Homework Assignment 2 solution - Algebras, Category Theory

Hopf algebras - Spring Semester 2018

Exercise 1 - Algebras, Category Theory

a) Let V and W be finite dimensional vector spaces over k. Show that there is an algebra isomorphism

$$\mathrm{End}_k(V \otimes_k W) \simeq \mathrm{End}_k(V) \otimes_k \mathrm{End}_k(W).$$

What does that imply for the algebra $M_n(k) \otimes_k M_m(k)$, $m, n \geq 1$?

Proof. The k-linear map

$$\varphi : \mathrm{End}_k(V) \otimes_k \mathrm{End}_k(W) \to \mathrm{End}_k(V \otimes_k W)$$

with

$$\varphi(f \otimes g) : V \otimes_k W \to V \otimes_k W, v \otimes w \mapsto f(v) \otimes f(w)$$

is an algebra homomorphism, because it preserves the unit element and is multiplicative on the $\mathbb{Z}$-span $(f \otimes g)_{f \in \mathrm{End}_k(V), g \in \mathrm{End}_k(W)}$ of $\mathrm{End}_k(V) \otimes_k \mathrm{End}_k(W)$.

It suffices to check that φ is injective, because both spaces have the same dimension $\dim_k(V)\dim_k(W)$. In order to verify $\ker \varphi = 0$, let $(b_i)_{1 \leq i \leq n}$ denote a basis of V. Then $f_{i,j} : V \to V$ with $f_{i,j}(b_k) = \delta_{i,k} b_j$ is a basis of $\mathrm{End}_k(V)$. Likewise, let $(c_s)_s$ be a basis of W and define a basis $(g_{s,t})_{s,t}$ of $\mathrm{End}_k(W)$ in this way. Thus $(f_{i,j} \otimes g_{s,t})_{i,j,s,t}$ is a basis of $\mathrm{End}_k(V) \otimes_k \mathrm{End}_k(W)$.

If $(\lambda_{i,j,s,t})$ is a family in k with

$$\varphi(\sum_{i,j,s,t} \lambda_{i,j,s,t} f_{i,j} \otimes g_{s,t}) = 0,$$

then it follows for all i and s that

$$0 = \varphi(\sum_{i,j,s,t} \lambda_{i,j,s,t} f_{i,j} \otimes g_{s,t})(b_i \otimes c_s) = \sum_{j,t} \lambda_{i,j,s,t} b_j \otimes c_t,$$

and hence $\lambda_{i,j,s,t} = 0$ for all j, t.

This implies that $M_n(k) \otimes_k M_m(k) \simeq M_{mn}(k)$. □

b) Let M be a finite abelian group. Show that there are integers $n \geq 1$, $m_1, \ldots, m_n \geq 1$ such that

$$k[M] \simeq k[X_1, \ldots, X_n]/(X_1^{m_1} - 1, \ldots, X_n^{m_n} - 1).$$

Hint: Show first that $k[G \times H] \simeq k[G] \otimes k[H]$ for any two monoids G and H.

Proof. Let $\iota : G \times H \to k[G] \otimes_k k[H], (g, h) \to g \otimes h$. By the universal property of $k[G \times H]$ there is a unique algebra homomorphism $\varphi : k[G \times H] \to k[G] \otimes_k k[H]$ such that:

$$\begin{array}{ccc} k[G \times H] & \xrightarrow{\varphi} & k[G] \otimes_k k[H] \\ {\scriptstyle \text{can}} \uparrow & \nearrow_{\iota} & \\ G \times H & & \end{array}$$

φ maps the basis $(g, h)_{g \in G, h \in H}$ to the basis $(g \otimes h)_{g \in G, h \in H}$, hence it is an algebra isomorphism.

Since M is a finite abelian group, there are integers $m_1, \ldots, m_n \geq 1$, $n \geq 1$ such that

$$M \simeq \prod_{1 \leq i \leq n} \mathbb{Z}/(m_i).$$

Consequently,

$$\begin{aligned} k[M] &\simeq \bigotimes_{1 \leq i \leq n} k[\mathbb{Z}/(m_i)] \\ &\simeq \bigotimes_{1 \leq i \leq n} k[X_i]/(X_i^{m_i} - 1) \\ &\simeq k[X_1, \ldots, X_n]/(X_1^{m_1} - 1, \ldots, X_n^{m_n} - 1). \end{aligned}$$

In the last isomorphism we have used that for all $f \in k[X]$, $g \in k[Y]$ it holds that

$$k[X]/(f) \otimes k[Y]/(g) \simeq k[X, Y]/(f, g).$$

To see this, note that the universal property of $k[X, Y]$ yields an algebra homomorphism

$$\psi : k[X, Y] \to k[X]/(f) \otimes k[Y]/(g) \quad \text{with} \quad \psi(X) = \bar{X} \otimes 1, \psi(Y) = 1 \otimes \bar{Y}.$$

The ideal $(f, g) \subset k[X, Y]$ gets mapped to zero, hence ψ induces an algebra homomorphism

$$\bar{\psi} : k[X, Y]/(f, g) \to k[X]/(f) \otimes k[Y]/(g) \quad \text{with} \quad \bar{\psi}(\bar{X}) = \bar{X} \otimes 1, \bar{\psi}(\bar{Y}) = 1 \otimes \bar{Y}.$$

The map

$$k[X]/(f) \otimes k[Y]/(g) \simeq k[X, Y]/(f, g), \quad (\bar{p}, \bar{q}) \mapsto \overline{pq}$$

is well-defined and inverse to ψ. □

Exercise 2 - Algebras and field extensions

Let $k \subset L$ be a field extension.

a) Let A be a k-algebra. Show that the L-algebra $A \otimes_k L$ has dimension

$$\dim_L(A \otimes_k L) = \dim_k(A).$$

Proof. If $(a_i)_{i \in I}$ is a k-basis of A, then $(a_i \otimes 1)_{i \in I}$ is an L-basis of $A \otimes_k L$. (It is straight-forward to verify that this family is an L-generating family of $A \otimes_k L$ and L-linear independent. □

b) Verify that $k[X] \otimes_k L \simeq L[X]$ as L-algebras.

Proof. The universal property of the polynomial algebra $L[X]$ yields an L-algebra homomorphism

$$\varphi : L[X] \to k[X] \otimes_k L \quad \text{with} \quad \varphi(X) = X \otimes 1.$$

The linear map $\psi : k[X] \otimes_k L \to L[X] \quad \text{with} \quad \psi(f \otimes x) = fx$ is the inverse of φ. □

c) Let $k \subset L$ be a finite Galois extension. Find an explicit description of the L-algebra $L \otimes_k L$.

Proof. Since $k \subset L$ is a finite Galois extension it has a primitive element $a \in L$. The irreducible, separable, monic minimal polynomial f of a satisfies

$$L = k(a) \simeq k[X]/(f)$$

and hence

$$L \otimes_k L \simeq k[X]/(f) \otimes_k L \tag{1}$$

as L-algebras. The sequence

$$0 \to (f) \to k[X] \to k[X]/(f) \to 0$$

is exact. By the exactness of the tensor product, it follows that

$$0 \to (f) \otimes_k L \to k[X] \otimes_k L \to k[X]/(f) \otimes_k L \to 0$$

is exact. Hence

$$k[X]/(f) \otimes_k L \simeq (k[X] \otimes L)/((f) \otimes L). \tag{2}$$

Moreover:

$$\begin{array}{ccc} k[X] \otimes_k L & \xrightarrow{\simeq} & L[X] \\ \big\uparrow & & \big\uparrow \\ \text{can} \quad (f) \otimes L & \longrightarrow & (f) \quad \text{can} \\ \big\downarrow & & \big\downarrow \\ (k[X] \otimes L)/((f) \otimes L) & \xrightarrow{\simeq} & L[X]/(f) \end{array}$$

Hence:

$$(k[X] \otimes L)/((f) \otimes L) \simeq L[X]/(f). \tag{3}$$

Let $n = \dim_k(L)$. Since f is separable and monic, we can find n distinct elements $a_1, \dots, a_n \in L$ such that $f = \prod_{i=1}^{n}(X - a_i)$ in $L[X]$. Using the Chinese remainder theorem, it follows that

$$L[X]/(f) \simeq L[X]/(\prod_{i=1}^{n}(X - a_i)) \simeq \prod_{i=1}^{n} L[X]/(X - a_i) \simeq L^n. \tag{4}$$

Thus

$$L \otimes_k L \simeq L^n.$$

□

Exercise 3 - Morita equivalence

a) Let R be a ring and $S = M_n(R)$ the ring of $n \times n$ matrices with coefficients in R. Let P be the space of all matrices in $M_n(R)$ with the property, that the coefficients in the rows $2, \dots, n$ are equal to zero. Let Q be the space of all matrices in $M_n(R)$ with the property, that the coefficients in the columns $2, \dots, n$ are equal to zero. Show that

$$P \otimes_S Q \simeq R \text{ in } {}_R\mathcal{M}_R \qquad \text{and} \qquad Q \otimes_R P \simeq S \text{ in } {}_S\mathcal{M}_S \ .$$

Proof. Let p_{11} be the projection that sends a matrix to its coefficient in the first row and first column. Then

$$P \times Q \to R, \quad (A, B) \to p_{11}(AB)$$

is $M_n(R)$ middle-linear. The induced additive map

$$f : P \otimes_S Q \to R$$

is (R, R)-linear. f is clearly surjective. It is also injective, because if $t = \sum_i A_i \otimes B_i \in \ker(f)$ then

$$t = \sum_i A_i \otimes B_i = (\sum_i A_i B_i) \otimes I = 0.$$

Hence f is an isomorphism.

The multiplication

$$Q \times P \to S, \quad (A,B) \to AB$$

is clearly R middle-linear. The induced map

$$g : Q \otimes_R P \to S$$

is (S,S)-linear. For all $1 \leq i,j \leq n$ let $I_{i,j}$ denote the matrix with the coefficient in the i-th row and j-th column being equal to 1 and the remaining coefficients being equal to 0. Then $Q \otimes_R P$ has the basis $(E_{i,1} \otimes E_{1,j})_{1 \leq i,j \leq n}$ and g maps this basis to the basis $(E_{i,j})_{i,j}$ of S. Hence g is an isomorphism. □

b) Let R, S be rings, P an (R,S)-bimodule, Q an (S,R)-bimodule. Suppose that

$$P \otimes_S Q \simeq R \text{ in } {}_R\mathcal{M}_R \qquad \text{and} \qquad Q \otimes_R P \simeq S \text{ in } {}_S\mathcal{M}_S .$$

Show that the functor $Q \otimes_R - : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and $P \otimes_S - : {}_S\mathcal{M} \to {}_R\mathcal{M}$ are quasi-inverse equivalences of categories.

Proof. Let $\varphi : R \to P \otimes_S Q$ in ${}_R\mathcal{M}_R$ and $\psi : S \to Q \otimes_R P$ in ${}_S\mathcal{M}_S$. Let $f : V \to W$ be a morphism in ${}_R\mathcal{M}$. Then

$$\begin{array}{ccccccc} V & \xrightarrow{\text{can}} & R \otimes_R V & \xrightarrow{\varphi \otimes \text{id}} & (P \otimes_S Q) \otimes_R V & \xrightarrow{\text{can}} & P \otimes_S (Q \otimes_R V) \\ \downarrow f & & & & & & \downarrow \text{id} \otimes \text{id} \otimes f \\ W & \xrightarrow{\text{can}} & R \otimes_R W & \xrightarrow{\varphi \otimes \text{id}} & (P \otimes_S Q) \otimes_R W & \xrightarrow{\text{can}} & P \otimes_S (Q \otimes_R W). \end{array}$$

Hence $F = Q \otimes_R -$ and $G = P \otimes_S -$ satisfy $GF \simeq \text{id}_{{}_R\mathcal{M}}$. Analogously we may show that $FG \simeq \text{id}_{{}_S\mathcal{M}}$ □

Applications: Equivalences of categories preserve category theoretic notions such as co-products. For example, if $R = k$ is a field, then there is exists a module $U \in {}_k\mathcal{M}$ such that for all $V \in {}_k\mathcal{M}$ there is an index set I such that $V \simeq \coprod_{i \in I} U$. Since ${}_k\mathcal{M} \simeq {}_{M_n(k)}\mathcal{M}$ an analogous statement holds for left modules over $M_n(k)$.

Exercise 4 - Exact functors

Let R and S be rings. A covariant functor $F : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is termed left exact, if for any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ in ${}_R\mathcal{M}$ the sequence

$$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact as well. It is termed right-exact, if for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$$

is exact as well. We say F is exact, if it is both left- and right-exact. A contravariant functor $F : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is left-exact, if for any exact sequence $0 \to C^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} B^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} A^{\mathrm{op}}$ in ${}_R\mathcal{M}^{\mathrm{op}}$ the sequence

$$0 \to F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$$

is exact. We have seen that for all left R-modules M, N the functors $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, N)$ are left-exact.

a) Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be group homomorphisms of abelian groups. Show that if

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(Z, A) \xrightarrow{\mathrm{Hom}(g,\mathrm{id})} \mathrm{Hom}_{\mathbb{Z}}(Y, A) \xrightarrow{\mathrm{Hom}(f,\mathrm{id})} \mathrm{Hom}_{\mathbb{Z}}(X, A)$$

is exact for every abelian group A, then

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is exact as well.

Proof. Let $\tilde{g} = \mathrm{Hom}(g, \mathrm{id})$ and $\tilde{f} = \mathrm{Hom}(f, \mathrm{id})$.

In the special case $A = Z$, it follows that $\tilde{f}(\tilde{g}(\mathrm{id}_Z)) = 0$. That is $gf = 0$ and hence $\mathrm{im}(f) \subset \ker(g)$.

In the special case $A = Y/\mathrm{im}(f)$ it holds that $\mathrm{can}_2 : Y \to Y/\mathrm{im}(f)$ satisfies $\tilde{f}(\mathrm{can}_2) = 0$. As $\ker(\tilde{f}) = \mathrm{im}(\tilde{g})$ there is a morphism $h : Z \to Y/\mathrm{im}(f)$ with $hg = \mathrm{can}_2$. In particular $\ker(hg) = \ker(\mathrm{can}_2) = \mathrm{im}(f)$. This implies $\ker(g) \subset \mathrm{im}(f)$.

In the special case $A = Z/\mathrm{im}(g)$ it holds that $\mathrm{can}_1 : Z \to Z/\mathrm{im}(g)$ satisfies $\tilde{g}(\mathrm{can}_1) = 0$. As $\tilde{g}$ is injective, it follows that $\mathrm{can}_1 = 0$. That is, $\mathrm{im}(g) = Z$. □

b) Let R and S be rings, and let ${}_RX_S$ be an (R, X)-bimodule. Recall that the functor

$${}_RX_S \otimes_S - : {}_S\mathcal{M} \to {}_R\mathcal{M}$$

is left adjoint to

$$\mathrm{Hom}_R({}_RX_S, -) : {}_R\mathcal{M} \to {}_S\mathcal{M}.$$

Combine this fact with a) to deduce that for any right R-module M_R the functor

$$M \otimes_R - : {}_R\mathcal{M} \to {}_{\mathbb{Z}}\mathcal{M}$$

is right-exact.

Proof. Let $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \to 0$ be an exact sequence in ${}_R\mathcal{M}$. Let A be an arbitrary abelian group. Since $M \otimes_R -$ is left-adjoint to $\operatorname{Hom}_R(M, -)$, it follows that:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R N_3, A) & \xrightarrow{\simeq} & \operatorname{Hom}_R(N_3, {}_R\operatorname{Hom}_{\mathbb{Z}}(M, A)) \\
\downarrow{\scriptstyle \operatorname{Hom}(\operatorname{id} \otimes g, \operatorname{id})} & & \downarrow{\scriptstyle \operatorname{Hom}(g, \operatorname{id})} \\
\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R N_2, A) & \xrightarrow{\simeq} & \operatorname{Hom}_R(N_2, {}_R\operatorname{Hom}_{\mathbb{Z}}(M, A)) \\
\downarrow{\scriptstyle \operatorname{Hom}(\operatorname{id} \otimes f, \operatorname{id})} & & \downarrow{\scriptstyle \operatorname{Hom}(f, \operatorname{id})} \\
\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R N_1, A) & \xrightarrow{\simeq} & \operatorname{Hom}_R(N_1, {}_R\operatorname{Hom}_{\mathbb{Z}}(M, A))
\end{array}$$

The right-column is exact because the functor $\operatorname{Hom}_R(-, \operatorname{Hom}_{\mathbb{Z}}(M, A))$ is left-exact. It follows that the left column is exact. As this holds for arbitrary abelian groups A, it follows from a) that

$$M \otimes_R N_1 \xrightarrow{\operatorname{id} \otimes f} M \otimes_R N_2 \xrightarrow{\operatorname{id} \otimes g} M \otimes_R N_3 \to 0$$

is exact. □

c) An R-module M is termed free, if there is an index set I with $M \simeq R^{(I)}$. M is termed projective if there is an R-module N such that $M \oplus N$ is free. Show that if M is a projective right R-module, then the functor $M \otimes_R -$ is exact.

Comment: It is clear that the same holds also for the functor $- \otimes_R N$ if N is a projective left R-module. In particular, the tensor product over a skew field is always exact.

Proof. It remains to verify that $M \otimes_R -$ is left-exact. Let us first consider the case where M is free. That is, without loss of generality we may assume that $M = R^{(I)} = \coprod_{i \in I} R_i$ with $R_i = R$. Let

$$0 \to N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$$

be an exact sequence of left R-modules. Then for all $i \in I$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_R N_1 & \xrightarrow{\operatorname{id} \otimes f} & M \otimes_R N_2 & \xrightarrow{\operatorname{id} \otimes g} & M \otimes_R N_3 \\
& & \simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow \\
0 & \longrightarrow & \coprod_{i \in I} R_i \otimes_R N_1 & \xrightarrow{\operatorname{id} \otimes f} & \coprod_{i \in I} R_i \otimes_R N_2 & \xrightarrow{\operatorname{id} \otimes g} & \coprod_{i \in I} R_i \otimes_R N_3 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & R_i \otimes_R N_1 & \xrightarrow{\operatorname{id} \otimes f} & R_i \otimes_R N_2 & \xrightarrow{\operatorname{id} \otimes g} & R_i \otimes_R N_3 \\
& & \simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow \\
0 & \longrightarrow & N_1 & \xrightarrow{f} & N_2 & \xrightarrow{g} & N_3
\end{array}$$

Hence

$$0 \longrightarrow M \otimes_R N_1 \xrightarrow{\mathrm{id} \otimes f} M \otimes_R N_2 \xrightarrow{\mathrm{id} \otimes g} M \otimes_R N_3$$

is exact.

It remains to consider the projective case. Suppose that $M \oplus M'$ is free. It follows that $M \otimes_R N_i \to (M \oplus M') \otimes_R N_i$ is injective. (Here we really need to use that $M \oplus M'$ is free. The fact that $M \subset M \oplus M'$ is a submodule is not enough.) The exactness of the first row in

$$\begin{array}{ccccccc}
0 \longrightarrow & (M \oplus M') \otimes_R N_1 & \xrightarrow{\mathrm{id} \otimes f} & (M \oplus M') \otimes_R N_2 & \xrightarrow{\mathrm{id} \otimes g} & (M \oplus M') \otimes_R N_3 \\
& \big\uparrow & & \big\uparrow & & \big\uparrow \\
0 \longrightarrow & M \otimes_R N_1 & \xrightarrow{\mathrm{id} \otimes f} & M \otimes_R N_2 & \xrightarrow{\mathrm{id} \otimes g} & M \otimes_R N_3
\end{array}$$

implies the exactness of the second row. □