Homework Assignment 3 - Coalgebras

Hopf algebras - Spring Semester 2018

Exercise 1 - Examples for duals of coalgebras

a) Recall that if G is a set, then $k^{(G)}$ is a coalgebra with $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. Show that if G is finite, then $(k^{(G)})^* \simeq k^G$ as k-algebras.

   Hint: Let $(e_g)_{g\in G}$ be the dual basis of $(g)_{g\in G}$. Then $\varphi : (k^{(G)})^* \to k^G$ with $\varphi(e_g) = (e_g(h))_{h\in G}$ is an algebra isomorphism.

b) Let C be a vector space over the field k with basis $(x_{i,j})_{1\leq i,j\leq n}$. We saw in the lecture that C is a coalgebra with $\Delta(x_{i,j}) = \sum_{k=1}^n x_{i,k} \otimes x_{k,j}$, $\epsilon(x_{i,j}) = \delta_{i,j}$. Show that $C^* \simeq M_n(k)$ as k-algebras.

c) Let C be a vector space over k with basis $(x_i)_{i\geq 0}$. We saw in the lecture that C is a coalgebra with $\Delta(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i}$ and $\epsilon(x_n) = \delta_{0,n}$. Show that its dual C^* is isomorphic to the power series algebra $k[[T]]$ in one indeterminate.

Exercise 2 - Coalgebra filtrations

Let (C, Δ, ϵ) be a coalgebra. A family $(C_n)_{n\geq 0}$ of subset $C_n \subset C$ is a coalgebra filtration, if $C = \bigcup_{n\geq 0} C_n$, $C_n \subset C_{n+1}$ and $\Delta(x) \in \sum_{i+j=n} C_i \otimes C_j$ for all $n \geq 0$ and $x \in C_n$. Note that this implies that $C_0 \subset C$ is a subcoalgebra.

Let (C, Δ, ϵ) be a k-coalgebra with filtration $(C_n)_{n\geq 0}$ and let (A, μ, η) be a k-algebra. Show that an element f of the algebra $\mathrm{Hom}(C, A)$ is $*$-invertible, if and only if its restriction $f|_{C_0} : C_0 \to A$ is $*$-invertible in the algebra $\mathrm{Hom}(C_0, A)$. Hints:

a) Let $g : C \to A$ be a k-linear map such that the restriction $g|_{C_0} : C_0 \to A$ is $*$-inverse to $f|_{C_0}$. Verify that $\psi := \eta \circ \epsilon - g * f \in \mathrm{Hom}(C, A)$ satisfies $\psi^{*n}(C_k) = 0$ for $k < n$.

b) We that ψ^i denotes the ith power of the element ψ of the algebra $\mathrm{Hom}(C, A)$, and use the convention that $\psi^0 = \eta \circ \epsilon$. Show that $\phi := \sum_{n\geq 0} \psi^n \in \mathrm{Hom}(C, A)$ is a well-defined linear map.

c) Show that ϕ is $*$-inverse to $g * f$.

Exercise 3 - An application of Dedekind's Lemma

Let k be a field that contains a primitive nth root of unity ζ. Let G be a finite cyclic group or order n. Recall that the group algebra $k[G]$ is an algebra with basis $(g)_{g\in G}$ that is also a coalgebra with $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. Show that there is a vector space isomorphism $\varphi : k[G] \to k[G]^*$ that preserves both the coalgebra and algebra structures.

Hint: Use ζ to determine the group-like elements of $k[G]^*$.

Exercise 4 - Examples of adjoint functors

a) Let R be a ring, and consider $\mathrm{Fo} : \mathcal{M}_R \to \mathrm{Set}$ the forgetful functor that sends an R-module to its underlying set. Find a functor F that is left-adjoint to Fo.

b) Let $\mathcal{M}_{\mathbb{Z}}^2$ be the category of ordered pairs of abelian groups with componentwise morphisms. Let $\oplus : \mathcal{M}_{\mathbb{Z}}^2 \to \mathcal{M}_{\mathbb{Z}}$ be the functor that maps a pair (X, Y) to its direct sum $X \oplus Y$. Find a functor H such that $\oplus$ is left-adjoint to H.

Exercise 5 - Uniqueness of adjoint functors

Let $F : \mathcal{C} \to \mathcal{D}$ and $G_1, G_2 : \mathcal{D} \to \mathcal{C}$ be functors such that F is left-adjoint to both G_1 and G_2. Show that there is a natural isomorphism between G_1 and G_2, that is for each object D in $\mathcal{D}$ we have an isomorphism ϕ_D such that the following commutes for any morphism $f : D_1 \to D_2$:

$$\begin{array}{ccc} G_1(D_1) & \xrightarrow{\phi_{D_1}} & G_2(D_1) \\ \downarrow{\scriptstyle G_1(f)} & & \downarrow{\scriptstyle G_2(f)} \\ G_1(D_2) & \xrightarrow{\phi_{D_2}} & G_2(D_2) \end{array} \tag{1}$$

Dually, if $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are both left-adjoint to $G : \mathcal{D} \to \mathcal{C}$ it holds that there is a natural isomorphism between F_1, F_2.

Hint: Yoneda's Lemma