Homework Assignment 3 - Coalgebras

Hopf algebras - Spring Semester 2018

Exercise 1 - Examples for duals of coalgebras

a) Recall that if S is a set, then $k^{(S)}$ is a coalgebra with $\Delta(s) = s \otimes s$ and $\epsilon(s) = 1$ for all $s \in S$. Show that if S is finite, then $(k^{(S)})^* \simeq k^S$ as k-algebras.

Hint: Let $(e_s^*)_{s\in S}$ be the dual basis of $(s)_{s\in S}$. Then $\varphi : (k^{(S)})^* \to k^S$ with $\varphi(e_s^*) = (e_s(h))_{h\in S}$ is an algebra isomorphism.

b) Let C be a vector space over the field k with basis $(x_{i,j})_{1\leq i,j\leq n}$. We saw in the lecture that C is a coalgebra with $\Delta(x_{i,j}) = \sum_{k=1}^n x_{i,k} \otimes x_{k,j}$, $\epsilon(x_{i,j}) = \delta_{i,j}$. Show that $C^* \simeq M_n(k)$ as k-algebras.

c) Let C be a vector space over k with basis $(x_i)_{i\geq 0}$. We saw in the lecture that C is a coalgebra with $\Delta(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i}$ and $\epsilon(x_n) = \delta_{0,n}$. Show that its dual C^* is isomorphic to the power series algebra $k[[T]]$ in one indeterminate.

Proof. We will present the product structure in the dual algebra. Recall that the unit in C^* for a coalgebra C is the counit $\epsilon \in C^*$. That the product structure is preserved by the given isomorphisms is elementary.

a) The map in the hint sends a basis set to a basis set, so it's an isomorphism of vector spaces. The multiplication in $(k^{(S)})^*$ is as follows in the basis elements:

$$e_{s_1}^* e_{s_2}^* = e_{s_1}\delta_{s_1,s_2}\,.$$

It's immediate that the algebra structure is preserved.

b) Take the dual basis $\{x_{i,j}^* | 1 \leq i,j \leq n\}$ and map $x_{i,j}^* \mapsto E_{i,j}$ to the matrix with only one non-zero entry, the entry $(E_{i,j})_{i,j} = 1$. The multiplication structure in C^* is given in the basis elements by :

$$x_{i_1,j_1}^* x_{i_2,j_2}^* = \delta i_2, j_1 x_{i_1,j_2}^*\,.$$

It's immediate that the algebra structure is preserved.

c) If $f \in C^*$ such that $f(x_i) = a_i$, then define $\phi(f) = \sum_{i\geq 0} a_i T^i$. The product of two elements $f_1, f_2 \in C^*$ is a map defined on the basis elements as:

$$f_1 f_2(x_i) = \sum_{a+b=i} f_1(x_a) f_2(x_b)\,.$$

It is immediate that this preserves the algebra structure.

□

Exercise 2 - Coalgebra filtrations

Let (C, Δ, ϵ) be a coalgebra. A family $(C_n)_{n\geq 0}$ of subset $C_n \subset C$ is a coalgebra filtration, if $C = \bigcup_{n\geq 0} C_n$, $C_n \subset C_{n+1}$ and $\Delta(x) \in \sum_{i+j=n} C_i \otimes C_j$ for all $n \geq 0$ and $x \in C_n$. Note that this implies that $C_0 \subset C$ is a subcoalgebra.

Let (C, Δ, ϵ) be a k-coalgebra with filtration $(C_n)_{n\geq 0}$ and let (A, μ, η) be a k-algebra. Show that an element f of the algebra $\mathrm{Hom}(C, A)$ is $*$-invertible, if and only if its restriction $f|_{C_0} : C_0 \to A$ is $*$-invertible in the algebra $\mathrm{Hom}(C_0, A)$. Hints:

a) Let $g : C \to A$ be a k-linear map such that the restriction $g|_{C_0} : C_0 \to A$ is $*$-inverse to $f|_{C_0}$. Verify that $\psi := \eta \circ \epsilon - g * f \in \mathrm{Hom}(C, A)$ satisfies $\psi^{*n}(C_k) = 0$ for $k < n$.

b) We let ψ^i denote the ith power of the element ψ of the algebra $\mathrm{Hom}(C, A)$, and use the convention that $\psi^0 = \eta \circ \epsilon$. Show that $\phi := \sum_{n\geq 0} \psi^n \in \mathrm{Hom}(C, A)$ is a well-defined linear map.

c) Show that ϕ is $*$-inverse to $g * f$.

Proof. a) First we pick an extension of g from $C|_0$ to C. This is possible because over vector spaces we can pick a basis in $C|_0$ and extend such basis to C. Now suppose that $x \in C_m$, so by the filtration property we have that

$$\Delta^{(n-1)} x \in \sum_{a_1+\cdots+a_n=m} C_{a_1} \otimes \cdots \otimes C_{a_n}\,.$$

From $a_1 + \cdots + a_n = k$ we have immediately that $a_i = 0$ for some i. However, ψ is clearly zero in $C|_0$ (because g is the inverse of f in $C|_0$), so ψ^n is zero in

$$\sum_{a_1+\cdots+a_n=k} C_{a_1} \otimes \cdots \otimes C_{a_n}\,.$$

It implies that $\psi^n(x) = \mu^{(n-1)} \circ \psi^{\otimes n} \circ \Delta^{(n-1)} x = 0$, as desired.

b) From the previous item, ϕ is a k-linear map in each $C|_n$, so it is also k-linear in the linear span.

c) Just note that

$$\phi * g * f = \phi * \psi - \phi * (\eta \circ \epsilon) = \phi * \psi - \phi = \sum_{n\geq 0} \psi^{*n} - \sum_{n\geq 1} \psi^{*n} = \eta \circ \epsilon\,.$$

We conclude that f is invertible on both sides, so it is invertible.

□

Exercise 3 - An application of Dedekind's Lemma

Let k be a field that contains a primitive nth root of unity ζ. Let G be a finite cyclic group or order n. Recall that the group algebra $k[G]$ is an algebra with basis $(g)_{g\in G}$ that is also a coalgebra with $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. Show that there is a vector space isomorphism $\varphi : k[G] \to k[G]^*$ that preserves both the coalgebra and algebra structures.

Hint: Use ζ to determine the group-like elements of $k[G]^*$.

Proof. Write $G = \{\bar{0}, \bar{1}, \cdots, \overline{n-1}\}$. Recall that the algebra structure in $k[G]^*$ is given in the dual basis as $g_1^* g_2^* = \delta_{g_1,g_2} g_1^*$, and the coalgebra structure is given by $\Delta g^* = \sum_{g=h_1h_2} h_1^* \otimes h_2^*$. Take a generic element $f = \sum_{g\in G} a_g g^* \in k[G]^*$.

We see that f is a group like element if and only if there is an integer $k0$ such that $a_{\bar{m}} = \zeta^{km}$. Indeed, since $\epsilon(f) = a_{\bar{0}}$ and $\Delta(f) = \sum_{h_1,h_2} a_{h_1h_2} h_1^* \otimes h_2^*$, f is group-like if and only if

$$a_{\bar{m}} = a_{\bar{1}}^m \text{ and } a_{\bar{0}} = 1\,.$$

So, $a_{\bar{1}}$ is a root of unity, say ζ^k, and the group-like elements are of the form

$$f = \sum_{m=0}^{n-1} \zeta^{mk} \bar{m}^* \in k[G]^*\,.$$

It can then be seen that $k[G]$ and $k[G]^*$ have the same group structure on the group-like elements, so it generates isomorphic coalgebras. By Dedekind's Lemma and a dimension argument, the group-like elements are linearly independent, so they form a basis, and the bialgebras $k[G]$ and $k[G]^*$ are isomorphic. □

Exercise 4 - Examples of adjoint functors

a) Let R be a ring, and consider $\mathrm{Fo} : \mathcal{M}_R \to \mathrm{Set}$ the forgetful functor that sends an R-module to its underlying set. Find a functor F that is left-adjoint to Fo.

b) Let $\mathcal{M}_{\mathbb{Z}}^2$ be the category of ordered pairs of abelian groups with componentwise morphisms. Let $\oplus : \mathcal{M}_{\mathbb{Z}}^2 \to \mathcal{M}_{\mathbb{Z}}$ be the functor that maps a pair (X, Y) to its direct sum $X \oplus Y$. Find a functor H such that $\oplus$ is left-adjoint to H.

Proof. We present the adjoint functors and the proof of the adjoincy is elementary.

a) The functor G is the free functor, i.e. $G(A) = AR$ is the free R-module with basis $\{a \in A\}$.

b) The functor H is the diagonal functor $H(A) = (A, A)$.

□

Exercise 5 - Uniqueness of adjoint functors

Let $F : \mathcal{C} \to \mathcal{D}$ and $G_1, G_2 : \mathcal{D} \to \mathcal{C}$ be functors such that F is left-adjoint to both G_1 and G_2. Show that there is a natural isomorphism between G_1 and G_2, that is for each object D in $\mathcal{D}$ we have an isomorphism ϕ_D such that the following commutes for any morphism $f : D_1 \to D_2$:

$$\begin{array}{ccc} G_1(D_1) & \xrightarrow{\phi_{D_1}} & G_2(D_1) \\ \downarrow{\scriptstyle G_1(f)} & & \downarrow{\scriptstyle G_2(f)} \\ G_1(D_2) & \xrightarrow{\phi_{D_2}} & G_2(D_2) \end{array} \tag{1}$$

Dually, if $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are both left-adjoint to $G : \mathcal{D} \to \mathcal{C}$ it holds that there is a natural isomorphism between F_1, F_2.

Hint: Yoneda's Lemma

Proof. The natural isomorphism $\mathrm{Hom}(A, G_1(B)) \simeq \mathrm{Hom}(F(A), B) \simeq \mathrm{Hom}(A, G_2(B))$ gives a natural isomorphism $G_1(B) \simeq G_2(B)$ from Yoneda's lemma. $\square$