Homework Assignment 4 - Bialgebras and Hopf algebras

Hopf algebras - Spring Semester 2018

Exercise 1

Fix $n \geq 0$ and for an algebra A consider the functor $O_n(A) := \{M \in M_n(A) | MM^T = Id\}$. Find a commutative Hopf algebra H_n such that we have the natural isomorphism

$$O_n(-) \simeq Alg_k(H, -) \,.$$

Proof. Take $H_n = k[(T_{i,j})_{1 \leq i,j \leq n} \mid (T_{i,j})_{i,j}(T_{j,i})_{i,j} = I]$ with $\Delta(T_{i,j}) = \sum_\ell T_{i,\ell} \otimes T_{\ell,j}$ and $\epsilon(T_{i,j}) = \delta_{i,j}$.

□

Exercise 2

a) Let $A \subset B$ be k-algebras, and let $A^\times, B^\times$ the sets of invertible elements in the respective algebras.

Suppose that A is finite dimensional. Show that $A^\times = B^\times \cap A$.

b) Suppose that $H \subset B$ are bialgebras over k, H is a Hopf algebra and B finite dimensional.

Then B is a Hopf algebra.

c) Suppose that H, B are bialgebras over k, H is a Hopf algebra and B finite dimensional. Suppose that there is a surjective map $\phi : H \to B$.

Then B is a Hopf algebra.

Proof. a) Clearly $A^\times \subset B^\times \cap A$. Take now an element $a \in B^\times \cap A$ and consider the isomorphism $\phi : B \to B$ given by $\phi : x \mapsto ax$.

We have $\phi(A) \subset A$, and $\phi|_A$ is injective. By a dimension argument, $\phi(A) = A$, so in particular, $\exists b \in A$ s.t. $ab = 1$, so $a \in A^\times$, as desired.

b) We just need to find an antipode in B. Take the antipode $s : H \to H$ in the Hopf algebra H, and the inclusion $\iota : B \to H$. Since $\mathrm{Hom}_k(\iota, H) : \mathrm{Hom}_k(H, H) \to \mathrm{Hom}_k(B, H)$ is an algebra homomorphism, $\iota = \mathrm{Hom}_k(\iota, H)(\mathrm{id}_H)$ and $\iota \circ s = \mathrm{Hom}_k(\iota, H)(s)$ are *-inverses in the k-algebra $\mathrm{Hom}_k(B, H)$.

$$
\begin{array}{ccccc}
 & & \mathrm{Hom}_k(H,H) & & \mathrm{id}_H \\
 & & \big\downarrow{\scriptstyle \mathrm{Hom}_k(\iota,H)} & & \big\downarrow \\
\mathrm{Hom}_k(B,B) & \xhookrightarrow{\mathrm{Hom}_k(B,\iota)} & \mathrm{Hom}_k(B,H) & & \\
 & & & & \\
\mathrm{id}_B & \longmapsto & & & \iota \in \mathrm{Hom}_k(B,H)
\end{array}
\tag{1}
$$

Now $\mathrm{Hom}_k(B,B)$ is identified with a finite dimensional subalgebra of $\mathrm{Hom}_k(B,H)$, where $\mathrm{id}_B = \iota \in \mathrm{Hom}_k(B,H)\times \cap \mathrm{Hom}_k(B,B)$. By the previous exercise, id_B is invertible in $\mathrm{Hom}_k(B,B)$, and B is a Hopf algebra.

c) Again we need only to find an antipode in B.

The map ϕ is surjective, hence $\mathrm{Hom}_k(\phi,B)$ is an injective algebra homomorphism, so we can identify $\mathrm{Hom}_k(B,B) \subset \mathrm{Hom}_k(H,B)$. Note that $\phi = \mathrm{Hom}_k(\phi,B)(\mathrm{id}_B)$, so $\phi \in \mathrm{Hom}_k(B,B)$.

$$
\begin{array}{ccccc}
 & & \mathrm{Hom}_k(H,H) & & \mathrm{id}_H \\
 & & \big\downarrow{\scriptstyle \mathrm{Hom}_k(H,\phi)} & & \big\downarrow \\
\mathrm{Hom}_k(B,B) & \xhookrightarrow{\mathrm{Hom}_k(\phi,B)} & \mathrm{Hom}_k(H,B) & & \\
 & & & & \\
\mathrm{id}_B & \longmapsto & & & \phi \in \mathrm{Hom}_k(B,H)
\end{array}
\tag{2}
$$

As in the previous exercise, ϕ is invertible in $\mathrm{Hom}_k(B,H)$ and its inverse is given as $\mathrm{Hom}_k(H,\phi)(s) = \phi \circ s$. So $\phi \in \mathrm{Hom}_k(B,H)^\times \cap \mathrm{Hom}_k(B,B) = \mathrm{Hom}_k(B,B)^\times$ by exercise a), as desired.

□

Exercise 3 - Kernel of counit

Suppose that B is a bialgebra, and denote $B^+ = \ker(\epsilon)$ the augmentation ideal. Show that if $x \in B^+$, then

$$\Delta(x) \in x \otimes 1 + 1 \otimes x + H^+ \otimes H^+ \,.$$

Proof. It is easy to see that $\Delta(x) - 1 \otimes x - x \otimes 1$ is in the kernel of the maps $\epsilon \otimes \mathrm{id}_H$ and $\mathrm{id}_H \otimes \epsilon$, by using the counit property and that $\epsilon(x) = 0$.

Now the following is exact

$$0 \to B^+ \to B \xrightarrow{\epsilon} k \to 0\,,$$

so tensoring with H along the sequence in each side we obtain that $\ker(\epsilon \otimes \mathrm{id}_B) = B^+ \otimes B$ and that $\ker(\mathrm{id}_B \otimes \epsilon) = B \otimes B^+$.

So $\Delta(x) - 1 \otimes x - x \otimes 1 \in (B^+ \otimes B) \cap (B \otimes B^+) = B^+ \otimes B^+$ holds in vector spaces, as desired. □

Exercise 4 - Dedekind's argument

Take a bialebra B and $x \in B$ a non-zero primitive element, i.e. $\Delta x = a \otimes x + x \otimes 1$.

Suppose that $\mathrm{char}(k) = 0$. Show that $\{1, x, x^2, \dots\}$ in l.i.

Proof. Suppose that $\sum_{k=0}^n p_k x^k = 0$ such that $p_n \neq 0$, and suppose that n is minimal in this way. Note that since $x \neq 0$, we have that $n \geq 2$. In particular, $1, x, \cdots, x^{n-1}$ from a linearly independent set. Consequently, $\{x^r \otimes x^s\}_{0 \leq r,s, \leq n-1}$ is a linearly independent set.

So, from $\Delta x = 1 \otimes x + x \otimes 1$ we see that $\Delta x^k = \sum_{r+s=k} \binom{k}{r} x^r \otimes x^s$, and consequently:

$$\begin{aligned}
0 \otimes 0 =& \Delta(\sum_{k=0}^n p_k x^k) = \sum_{k=0}^n p_k \sum_{r+s=k} \binom{k}{r} x^r \otimes x^s \\
=& 1 \otimes x^n + x^n \otimes 1 \sum_{k=0}^n p_k \sum_{\substack{r+s=k \\ r,s<n}} \binom{k}{r} x^r \otimes x^s \\
=& 1 \otimes \left(\sum_{k=0}^{n-1} -\frac{p_k}{p_n} x^k \right) + \left(\sum_{k=0}^{n-1} -\frac{p_k}{p_n} x^k \right) \otimes 1 + \sum_{k=0}^n p_k \sum_{\substack{r+s=k \\ r,s<n}} \binom{k}{r} x^r \otimes x^s \\
=& \left(\sum_{k=0}^{n-1} -\frac{p_k}{p_n} 1 \otimes x^k - \frac{p_k}{p_n} x^k \otimes 1 \right) + \sum_{k=0}^n p_k \sum_{\substack{r+s=k \\ r,s<n}} \binom{k}{r} x^r \otimes x^s \,.
\end{aligned} \tag{3}$$

This contradicts the linear independence of $\{x^r \otimes x^s\}_{0 \leq r,s, \leq n-1}$, and we conclude that $\{x^n\}_{n \geq 0}$ form a linear independent set. □

Exercise 5 - Primitive elements

For a Hopf algebra H, let $P(H) = \{x \in H | \Delta x = 1 \otimes x + x \otimes 1\}$ be the set of primitive elements.

a) Suppose that G is a group, and take the Hopf algebra $H = k[G]$, where $\Delta(g) = g \otimes g$. Then $P(H) = 0$.

b) If G is a finite group, then $P(k^G) = \mathrm{Hom}_{Ab}(G, k)$.

c) For a variable T, compute $P(k[T])$ for $\mathrm{char}\, k = 0$ and $\mathrm{char}\, k = p > 0$.

Proof. a) Take a generic element $x = \sum_{g \in G} a_g g$ such that $\Delta x = 1 \otimes x + x \otimes 1$. Recall that $1 = \mathrm{id}_G$.

Note that

$$\begin{gathered}
1 \otimes x + x \otimes 1 = \sum_{g \in G} a_g (\mathrm{id}_{\otimes} g + g \otimes \mathrm{id}_G) \\
\\
\Delta(x) = \sum_{g,h \in G} a_g g \otimes g \,.
\end{gathered} \tag{4}$$

So, by linear independence of $\{g \otimes h\}_{g,h\in G}$ we obtain that $a_g = 0$ for every $g \neq id_G$, and $a_{\mathrm{id}_G} = 2a_{\mathrm{id}_G}$. Hence, $x = 0$ and $P(H) = 0$ is the zero vector space.

b) We have that x is primitive if and only if

$$\Delta(x)(g \otimes h) = (1 \otimes x + x \otimes 1)(g \otimes h) \ \forall g, h \in G\,,$$

or equivalently, if and only of

$$x(gh) = x(g) + x(h)\,,$$

i.e., if and only if x is a group homomorphism.

c) First, note that $\Delta T = 1 \otimes T + T \otimes 1$, so $T \in P(k[T])$. Suppose that $x = \sum_{n=0}^{m} a_n T^n$. Note that:

$$\begin{aligned} 1 \otimes x + x \otimes 1 =& 2a_0 1 \otimes 1 + \sum_{n=1}^{m} a_n T^n \otimes 1 + \sum_{n=1}^{m} a_n 1 \otimes T^n \\ \Delta(x) =& \sum_{0\leq r,s\leq m} a_{r+s} \binom{r+s}{r} T^r \otimes T^s\,. \end{aligned} \tag{5}$$

By linear independence, we obtain that x is primitive if and only if

$$\begin{aligned} 2a_0 =& a_0 \\ a_n =& \binom{n}{0} a_n \ \forall n > 0 \\ 0 =& \binom{r+s}{r} a_{r+s} \ \forall r, s > 0\,. \end{aligned} \tag{6}$$

This readily implies that $a_0 = 0$. Now we study two cases separately: call $p = \operatorname{char} k$.

(a) **Case $p = 0$:** If $p = 0$ then $0 = \binom{r+s}{r} a_{r+s} \Rightarrow a_{r+s} = 0$, so we obtain that $a_n = 0$ for each $n \geq 2$. Consequently, $P(k[T]) =$ span $\{T\}$.

(b) **Case $p > 0$ prime number:** If $p > 0$ then we only have $0 = \binom{r+s}{r} p_{r+s} \Rightarrow a_{r+s} = 0$ whenever $\binom{r+s}{r}$ is not a multiple of p.
If n is not a power of p, take $r = p^k$ the biggest power of p smaller than n, and $s = n - r$. Then it follows from Kummer's theorem that $\binom{n}{r}$ is not a multiple of p, and so $a_n = 0$.
Also, if $n = p^k$ is a power of p, then, from Kummer's theorem, $\binom{n}{s}$ is a multiple of p for every positive $s < n$. We conclude that T^n is a primitive element of $k[T]$, and so we conclude that

$$P(k[T]) = \text{ span } \{T^n | n \text{ is a power of } p\}\,.$$

□