Homework Assignment 6 - Solution

Hopf algebras - Spring Semester 2018

Exercise 1

Let H be a bialgebra.

a) Show that H^{op} is a bialgebra. (Recall that for any algebra A we let A^{op} denote the algebra with $A^{\mathrm{op}} := \{a^{\mathrm{op}} \mid a \in A\}$ and $a^{\mathrm{op}}b^{\mathrm{op}} = (ba)^{\mathrm{op}}$ for all $a^{\mathrm{op}}, b^{\mathrm{op}} \in A^{\mathrm{op}}$.)

b) Show that H^{cop} is a bialgebra. (Recall that for any coalgebra C we let C^{cop} denote the coalgebra with $C^{\mathrm{cop}} := \{x^{\mathrm{cop}} \mid x \in C\}$ and $\Delta_{C^{\mathrm{cop}}}(x^{\mathrm{cop}}) = x_2^{\mathrm{cop}} \otimes x_1^{\mathrm{cop}}$ for all $x^{\mathrm{cop}} \in C^{\mathrm{cop}}$.)

c) Show that if H is a Hopf algebra then so is H^{opcop}.

d) Show that if H is a Hopf algebra with a bijective antipode, then so are H^{op} and H^{cop}.

Proof. These can be observed immediately by diagrams, but also by checking algebraically. Here we do the latter:

a) We need to show that μ^{op} and ι are comultiplicative for the opposite product, note that the unit is the same one so there is no need to check that ι is a comultiplicative.

$$\Delta(a) \cdot^{\mathrm{op}} \Delta(b) = \Delta b \cdot \Delta a = \Delta(ba) = \Delta(a \cdot^{\mathrm{op}} b)\,.$$
$$\epsilon(a \cdot^{\mathrm{op}} b) = \epsilon(b \cdot a) = \epsilon(b)\epsilon(a) = \epsilon(a)\epsilon(b)\,.$$

b) Similarly, we need only to show that Δ^{cop} and ϵ are multiplicative for the opposite product, and note again that the counit is the same so there is no need to check that it is multiplicative.

$$\Delta^{cop} a \cdot \Delta^{\mathrm{cop}} b = a_2 b_2 \otimes a_1 b_1 = (a \cdot b)_2 \otimes (a \cdot b)_1 = \Delta^{\mathrm{cop}}(a \cdot b)\,.$$
$$\Delta^{\mathrm{cop}}(1) = 1 \otimes 1\,.$$

c) We know that H^{opcop} is a bialgebra. We claim that if S is the antipode of H, then it is also the antipode of H^{opcop}. Indeed, it is trivial that

$$\mu^{\mathrm{op}} \circ (\mathrm{id} \otimes S) \circ \Delta^{\mathrm{cop}} = \mu \circ (S \otimes \mathrm{id}) \circ \Delta = \iota \circ \epsilon\,.$$
$$\mu^{\mathrm{op}} \circ (S \otimes \mathrm{id}) \circ \Delta^{\mathrm{cop}} = \mu \circ (\mathrm{id} \otimes S) \circ \Delta = \iota \circ \epsilon\,.$$

d) Now we claim that S^{-1} is the antipode of H^{op}. Indeed, recall that S is an algebra antihomomorphism, so

$$S \circ \mu^{\mathrm{op}} \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta = \mu \circ S \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes S) \circ \Delta = \iota \circ \epsilon \,,$$

and the desired is concluded after applying S^{-1} on both sides.

Similarly to show that $\mu^{\mathrm{op}} \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta = \iota \circ \epsilon$.

□

Exercise 2

Let H be a Hopf algebra and (A, δ) an H right comodule algebra. The elements of the subalgebra

$$A^{\mathrm{co}\ H} = \{a \in A \mid a_0 \otimes a_1 = a \otimes 1\}$$

are termed H-coinvariant. If the map

$$\mathrm{can} : A \otimes_{A^{\mathrm{co}\ H}} A \to A \otimes_{A^{\mathrm{co}\ H}} H, \quad x \otimes y \mapsto xy_0 \otimes y_1$$

is bijective, we say $A^{\mathrm{co}\ H} \subset A$ is an H Galois extension and A is H-Galois.

Now, let A be an H left module algebra. Recall that the smash product $A\#H$ is an H right comodule algebra via $\mathrm{id} \otimes \Delta$. Show that $A \subset A\#H$ is the subalgebra of H-coinvariant elements and that $A \subset A\#H$ is an H Galois extension.

Proof. First we observe that $A\#H^{\mathrm{co}H} = A$, note that $\delta(a\#1) = a\#\Delta 1 = a\#1 \otimes 1$. On the other hand, pick a basis $\{a_k\}_{k\in K}$ of A, and suppose that $\sum_{k\in K} a_k\#h_k \in A\#H^{\mathrm{co}H}$, then by hypothesis

$$\sum_{k\in K} a_k\#h_k \otimes 1 = \sum_{k\in K} a_k\#\Delta h_k \,,$$

and consequently, by linear independence, we have that $\Delta h_k = h_k \otimes 1$. Applying $(\epsilon \otimes \mathrm{id})$ on both sides yields $h_k = \epsilon(h_k)1$ so we conclude that

$$\sum_{k\in K} a_k\#h_k = \left(\sum_{k\in K} a_k \epsilon(h_k)\right) \otimes 1 \in A \,,$$

as desired.

To show that this is in fact a Galois extension, we will find the inverse of the map

$$\mathrm{can} : A\#H \otimes_A A\#H \to A\#H \otimes_A H, \quad \mathrm{can} : a\#g \otimes 1\#h \mapsto (a\#g) \cdot (1\#h_1) \otimes h_2$$

Note that we have

$$(a\#g) \cdot (1\#h_1) \otimes h_2 = (a(g_1 \cdot 1)\#g_2 h_1 \otimes h_2 = a\#\epsilon(g_1) g_2 h_1 \otimes h_2 = a\#gh_1 \otimes h_2 \,,$$

so

$$\text{can} : a\#g \otimes 1\#h \mapsto a\#gh_1 \otimes h_2$$

With this, the inverse that we propose is the following

$$\alpha : a\#g \otimes h \mapsto a\#gS(h_1) \otimes 1\#h_2$$

Indeed, note that

$$\begin{aligned} \alpha(\text{can}(a\#g \otimes 1\#h)) &= a\#gh_1S((h_2)_1) \otimes 1\#(h_2)_2 = a\#gh_1S(h_2) \otimes 1\#h_3 \\ &= a\#g1\epsilon(h_1) \otimes 1\#h_2 = a\#g \otimes 1\#\epsilon(h_1)h_2 \\ &= a\#g \otimes 1\#h\,. \end{aligned} \tag{1}$$

And also

$$\begin{aligned} (\alpha(a\#g \otimes h)) &= a\#gS(h_1)(h_2)_1 \otimes (h_2)_2 = a\#gS(h_1)h_2 \otimes h_3 \\ &= a\#g1\epsilon(h_1) \otimes h_2 = a\#g \otimes \epsilon(h_1)h_2 \\ &= a\#g \otimes h\,, \end{aligned} \tag{2}$$

concluding the proof. □

Exercise 3

Let $k \subset L$ be a Galois extension with Galois group $G = \mathrm{Aut}_k(L)$. Clearly G operates on L, making L a $k[G]$ left module algebra and hence a $k[G]^* = k^G$ right comodule algebra. Show that $k \subset L$ is a k^G Galois extension.

Proof. Let us first recall the Hopf algebra structures on $k[G] \cong k[G]^{**}$ and $k[G]^*$. Let $\{e_g\}_{g\in G}$ be the canonical basis of $k[G]$, so that $e_g e_h = e_{gh}$ and $\Delta e_g = e_g \otimes e_g$. Take $\{f_g\}_{g\in G}$ the dual basis of $\{e_g\}_{g\in G}$, so that $f_g(e_h) = \delta_{g,h}$, and note that

$$f_g f_h = \delta_{g,h} f_g\,.$$

$$\Delta f_g = \sum_{h_1h_2=g} f_{h_1} \otimes f_{h_2}\,.$$

Remark that if we take the dual basis of $\{f_g\}_{g\in G}$ we obtain again $\{e_g\}_{g\in G}$, so we can write $e_g(f_h) = \delta_{g,h}$.

The left $k[G]^{**}$-module algebra structure on L is exactly $g \cdot \alpha = g(\alpha)$, and to find it's adjungated $k[G]^*$-module coalgebra structure (L, δ) it needs to satisfy

$$e_g \cdot v = v_0 e_g(v_1)\,,$$

we note that $\delta(v) = \sum_{g\in G} g(v) \otimes f_g$ is the unique such structure.

Now we wish to show that $k \subset L$ is a $k[G]^*$-Galois extension. First, let's observe that $L^{\mathrm{co}\,k[G]^*} = k$. Indeed, note that $v \in L^{\mathrm{co}\,k[G]^*} \Leftrightarrow g(v) = v\ \forall g \in G$, and the only fixed points of all automorphisms is exactly k (this is the fundamental Galois theorem for the subgroup $G \subset G$ identified with the field extension $k \subset k \subset L$).

Now to show that

$$\mathrm{can}: L \otimes_k L \to L \otimes_k H, \quad v \otimes w \mapsto \sum_{g \in G} vg(w) \otimes f_g \,,$$

note first that both sides are $|G|^2$-dimensional k-vector spaces, so it is enough to establish injectivity.

Take $\sum_i v_i \otimes w_i \in \ker \mathrm{can}$, and let's recall that $\mathrm{Hom}_L(L \otimes_k L, L)$ has basis $\{v \otimes w \mapsto vg(w) = \mathrm{id} \odot g\}_{g \in G}$. Note that $\mathrm{can} \sum_i v_i \otimes w_i = \sum_{g \in G} (\sum_i v_i g(w_i)) \otimes f_g$. So $\sum_i v_i \otimes w_i \in \ker \mathrm{can} \Rightarrow \sum_i v_i \otimes w_i \mathrm{id} \odot g \Rightarrow \sum_i v_i \otimes w_i = 0$ since $\{\mathrm{id} \odot g\}_{g \in G}$ is a basis of $(L \otimes_k L)^*$. This concludes the proof. □

Exercise 4

Suppose that $\mathrm{char} k = p > 0$ and let $m, n \geq 1$, $\alpha, \beta \in k$. Show that

$$H = k < t \mid t^{p^{n+m}} = 0 >$$

is a commutative Hopf algebra with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \alpha t^{p^n} \otimes t^{p^m} + \beta t^{p^m} \otimes t^{p^n}.$$

Describe the affine algebraic group $\mathrm{Sp}(H)$.

Proof. We will see that H is in fact a bialgebra. First define $\epsilon(t^n) = \delta_{n,0}$ and $\Delta(t^n) = (\Delta(t))^n$. Recall that $\alpha \mapsto \alpha^p$ is a linear map, hence both functions are well defined in H, as we have

$$\epsilon(t^{p^{n+m}}) = 0 \,,$$

$$\Delta(t^{p^{n+m}}) = (\Delta(t))^{p^{n+m}} = t^{p^{n+m}} \otimes 1 + 1 \otimes t^{p^{n+m}} + \alpha t^{p^{2n+m}} \otimes t^{p^{2m+n}} + \beta t^{p^{2m+n}} \otimes t^{p^{2n+m}} = 0 \,.$$

So Δ and ϵ are well defined algebra homomorphisms, which endows H with a bialgebra structure.

It is easy to see, since H is the quotient of a free algebra, that

$$\mathrm{Sp}(H)(A) = Alg_k(H, A) \cong B \,,$$

where $B \subset A$ is the subalgebra of elements $a \in A$ such that $a^{p^{n+m}} = 0$.

Note that this is an affine group with respect to the addition, as for $a, b \in SP(H)(A)$ we have that $(a+b)^{p^{m+n}} = a^{p^{n+m}} + b^{p^{n+m}} = 0$. Also, H is a commutative bimonoid, so H is indeed a Hopf algebra. □