1. *Let $\mathcal{C}$ be a block-stable class of connected graphs. Suppose that $\mathcal{C}$ satisfies $B'^\bullet(y) = 1$ and the offspring distribution ξ has finite variance, i.e. $B'''(y) < 1$. Then there are $C, c > 0$ such that for all $n, x \ge 0$*

$$\mathbb{P}\big(\mathrm{D}(\mathsf{C}_n) \ge x\big) \le C\exp\big(-cx^2/n\big) \quad \textit{and} \quad \mathbb{P}\big(\mathrm{H}(\mathsf{C}_n^\bullet) \ge x\big) \le C\exp\big(-cx^2/n\big).$$

As $\mathrm{H}(\mathsf{T}_n) \le \mathrm{H}(\mathsf{C}_n^\bullet)$, inequality (1) also yields a lower tail bound for the height of $\mathsf{C}_n^\bullet$.

COROLLARY 6.2. *There are constants $C, c > 0$ such that for all $x \ge 0$ and n*

$$\mathbb{P}\big(\mathrm{H}(\mathsf{C}_n^\bullet) \le x\big) \le C\exp\big(-c(n-2)/x^2\big).$$

As a main ingredient in our proof, we consider the *lexicographic depth-first-search (DFS)* of the plane tree T_n by labeling the vertices in the usual way (as a subtree of the Ulam–Harris tree) by finite sequences of integers and listing them in lexicographic order $v_0, v_1, \dots, v_{n-1}$. The search keeps a queue of Q_i^d nodes with $Q_0^d = 1$ and the recursion

$$Q_i^d = Q_{i-1}^d - 1 + d_{\mathsf{T}_n}^+(v_{i-1}).$$

The mirror-image of T_n is obtain by reversing the ordering on each offspring set and the *reverse* DFS Q_i^r is defined as the DFS of the mirror-image. Then $(Q_i^d)_{0\le i\le n}$ and $(Q_i^r)_{0\le i\le n}$ are identically distributed and satisfy the following bound given in [1], inequality (4.4):

$$\mathbb{P}\Big(\max_j Q_j^d \ge x\Big) \le C\exp\big(-cx^2/n\big) \tag{22}$$

with $C, c > 0$ denoting some constants that do not depend on x or n.

PROOF OF THEOREM 6.1. Since $\mathrm{D}(\mathsf{C}_n) \le 2\mathrm{H}(\mathsf{C}_n^\bullet)$, it suffices to show the bound for the height. Let $h \ge 0$. If $\mathrm{H}(\mathsf{C}_n^\bullet) \ge h$, then there exists a vertex whose height equals h. Consequently, we may estimate $\mathbb{P}(\mathrm{H}(\mathsf{C}_n^\bullet) \ge h) \le \mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_2)$ with $\mathcal{E}_1$ (resp., $\mathcal{E}_2$) denoting the event that there is a vertex v such that $h_{\mathsf{C}_n^\bullet}(v) = h$ and $h_{\mathsf{T}_n}(v) \ge h/2$ [resp., $h_{\mathsf{T}_n}(v) \le h/2$]. By the tail bound (1) for the height of Galton–Watson trees, we obtain

$$\mathbb{P}(\mathcal{E}_1) \le \mathbb{P}\big(\mathrm{H}(\mathsf{T}_n) \ge h/2\big) \le C_2\exp\big(-c_2h^2/(4n)\big)$$

for some constants $C_2, c_2 > 0$. In order to bound $\mathbb{P}(\mathcal{E}_2)$, suppose that there is a vertex v with height $h_{\mathsf{C}_n^\bullet}(v) = h$ and $h_{\mathsf{T}_n}(v) \le h/2$. If a is a vertex of T_n and b one of its offspring, then $d_{\mathsf{C}_n^\bullet}(a, b) \le d^+_{\mathsf{T}_n}(a)$. Hence,

$$\sum_{u\succ v} d^+_{\mathsf{T}_n}(u) \ge h_{\mathsf{C}_n^\bullet}(v) = h$$

with the sum index u ranging over all ancestors of v. Consider the lexicographic depth-first-search $(Q_i^d)_i$ and reverse depth-first-search $(Q_i^r)_i$ of T_n. Let j (resp., k) denote the index corresponding to the vertex v in the lexicographic (resp., reverse lexicographic) order. It follows from the definition of the queues that if $\mathcal{E}_2$ occurs

$$Q_j^d + Q_k^r = 2 + \sum_{u\succ v} d^+_{\mathsf{T}_n}(u) - h_{\mathsf{T}_n}(v) \ge h/2$$

and hence $\max(Q_j^d, Q_k^r) \ge h/4$. Since Q_j^d and Q_k^r are identically distributed, by (22)

$$\begin{aligned}\mathbb{P}(\mathcal{E}_2) &\le \mathbb{P}\Big(\max_i\big(Q_i^d\big) \ge h/4\Big) + \mathbb{P}\Big(\max_i\big(Q_i^r\big) \ge h/4\Big)\\ &\le 2\mathbb{P}\Big(\max_i\big(Q_i^d\big) \ge h/4\Big)\\ &\le 2C\exp\big(-ch^2/(16n)\big).\end{aligned}$$

This completes the proof. $\square$

7. Extensions.

In the following, we use the notation from Section 5.

7.1. *First passage percolation.* Let $\omega > 0$ be a given random variable with finite exponential moments, that is, such that there is a $\delta > 0$ with $\mathbb{E}[e^{\theta\omega}] < \infty$ for all θ with $|\theta| \le \delta$. For any connected graph G, we may consider the random graph $\hat{G}$ obtained by assigning to each edge $e \in E(G)$ a weight ω_e that is an independent copy of ω. The $d_{\hat{G}}$-distance of two vertices a and b is then given by

$$d_{\hat{G}}(a,b) = \inf\left\{ \sum_{e \in E(P)} \omega_e \,|\, P \text{ a path connecting } a \text{ and } b \text{ in } G \right\}.$$

We may extend our main result to random graphs with link-weights.

THEOREM 7.1. *Let $\mathcal{C}$ be a subcritical class of connected graphs and $\mathcal{B}$ its subclass of graphs that are 2-connected or a single edge. Let $\mathsf{C}_n \in \mathcal{C}_n$ and $\mathsf{C}_n^\bullet \in \mathcal{C}_n^\bullet$ denote the uniform (rooted) random graphs. Form the link-weighted versions $\hat{\mathsf{C}}_n$ and $\hat{\mathsf{C}}_n^\bullet$ by assigning to each edge an independent copy of a random variable $\omega > 0$ having finite exponential moments. Then*

$$\frac{\sigma}{2\hat{\kappa}\sqrt{n}} \hat{\mathsf{C}}_n^\bullet \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}} \quad \textit{and} \quad \frac{\sigma}{2\hat{\kappa}\sqrt{n}} \hat{\mathsf{C}}_n \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}}$$

with respect to the (pointed) Gromov–Hausdorff metric. The scaling constant $\hat{\kappa}$ is given by $\hat{\kappa} := \mathbb{E}[\mathsf{sp}(\hat{\mathsf{B}})]$ with B drawn according to the Boltzmann sampler $\Gamma B'^\bullet(y)$ and $\mathsf{sp}(\hat{\mathsf{B}})$ denoting the $d_{\hat{\mathsf{B}}}$-distance from the $*$-vertex to the root vertex.

PROOF. For any n, let K_n denote the complete graph with n vertices. The idea is to generate $\hat{\mathsf{C}}_n$ by drawing C_n and $\hat{K}_n$ independently and assign the weights via the inclusion $E(\mathsf{C}_n) \subset E(K_n)$. By considering subsets $\mathcal{E} \subset \mathcal{C}^{\bullet\bullet} \times \mathbb{R}^{\bigcup_n E(K_n)}$, we may easily prove a weighted version of Lemma 4.2, that is, the probability that the random pair $(\mathsf{C}_n^{\bullet\bullet}, \hat{K}_n)$ has some property $\mathcal{E}$ is bounded by

$$O(n^{5/2}) \sum_{\ell=0}^{n-1} \mathbb{P}\big(\big|\Gamma C^{\bullet(\ell)}(\rho)\big| = n, \big(\Gamma C^{\bullet(\ell)}(\rho), \hat{K}_n\big) \in \mathcal{E}\big).$$

This implies that the blocks along sufficiently long paths in $\hat{\mathsf{C}}_n$ behave like independent copies of the weighted block $\hat{\mathsf{B}}$ with B drawn according to the Boltzmann sampler $\Gamma B'^\bullet(y)$. Hence, weighted versions of Lemma 5.5 and Proposition 5.6 may be deduced analogously to their original proofs with $\hat{\kappa}$ replacing κ and only minor modifications otherwise. Thus, the scaling limit follows in the same fashion. □

7.2. *Random graphs given by their connected components.* We study the case of an arbitrary graph consisting of a set of connected components. Let $\mathcal{G} \simeq \mathrm{SET} \circ \mathcal{C}$ denote a subcritical graph class given by its subclass $\mathcal{C}$ of connected graphs. For simplicity, we are going to assume that all trees belong to the class $\mathcal{C}$.

Consider the uniform random graph $\mathsf{G}_n \in \mathcal{G}_n$. Of course, we cannot expect G_n to converge to the continuum random tree since it may be disconnected with a probability that is bounded away from zero. Instead we study a uniformly chosen component H_n of maximal size. We are going to show that $\mathsf{H}_n/\sqrt{n}$ converges to a multiple of the CRT.

THEOREM 7.2. *Suppose $\mathcal{C}$ is a subcritical class of connected graphs containing all trees. Then $\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{H}_n \xrightarrow{(d)} \mathcal{T}_{\mathsf{e}}$ with respect to the Gromov–Hausdorff metric, where σ, κ are as in Theorem 5.1.*

We are going to use the known fact that with a probability that tends to 1 as n becomes large the random graph G_n has a unique giant component with size $n + O_p(1)$. This follows for example from [39], Theorem 6.4.

LEMMA 7.3. *If $\mathcal{C}$ contains all trees, then the size of a largest component satisfies $|\mathsf{H}_n| = n + O_p(1)$.*

PROOF OF THEOREM 7.2. Let $f : \mathbb{K} \to \mathbb{R}$ be a bounded Lipschitz-continuous function defined on the space of isometry classes of compact metric spaces. We will show that $\mathbb{E}[f(\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{H}_n)] \to \mathbb{E}[f(\mathcal{T}_{\mathsf{e}})]$ as n tends to infinity. Set $\Omega_n := \log n$. By Lemma 7.3, we know that $|\mathsf{H}_n| = n + O_p(1)$. Hence, with a probability that tends to 1 as n becomes large we have $n - |H_n| \leq \Omega_n$, and thus

$$\begin{aligned}
&\mathbb{E}\left[f\left(\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{H}_n\right)\right] \\
&\qquad = o(1) + \sum_{0\leq k\leq \Omega_n} \mathbb{E}\left[f\left(\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{H}_n\right)\Big| |\mathsf{H}_n| = n-k\right]\mathbb{P}(|\mathsf{H}_n| = n-k).
\end{aligned}$$

The conditional distribution of G_n given the sizes $(s_i)_i$ of its components is given by choosing components $K_i \in \mathcal{C}[s_i]$ independently uniformly at random and distributing labels uniformly at random. In particular, as a metric space, H_n conditioned on $|\mathsf{H}_n| = n - k$ is distributed like the uniform random graph C_{n-k}. Thus, given $\varepsilon > 0$ we have for n sufficiently large by Lipschitz-continuity

$$\mathbb{E}\left[f\left(\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{H}_n\right)| |\mathsf{H}_n| = n-k\right] = \mathbb{E}\left[f\left(\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{C}_{n-k}\right)\right] \in \mathbb{E}[f(\mathcal{T}_{\mathsf{e}})] \pm \varepsilon$$

for all $0 \leq k \leq \Omega_n$. Thus, $|\mathbb{E}[f(\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{H}_n)] - \mathbb{E}[f(\mathcal{T}_{\mathsf{e}})]| \leq \varepsilon$ for sufficiently large n. Since $\varepsilon > 0$ was chosen arbitrarily it follows that $\mathbb{E}[f(\frac{\sigma}{2\kappa\sqrt{n}}\mathsf{H}_n)] \to \mathbb{E}[f(\mathcal{T}_{\mathsf{e}})]$ as n tends to infinity. □

8. The scaling factor of specific classes. In this section, we apply our main results to several specific examples of subcritical graph classes. The notation that will be fixed throughout this section is as follows: $\mathcal{C}$ denotes a subcritical class of connected graphs and $\mathcal{B}$ its subclass of 2-connected graphs and edges. The radius of convergence of $C(z)$ is denoted by ρ. The constant $y = \mathcal{C}^\bullet(\rho)$ is the unique positive solution of the equation

$$yB''(y) = 1.$$

By Lemma 3.2, this determines $\rho = y\exp(-B'(y))$. Moreover, we set

$$\kappa = \mathbb{E}\big[\mathsf{sp}\big(\Gamma B'^\bullet(y)\big)\big],$$

that is, the expected distance from the $*$-vertex to the root in a random block chosen according to the Boltzmann distribution with parameter y. We call κ the *scaling factor* for $\mathcal{C}$. The offspring distribution ξ of the random tree corresponding to the sampler $\Gamma C^\bullet(y)$ has probability generating function $\varphi(z) = \exp(B'(yz) - \lambda)$ with $\lambda = B'(y)$; see Proposition 3.6. Its variance is given by

$$\sigma^2 = 1 + B'''(y)y^2 = \mathbb{E}\big[\big|\Gamma B'^\bullet(y)\big|\big].$$

We let d denote the span of the offspring distribution. By applying Corollary 5.2 we obtain

$$\mathbb{E}\big[\mathsf{H}\big(\mathsf{C}_n^\bullet\big)\big]/\sqrt{n} \to \kappa\sqrt{2\pi/\sigma^2} =: H \qquad \text{as } n \to \infty \text{ with } n \equiv 1 \bmod d$$

with $\mathsf{C}_n^\bullet \in \mathcal{C}_n^\bullet$ drawn uniformly at random. We call H the *expected rescaled height*. Moreover, Corollary 3.9 yields that

$$|\mathcal{C}_n| \sim cn^{-5/2}\rho_C^{-n}n! \qquad \text{as } n \to \infty \text{ with } n \equiv 1 \bmod d$$

with $c = yd/\sqrt{2\pi\sigma^2}$. In this section, we derive analytical expressions for the relevant constants $\kappa, H, c, \rho, y, \lambda, \sigma^2$ for several graph classes; Table 3 provides numerical approximations. For a set of graphs M, we denote by $\mathsf{Forb}(M)$ the class of all connected graphs that contain none of the graphs in M as a topological minor; if M contains only 2-connected graphs, then it is easy to see that $\mathsf{Forb}(M)$ is block-stable, cf. [17]. For $n \geq 3$, we denote by C_n a graph that is isomorphic to a cycle with n vertices.

8.1. *Trees.* Let $\mathcal{C}$ be the class of trees, that is, $\mathcal{B}$ consists only of the graph K_2. It is easy to see that the offspring distribution follows a Poisson distribution with parameter one. We immediately obtain the following.

PROPOSITION 8.1. *For the class of trees, we have $\kappa = 1$ and $\sigma^2 = 1$.*

TABLE 3
Numerical approximations of constants for examples of subcritical classes of connected graphs

Graph class	κ	H	c	ρ	y	λ	σ^2
Trees = Forb(C_3)	1	2.50662	0.39894	0.36787	1	1	1
Forb(C_4)	1	2.13226	0.20973	0.23618	0.27520	0.80901	1.38196
Forb(C_5)	1.10355	1.88657	0.10987	0.06290	0.40384	1.85945	2.14989
Cacti Graphs	1.20297	1.99021	0.12014	0.23874	0.45631	0.64779	2.29559
Outerplanar Graphs	5.08418	1.30501	0.00697	0.13659	0.17076	0.22327	95.3658

8.2. Forb(C_4). Let $\mathcal{C}$ denote the connected graphs of the class Forb(C_4). Then each block is either isomorphic to K_2 or K_3. Hence, $B(z) = z^2/2 + z^3/6$. Moreover, for any $B \in \mathcal{B}$ and any two distinct vertices in B their distance is one. A simple computation then yields the following.

PROPOSITION 8.2. *For the class* Forb(C_4), *we have* $\kappa = 1$ *and* $\sigma^2 = (5 - \sqrt{5})/2$.

8.3. Forb(C_5). Recall that the class Forb(C_5) consists of all graphs that do not contain a cycle with five vertices as a topological minor. Hence, a graph belongs to this class if and only if it contains no cycle of length at least five as subgraph.

PROPOSITION 8.3. *For the class* Forb(C_5), *the constant y is the unique positive solution to $zB''(z) = 1$, where B' is given in* (25). *Moreover, we have*

$$\kappa = (2y^2 + 4y + 3)ye^y - (3y^2 + 12y + 4)y/2 \approx 1.10355$$

and $\sigma^2 = 1 + B'''(y)y^2 \approx 2.14989$.

Before proving Proposition 8.3, we identify the unlabeled blocks of this class. A similar result is given in [25]; we include a short proof for completeness.

PROPOSITION 8.4. *The unlabeled blocks of the class* Forb(C_5) *are given by*

$$K_2, K_4, (K_{2,m})_{m\geq 1}, (K_{2,m}^+)_{m\geq 2}. \tag{23}$$

Here, K_n denotes the complete graph and $K_{m,n}$ the complete bipartite graph with bipartition $\{[m], [n+m] \setminus [m]\}$. The graph $K_{2,n}^+$ is obtained from $K_{2,n}$ by adding an additional edge between the two vertices from $[m] = [2]$.

PROOF. We may verify (23) by considering the standard decomposition of 2-connected graphs: an arbitrary graph G is 2-connected if and only if it can be constructed from a cycle by adding H-paths to already constructed graphs H [17]. If $G \in$ Forb(C_5), then so do all the graphs along its decomposition. In particular we must start with a triangle or a square. Since every edge of a 2-connected graph lies on a cycle, we may only add paths of length at most two in each step. In particular, for $m \geq 3$ a $K_{2,m}$ may only become a $K_{2,m}^+$ or $K_{2,m+1}$, and a $K_{2,m}^+$ may only become a $K_{2,m+1}^+$. Thus, (23) follows by induction on the number of vertices. □

PROOF OF PROPOSITION 8.3. With foresight, we use the decomposition

$$\mathcal{B} = \mathcal{S} + \mathcal{H} + \mathcal{P} \tag{24}$$

with the classes of labeled graphs $\mathcal{S}$, $\mathcal{H}$ and $\mathcal{P}$ defined by their sets of unlabeled graphs $\tilde{\mathcal{S}} = \{K_2, K_3, K_4, C_4\}$, $\tilde{\mathcal{H}} = \{K_{2,m} | m \geq 3\}$ and $\tilde{\mathcal{P}} = \{K_{2,m}^+ | m \geq 2\}$. Any unlabeled graph from $\mathcal{H}$ or $\mathcal{P}$ with n vertices has exactly $\binom{n}{2}$ different labelings, since any labeling is determined by the choice of the two unique vertices with degree at least three. Hence,

$$S(x) = x^2/2 + x^3/6 + x^4/6, \qquad H(x) = \sum_{n \geq 5} \binom{n}{2} \frac{x^n}{n!} \quad \text{and}$$

$$P(x) = \sum_{n \geq 4} \binom{n}{2} \frac{x^n}{n!}$$

and thus

$$B'(x) = x(x+2)e^x - x(15x + 2x^2 + 6)/6. \tag{25}$$

Solving the equation $B'^{\bullet}(y) = 1$ yields

$$y \approx 0.40384.$$

First, let $\mathsf{H}_n \in \mathcal{H}_n'^{\bullet}$ with $n \geq 4$ be drawn uniformly at random. We say that a vertex lies on the left if it has degree at least three, otherwise we say it lies on the right. There are $n\binom{n+1}{2}$ graphs in the class $\mathcal{H}_n'^{\bullet}$ and precisely n^2 of those have the property that the $*$-vertex lies on the left. The distance of the root and the $*$-vertex equals two if they lie on the same side and one otherwise. Hence,

$$\mathbb{E}[\mathsf{sp}(\mathsf{H}_n)] = \frac{n}{\binom{n+1}{2}}\left(\frac{1}{n} \cdot 2 + \frac{n-1}{n} \cdot 1\right) + \left(1 - \frac{n}{\binom{n+1}{2}}\right)\left(\frac{2}{n} \cdot 1 + \frac{n-2}{n} \cdot 2\right).$$

Let $\mathsf{P}_n \in \mathcal{P}_n'^{\bullet}$ with $n \geq 3$ and $\mathsf{S}_n \in \mathcal{S}_n$ with $n = 1, 2, 3$ be drawn uniformly at random. Analogously to the above calculation, we obtain

$$\mathbb{E}[\mathsf{sp}(\mathsf{P}_n)] = \frac{n}{\binom{n+1}{2}} 1 + \left(1 - \frac{n}{\binom{n+1}{2}}\right)\left(\frac{2}{n} \cdot 1 + \frac{n-2}{n} \cdot 2\right)$$

and

$$\mathbb{E}[\mathsf{sp}(\mathsf{S}_1)] = \mathbb{E}[\mathsf{sp}(\mathsf{S}_2)] = 1, \qquad \mathbb{E}[\mathsf{sp}(\mathsf{S}_3)] = \tfrac{1}{4} \cdot 1 + \tfrac{3}{4}(\tfrac{2}{3} \cdot 1 + \tfrac{1}{3} \cdot 2) = \tfrac{5}{4}.$$

Since $B'^{\bullet}(y) = 1$, we have for any class $\mathcal{F} \in \{\mathcal{S}'^{\bullet}, \mathcal{H}'^{\bullet}, \mathcal{P}'^{\bullet}\}$ that

$$\mathbb{E}[\mathsf{sp}(\Gamma B'^{\bullet}(y)), \Gamma B'^{\bullet}(y) \in \mathcal{F}] = \sum_n ([z^n]F(yz))\mathbb{E}[\mathsf{sp}(\mathsf{F}_n)].$$

Summing up yields

$$\mathbb{E}[\mathsf{sp}(\Gamma B'^{\bullet}(y))] = (2y^2 + 4y + 3)ye^y - (3y^2 + 12y + 4)y/2 \approx 1.10355. \quad \square$$

8.4. *Cacti graphs*. A cactus graph is a graph in which each edge is contained in at most one cycle. Equivalently, the class of cacti graphs is the block-stable class of graphs where every block is either an edge or a cycle. In the following, $\mathcal{C}$ denotes the class of cacti graphs.

PROPOSITION 8.5. *For the class of cacti graphs, the constant y is the unique positive solution to $zB''(z) = 1$, where B' is given in* (26). *Moreover, we have*

$$\kappa = \frac{y^4 - 2y^3 + 2y - 2}{(y^2 - 2y + 2)(1 + y)(y - 1)} \approx 1.20297$$

and $\sigma^2 = 1 + B'''(y)y^2 \approx 2.29559$.

PROOF. By counting the number of labelings of a cycle, we obtain $|\mathcal{B}'_n| = n!/2$ for $n \geq 2$. It follows that

$$B'(z) = z + \frac{z^2}{2(1 - z)} \tag{26}$$

and hence $B'^{\bullet}(z) = z + \frac{1}{2}\sum_{n \geq 2} nz^n = \frac{z^3 - 2z^2 + 2z}{2(z-1)^2}$. Solving the equation $B'^{\bullet}(y) = 1$ yields

$$y = -\tfrac{1}{3}(17 + 3\sqrt{33})^{1/3} + \tfrac{2}{3}(17 + 3\sqrt{33})^{-1/3} + \tfrac{4}{3} \approx 0.45631.$$

Let $\Gamma B'^{\bullet}(y)$ denote a Boltzmann-sampler for the class $\mathcal{B}'^{\bullet}$ with parameter y and for any $n \geq 1$ let $\mathsf{B}_n \in \mathcal{B}'^{\bullet}_n$ be drawn uniformly at random. Since $B'^{\bullet}(y) = 1$, it follows that

$$\begin{aligned}\kappa &= \mathbb{E}[\mathbb{E}[\mathsf{sp}(\Gamma B'^{\bullet}(y))||\Gamma B'^{\bullet}(y)|]] \\ &= \sum_{n \geq 1} \mathsf{sp}(\mathsf{B}_n)[z^n]B'^{\bullet}(yz) = \mathsf{sp}(B_1)y + \frac{1}{2}\sum_{n \geq 2} \mathsf{sp}(\mathsf{B}_n)ny^n.\end{aligned}$$

Clearly, $\mathsf{sp}(\mathsf{B}_1) = 1$ and for $n \geq 2$ we have that $\mathsf{sp}(\mathsf{B}_n)$ is distributed like the distance from the $*$-vertex to a uniformly at random chosen root from $[n]$ in the cycle $(*, 1, 2, \ldots, n)$. Hence,

$$\mathsf{sp}(\mathsf{B}_n) = \begin{cases} \dfrac{2}{n}\displaystyle\sum_{i=1}^{n/2} i = \dfrac{n+2}{4}, & n \text{ is even}, \\[2ex] \dfrac{n+1}{2n} + \dfrac{2}{n}\displaystyle\sum_{i=1}^{(n-1)/2} i = \dfrac{(n+1)^2}{4n}, & n \text{ is odd}. \end{cases}$$

Summing up over all possible values of n yields the claimed value of κ. □

8.5. *Outer-planar graphs*. An outer-planar graph is a planar graph that can be embedded in the plane in such a way that every vertex lies on the boundary of the outer face. Any such embedding (considered up to continuous deformation) is termed an outer-planar map. The scaling limit of the model "all outer-planar maps with n vertices equally likely" was studied by Caraceni [13], who established convergence to the CRT using a bijection by Bonichon, Gavoille and Hanusse [11]. Our results allow us to study the model "all outer-planar graphs with n vertices equally likely," which is a different setting. Note also that the scaling factor obtained in the following differs from the one established for outer-planar maps.

Let $\mathcal{C}$ denote the class of connected outer-planar graphs and $\mathcal{B}$ the subclass consisting of single edges or 2-connected outer-planar graphs.

PROPOSITION 8.6. *For the class of outer-planar graphs the constant y is the unique positive solution to $zB''(z) = 1$, where $B'(z) = (z + D(z))/2$ and D is given in* (27). *Moreover,*

$$\kappa = \frac{y}{2} + \left(1 - \frac{y}{2}\right)\frac{8w^4 - 16w^3 + 4w - 1}{(4w^3 - 6w^2 - 2w + 1)(2w - 1)} \approx 5.0841$$

with $w = D(y)$ and $\sigma^2 = 1 + B'''(y)y^2 \approx 95.3658$.

Following [8], we develop a specification of $\mathcal{B}'^{\bullet}$ that eventually will enable us to prove the above expressions of the relevant constants. Any 2-connected outer-planar graph has a unique Hamilton cycle, which corresponds to the boundary of the outer face in any drawing having the property that all vertices lie on the outer face. The edge set of a 2-connected outer-planar graph can thus be partitioned in two parts: the edges of the Hamilton cycle, and all other edges, which we refer to as the set of *chords*. Let $\mathcal{D}$ denote the class obtained from $\mathcal{B}'$ by orienting the Hamilton cycle of each object of size at least three in one of the two directions and marking the oriented edge whose tail is the $*$-vertex. The block consisting of a single edge is oriented in the unique way such that the $*$-vertex is the tail of the marked edge. We start with some observations.

LEMMA 8.7. *We have that $B'(z) = (z + D(z))/2$ and*

$$\mathbb{E}[\mathsf{sp}(\Gamma B'^{\bullet}(x))] = \frac{x}{2B'^{\bullet}(x)} + \left(1 - \frac{x}{2B'^{\bullet}(x)}\right)\mathbb{E}[\mathsf{sp}(\Gamma D^{\bullet}(x))].$$

PROOF. We have an isomorphism

$$\mathcal{B}' + \mathcal{B}' =: 2\mathcal{B}' \simeq \mathcal{X} + \mathcal{D}.$$

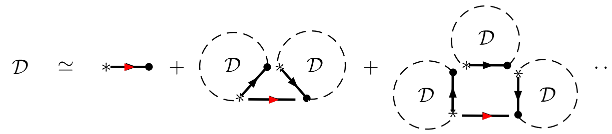

FIG. 7. *Recursive specification of the class $\mathcal{D}$.*

Consequently, the classes $\mathcal{B}'^{\bullet}$ and $\mathcal{D}^{\bullet}$ obtained by additionally rooting at a non-$*$-vertex satisfy

$$2\mathcal{B}'^{\bullet} \simeq \mathcal{X} + D^{\bullet}.$$

Hence, the following procedure is a Boltzmann sampler for the class $\mathcal{B}'^{\bullet}$ with parameter x.

$\Gamma B'^{\bullet}(x)$: $s \leftarrow \mathsf{Bern}(\frac{x}{2B'^{\bullet}(x)})$
if $s = 1$ **then return** a single edge $\{*, 1\}$ rooted at 1
else return $\Gamma D^{\bullet}(x)$ without the orientation

This completes the proof. □

Hence, it suffices to study the class $\mathcal{D}^{\bullet}$; see also Figures 7 and 8.

LEMMA 8.8. *The classes $\mathcal{D}$ and $\mathcal{D}^{\bullet}$ satisfy*

$$\mathcal{D} = \mathcal{X} + \mathcal{D}\times\mathcal{D} + \mathcal{D}\times\mathcal{D}\times\mathcal{D} + \cdots,$$
$$\mathcal{D}^{\bullet} = \mathcal{X} + (\mathcal{D}^{\bullet}\times\mathcal{D} + \mathcal{D}\times\mathcal{D}^{\bullet}) + (\mathcal{D}^{\bullet}\times\mathcal{D}\times\mathcal{D} + \mathcal{D}\times\mathcal{D}^{\bullet}\times\mathcal{D} + \mathcal{D}\times\mathcal{D}\times\mathcal{D}^{\bullet}) + \cdots.$$

Their exponential generating functions are given by

$$D^{\bullet}(z) = \frac{z(D(z)-1)^2}{2(D(z))^2 - 4D(z) + 1}, \tag{27}$$

$$D(z) = \frac{1}{4}\left(1 + z - \sqrt{z^2 - 6z + 1}\right). \tag{28}$$

PROOF. Let $B \in \mathcal{D}$ with $|B| \geq 2$ be a derived outer-planar block, rooted at an oriented edge $\overrightarrow{e}$ of its Hamilton cycle C such that the $*$-vertex is the tail of $\overrightarrow{e}$.

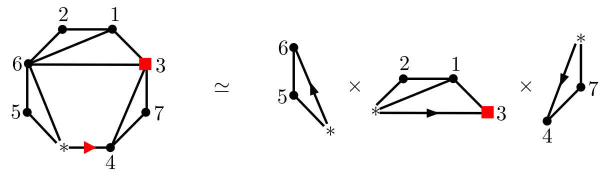

FIG. 8. *Decomposition of a $\mathcal{D}^{\bullet}$-object into a $\mathcal{D}\times\mathcal{D}^{\bullet}\times\mathcal{D}$-object. The root is marked with a square.*

Given a drawing of B such that C is the boundary of the outer face, the root face is defined to be the bounded face F whose border contains $\vec{e}$. Then B may be identified with the sequence of blocks along F, ordered in the reverse direction of the edge $\vec{e}$. This yields the decompositions illustrated in Figures 7 and 8. Solving the corresponding equations of generating functions yields (27). $\square$

The equation determining $y = C^\bullet(\rho)$ is

$$1 = B'^\bullet(y) = \big(y + D^\bullet(y)\big)/2.$$

We obtain that $y \approx 0.17076$ is the unique root of the polynomial

$$3z^4 - 28z^3 + 70z^2 - 58z + 8$$

in the interval [0, 1/2], and hence

$$\sigma^2 = 1 + B'''(y)y^2 \approx 95.3658.$$

It remains to compute κ.

LEMMA 8.9. *We have that*

$$\mathbb{E}[\mathsf{sp}(\Gamma D^\bullet(y))] = \frac{8w^4 - 16w^3 + 4w - 1}{(4w^3 - 6w^2 - 2w + 1)(2w - 1)} \approx 5.46545$$

with $w := D(y) \approx 0.27578$.

Since $B'^\bullet(y) = 1$ this implies with Lemma 8.7 that

$$\kappa = \mathbb{E}[\mathsf{sp}(\Gamma B'^\bullet(y))] = \frac{y}{2} + \left(1 - \frac{y}{2}\right)\mathbb{E}[\mathsf{sp}(\Gamma D^\bullet(y))] \approx 5.08418,$$

and the this completes the proof of Proposition 8.6.

PROOF OF LEMMA 8.9. The rules for Boltzmann samplers translate the specification

$$\begin{aligned}\mathcal{D}^\bullet = \mathcal{X} &+ (\mathcal{D}^\bullet \times \mathcal{D} + \mathcal{D} \times \mathcal{D}^\bullet) \\ &+ (\mathcal{D}^\bullet \times \mathcal{D} \times \mathcal{D} + \mathcal{D} \times \mathcal{D}^\bullet \times \mathcal{D} + \mathcal{D} \times \mathcal{D} \times \mathcal{D}^\bullet) + \cdots\end{aligned}$$

of $\mathcal{D}^\bullet$ given in Lemma 8.8 into the following sampling algorithm.

$\Gamma D^\bullet(x)$: $s \leftarrow$ drawn with $\mathbb{P}(s=2) = \frac{x}{D^\bullet(x)}$ and, for $i \geq 3$,
$\mathbb{P}(s=i) = (i-1)(D(x))^{i-2}$
if $s = 2$ **then**
return a single directed edge $(*, 1)$
else
$\gamma \leftarrow$ a cycle $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_s, v_1\}$ with $v_1 = *$
$t \leftarrow$ a number drawn uniformly at random from the set $[s-1]$
$\gamma \leftarrow$ identify (v_t, v_{t+1}) with the root-edge of $\gamma_t \leftarrow \Gamma D^\bullet(x)$
for each $i \in [s-1] \setminus \{t\}$
$\gamma \leftarrow$ identify (v_i, v_{i+1}) with the root-edge of $\gamma_i \leftarrow \Gamma D(x)$
end for
root γ at the directed edge $(*, v_s)$
return γ relabeled uniformly at random
end if

Given a graph H in $\mathcal{D}^\bullet$ let $S(H)$, $S'(H)$ denote the length of a shorted past in H from the root-vertex to the tail $v_1 = *$ or head v_s of the directed root-edge, respectively. Clearly, $S(H)$ and $S'(H)$ differ by at most one. It will be convenient to also consider their minimum $M(H)$. Let S, S' and M denote the corresponding random variables in the random graph D drawn according to the sampler $\Gamma D^\bullet(x)$. For any integers $\ell, k \geq 0$ with $\ell + k \geq 1$ let $\mathsf{D}_{\ell,k}$ be the random graph D conditioned on the event that the graph is not a single edge and that in the root face $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_s, v_1\}$ the length of the path $v_1 v_2 \cdots v_t$ equals ℓ and the length of the path $v_{t+1} v_{t+2} \cdots v_s$ equals k. Note that the probability for this event equals

$$p_{\ell,k} = \mathbb{P}(s = \ell + k + 2)\mathbb{P}(t = \ell + 1 | s = \ell + k + 2) = \big(D(x)\big)^{k+\ell}.$$

We denote by $S_{\ell,k}$, $S'_{\ell,k}$ and $M_{\ell,k}$ the corresponding distances in the conditioned random graph $\mathsf{D}_{\ell,k}$. Summing over all possible values for k and ℓ, we obtain

$$\mathbb{E}[S] = \frac{x}{D^\bullet(x)} + \sum_{k+\ell \geq 1} \mathbb{E}[S_{\ell,k}] p_{\ell,k}, \qquad \mathbb{E}[S'] = \sum_{k+\ell \geq 1} \mathbb{E}[S'_{\ell,k}] p_{\ell,k},$$
$$\mathbb{E}[M] = \sum_{k+\ell \geq 1} \mathbb{E}[M_{\ell,k}] p_{\ell,k}.$$

Any shortest path from $*$ or v_s to the root-vertex of a $\mathcal{B}'^\bullet$-graph H ($\neq$ a single edge) must traverse the boundary of the root-face in one of the two directions until it reaches the root-edge of the attached $\mathcal{B}'^\bullet$-object H'. From there, it follows a shortest path to the root in the graph H'. Hence, for all $k, \ell \geq 0$ with $k + \ell \geq 1$ the following equations hold:

$$S_{\ell,k} \stackrel{(d)}{=} \min\{\ell + S, k + 1 + S'\},$$
$$S'_{\ell,k} \stackrel{(d)}{=} \min\{\ell + 1 + S, k + S'\},$$
$$M_{\ell,k} \stackrel{(d)}{=} \min\{\ell + S, k + S'\}.$$

Since S and S' differ by at most one, this may be simplified further depending on the parameters k and ℓ as follows:

$$S_{\ell,k} \stackrel{(d)}{=} \begin{cases} \ell + S, & \ell \le k, \\ \ell + M, & \ell = k+1, \\ k + 1 + S', & \ell \ge k+2, \end{cases} \qquad S'_{\ell,k} \stackrel{(d)}{=} \begin{cases} k + S', & k \le \ell, \\ k + M, & k = \ell+1, \\ \ell + 1 + S, & k \ge \ell+2 \end{cases}$$

and

$$M_{\ell,k} \stackrel{(d)}{=} \begin{cases} \ell + S, & \ell \le k-1, \\ \ell + M, & \ell = k, \\ k + S', & \ell \ge k+1. \end{cases}$$

Using this and (27), we arrive at the system of linear equations with parameter $w = D(x)$ and variables $\mu_S = \mathbb{E}[S]$, $\mu_{S'} = \mathbb{E}[S']$ and $\mu_M = \mathbb{E}[M]$

$$\begin{aligned} \mu_S = {} & \frac{2w^2 - 4w + 1}{(w-1)^2} + \sum_{k \ge 1} \sum_{\ell=0}^{k} (\ell + \mu_S) w^{\ell+k} + \sum_{\ell \ge 1} (\ell + \mu_M) w^{2\ell-1} \\ & + \sum_{k \ge 0} \sum_{\ell \ge k+2} (k + 1 + \mu_{S'}) w^{\ell+k}, \\ \mu_{S'} = {} & \sum_{\ell \ge 1} \sum_{k=0}^{\ell} (k + \mu_{S'}) w^{\ell+k} + \sum_{k \ge 1} (k + \mu_M) w^{2k-1} \\ & + \sum_{\ell \ge 0} \sum_{k \ge \ell+2} (\ell + 1 + \mu_S) w^{\ell+k}, \\ \mu_M = {} & \sum_{k \ge 2} \sum_{\ell=0}^{k-1} (\ell + \mu_S) w^{\ell+k} + \sum_{\ell \ge 1} (\ell + \mu_M) w^{2\ell} + \sum_{k \ge 0} \sum_{\ell \ge k+1} (k + \mu_{S'}) w^{\ell+k}. \end{aligned}$$

Simplifying the equations yields the equivalent system

$$A \cdot \big(\mathbb{E}[S], \mathbb{E}[S'], \mathbb{E}[M]\big)^T = b$$

with

$$A = \begin{pmatrix} 2w^4 - 4w^3 + 3w - 1 & -w^3 + w^2 & w^3 - 2w^2 + w \\ -w^3 + w^2 & 2w^4 - 4w^3 + 3w - 1 & w^3 - 2w^2 + w \\ -w^2 + w & -w^2 + w & 2w^4 - 4w^3 + w^2 + 2w - 1 \end{pmatrix}$$

and

$$b^T = (\, 2w^4 - 4w^3 - w^2 + 3w - 1 \quad -w \quad -w^2 \,).$$

For $x = y \approx 0.17076$, we obtain $w \approx 0.27578$ and $\det(A) \approx -0.00799 \neq 0$. Solving the system of linear equations yields

$$\mathbb{E}[S] = \frac{8w^4 - 16w^3 + 4w - 1}{(4w^3 - 6w^2 - 2w + 1)(2w - 1)} \approx 5.46545,$$

and the proof is completed. □

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