FURTHER BAIRE RESULTS ON THE DISTRIBUTION OF SUBSEQUENCES

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Abstract. This paper presents results about the distribution of subsequences which are typical in the sense of Baire.

The first main part is concerned with sequences of the type $x_k = n_k\alpha$, $n_1 < n_2 < n_3 < \cdots$, mod 1. Improving a result of Šalát we show that, if the quotients $q_k = n_{k+1}/n_k$ satisfy $q_k \geq 1+\varepsilon$, then the set of α such that (x_k) is uniformly distributed is of first Baire category, i.e. for generic α we do not have uniform distribution. Under the stronger assumption $\lim_{k\to\infty} q_k = \infty$ one even has maldistribution for generic α, the strongest possible contrast to uniform distribution. Nevertheless, growth conditions on the n_k alone do not suffice to explain various interesting phenomena. In particular, for individual sequences the situation maybe quite diverse: For $n_k = 2^k$ there is a set M such that for generic α the set of all limit measures of (x_k) is exactly M, while for $n_k = 2^k + 1$ such an M does not exist.

For the rest of the paper we consider appropriately defined Baire spaces S of subsequences. For a fixed well distributed sequence (x_n) we show that there is a set M of measures such that for generic $(n_k) \in S$ the set of limit measures of the subsequence (x_{n_k}) is exactly M.

1. Introduction

1.1. Motivation. This paper is a continuation of topological investigations contained in [GSW 00]. Let X be a compact metric space and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ etc. denote sequences on X, and

$$A(\mathbf{x}) := \bigcap_{n_0\geq 1} \overline{\{x_n :\ n \geq n_0\}}$$

the set of accumulation points of the sequence $\mathbf{x}$. We are interested in subsequences of $\mathbf{x}$, therefore we write $\mathbf{n} = (n_k)_{k\in\mathbb{N}}$ for sequences $0 < n_1 < n_2 < \cdots$ of positive integers and $\mathbf{xn} = \mathbf{x}\circ\mathbf{n} = (x_{n_k})_{k\in\mathbb{N}}$ for the corresponding subsequence of $\mathbf{x}$ induced by $\mathbf{n}$.

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$\mathscr{M}(X)$ denotes the set of Borel probability measures on X, equipped with the compact and metrizable topology of weak convergence. For the special case $X = [0,1]$ we simply write $\mathscr{P} = \mathscr{M}([0,1])$. Sometimes we write $\mu(i)$ for $\mu(\{i\})$ ($i \in X$, $\mu \in \mathscr{M}(X)$). Let, as usual, $\delta_x \in \mathscr{M}(X)$ denote the point measure concentrated in $x \in X$, i.e. $\delta_x(B) = 1$ for $x \in B$, $\delta_x(B) = 0$ for $x \notin B$, $B \subseteq X$ Borel.

In order to describe the distribution behavior of sequences we introduce the discrete measures

$$\mu_{\mathbf{x},N} = \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}$$

and define

$$M(\mathbf{x}) := A((\mu_{\mathbf{x},N})_{N\in\mathbb{N}}) \subseteq \mathscr{M}(X),$$

the set of so-called limit measures of $\mathbf{x}$. $\mathbf{x} \sim \lambda$ means that $M(\mathbf{x}) = \{\lambda\}$, i.e. the sequence $\mathbf{x}$ is *uniformly distributed* with respect to the measure $\lambda \in \mathscr{M}(X)$.

The set S_0 of strictly increasing sequences $\mathbf{n} = (n_k)_{k\in\mathbb{N}}$ of positive integers carries (via $\mathbf{n} \mapsto \sum_k 2^{-n_k}$) a natural measure theoretic structure as well as a metric and topological one. Thus for any given sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ on some compact metric space X it makes sense so say that a *typical* subsequence $\mathbf{xn} = (x_{n_k})_{k\in\mathbb{N}}$ has a certain property if the set of exceptional $\mathbf{n} \in S_0$ is small in the sense that either it has measure zero (measure theoretic), or small Hausdorff dimension (depending on the metric), or that it is meager (of first Baire category). In this paper we will mainly focus on the last, i.e. the topological point of view. Thus we will say that a typical or generic element has some property P if the set of elements with property P is residual, i.e., if the set of exceptions is meager. In [GSW 00] we have investigated the situation with respect to the set $M(\mathbf{x})$ of limit measures of a sequence (for formal definitions see below) which is a natural object to describe the distribution behavior of $\mathbf{x}$.

In the measure theoretic context the typical distribution of a subsequence is the same as for the original one, i.e. $M(\mathbf{xn}) = M(\mathbf{x})$. In particular, if $\mathbf{x}$ is uniformly distributed w.r.t. some measure λ, then the same holds for almost all subsequences (cf. also [LT 86]). In the topological context the situation is quite different, namely: Provided all $x \in X$ are accumulation points of $\mathbf{x}$ then $M(\mathbf{xn}) = \mathscr{M}(X)$ for a generic $\mathbf{n} \in S_0$.

Note the analogy to the following facts. Consider the product space $X^{\mathbb{N}}$ of all sequences on X, equipped with the product measure $\lambda^{\mathbb{N}}$ induced by some fixed probability measure λ on X. As a consequence of the strong law of large numbers, $\lambda^{\mathbb{N}}$-almost all $\mathbf{x}$ are λ-uniformly distributed, i.e. $M(\mathbf{x}) = \{\lambda\}$, while $M(\mathbf{x}) = \mathscr{M}(X)$ for $\mathbf{x} \in X^{\mathbb{N}}$ generic in the Baire sense. Extending a concept from [M 93], sequences $\mathbf{x}$ with $M(\mathbf{x}) = \mathscr{M}(X)$ have been called *maldistributed* in [Wi 97].

Thus the situation is, roughly spoken, as follows: Almost all (sub)sequences are regularly (uniformly) distributed, but generic sequences are irregularly distributed (maldistributed). The topic of Section 3 is a refined analysis of this topological maldistribution phenomenon.

1.2. Kuratowski-Ulam's theorem and $n\alpha$-sequences. The theorem of Kuratowski-Ulam is the topological counterpart to the measure theoretic Fubini theorem on product spaces. Recall that a Polish space is a complete separable metric space.

Proposition 1.1. (Kuratowski-Ulam) *Let A, B be Polish spaces and let $M \subseteq A \times B$ be a Borel set. Furthermore let, for each $a \in A$, ${}_aM = \{b \in B : (a, b) \in M\}$ and for each $b \in B$, $M_b = \{a \in A : (a, b) \in M\}$. Then the following statements are equivalent:*

(1) *M is meager in $A \times B$.*
(2) *The set of all $b \in B$ such that M_b is not meager in A is meager in B.*
(3) *The set of all $a \in A$ such that ${}_aM$ is not meager in B is meager in A.*

For proofs and much more background we refer to [O 80].

For our context, think about the spaces $A = S_0$ and $B = X = \mathbb{R}/\mathbb{Z}$ (unit circle, one dimensional torus). For each point $(\mathbf{n}, \alpha) \in S_0 \times X$ we are interested in $M(\mathbf{n}\alpha)$ where $\mathbf{n}\alpha = (n_k\alpha)_{k\in\mathbb{N}}$. The sequence $\mathbf{n}\alpha$ is uniformly distributed w.r.t. Lebesgue (Haar) measure λ, hence dense in X for every irrational α. (Of course $\alpha \in X = \mathbb{R}/\mathbb{Z}$ is called irrational if it is a remainder class consisting of irrational numbers.) Theorem 1.3 in [GSW 00] says that the typical subsequence of a dense sequence is maldistributed, hence for each irrational α the equality $M(\mathbf{n}\alpha) = \mathscr{M}(X)$ holds for a generic $\mathbf{n}$. Since rationals, forming a countable set, are of first category, this shows that the third condition in Kuratowski-Ulam's theorem is satisfied. This yields that also the other two conditions hold. The first one translates to the statement that maldistribution holds for a generic $(\mathbf{n}, \alpha) \in S_0 \times X$. The second one, finally, reads as follows: The set $R \subseteq S_0$ of all $\mathbf{n}$ such that the sequence $\mathbf{n}\alpha$ is maldistributed for a generic α is residual. For a fixed $\mathbf{n} = (n_k)_{k\in\mathbb{N}}$ it might be much more difficult to decide whether it is in R. Section 2 will be devoted to this topic, in particular for sequences satisfying growth conditions.

1.3. Contents of the paper. The theorem of Kuratowski-Ulam motivates two types of questions. They correspond to the main sections of this paper, which can be read independently of each other.

Question 1: Given $\mathbf{n}$, can we make assertions on $M(\mathbf{n}\alpha)$ for generic α? (Section 2)
Question 2: Given α (or more generally $\mathbf{x}$ with certain known distribution properties), can we make assertions on $M(\mathbf{n}\alpha)$ for generic $\mathbf{n}$? (Section 3 treats a refinement of this question.)

Concerning Question 1, it is clear that for sequences $\mathbf{n}$ with positive lower density the distribution of $\mathbf{n}\alpha$ cannot be arbitrarily irregular. (However, see 2.7.)

This indicates that very strong irregularity results (stronger for instance than Theorem 2.7) can be expected only if the sequence $\mathbf{n}$ grows fast enough. A positive result into this direction is Theorem 1.1 from [S 00]: If $n_{k+1} = a_k n_k$ with $a_k \in \{2, 3, \ldots\}$ for all k then $\mathbf{n}\alpha$ is not uniformly distributed for generic α. Our Theorem 2.4 tells us that the same conclusion holds under the weaker assumption $\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$. Under the stronger growth condition $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty$ we can even obtain maldistribution for typical α (Theorem 2.6).

For arbitrary $\mathbf{n}$ the situation is not clear. We illustrate this by contrasting the cases $n_k = 2^k$ and $n_k = 2^k + 1$ (Theorem 2.8).

Thus the following very general problem might be an initial point for future research.

Problem 1: For which sequences $\mathbf{n}$ is there a set M of measures such that $M(\mathbf{n}\alpha) = M$ for generic α?

The rest of the paper (Section 3) is motivated by Question 2. To understand our approach, recall first that Theorem 1.3 in [GSW 00] gives a complete answer to the question as stated above: Given α, a generic subsequence takes as limit measures all Borel measures. To get deeper insights we look at appropriate closed subspaces S of the Baire space S_0 of all $\mathbf{n}$. Varying the subspace S one tries to get different sets $M(S)$ of measures such that

$$(*) \qquad M(\mathbf{xn}) = M(S) \text{ for generic } \mathbf{n} \in S.$$

This indeed works for all S from a certain class of subspaces, each of them induced by a given interval partition $(I_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ and a sequence $m_1, m_2, \ldots \in \mathbb{N}$ by the requirement that each I_j contains exactly m_j elements from $\mathbf{n} \in S$.

To save notation at this place we refer to Section 3 for more precise statements. (Note the analogy to stochastic processes as Markov chains where the probability measure on the space of sequences is not the mere product measure but may be supported on some small, i.e. nowhere dense closed subspace.)

The essential property we will use in the proof is that the sequences $n\alpha$ are not merely uniformly distributed but even well distributed (cf. [KN 74] or [DT 97]). Thus Section 3 will be presented in this more general context.

Our results (Theorems 3.1 and 3.2) are just first examples for a topic which might deserve further investigations in future research. To make such projects more concrete we pose the following problems:

Problem 2: Our results only depend on the well distribution property but do not make further use of the arithmetic structure of $n\alpha$-sequences. Thus it seems desirable to find interesting classes of subspaces S allowing results of the above type with more number theoretic impact.

Problem 3: Sets $M(S)$ as in $(*)$ cannot exist for arbitrary closed subspaces $S \subseteq S_0$. (Every disjoint union $S = S_1 \cup S_2$ with $M(S_1) \neq M(S_2)$ works as a counterexample.) Is it possible to characterize those S for which there is $\mathbf{x}$ such that $(*)$ holds?

2. Sparse subsequences of $(n\alpha)$

In this section λ denotes the Lebesgue (Haar) measure on $\mathbb{R}/\mathbb{Z}$.

2.1. Statement of the main results of this section. In this section we consider the distribution behavior of sparse subsequences of $(n\alpha)_{n\in\mathbb{N}}$. In [S 00], (essentially) the following has been proved:

Proposition 2.1. (Šalát) *Let $\mathbf{n} = (n_0, n_1, \ldots)$ be a sequence of natural numbers satisfying $n_{k+1} \geq 2n_k$ for all k. Then the set*

$$\mathscr{U} := \{\, \alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is uniformly distributed w.r.t. } \lambda \,\}$$

is meager.

We will improve this result by weakening the growth condition on the sequence $\mathbf{n}$.

Definition 2.2. For any sequence $\mathbf{x} = (x_n)$ and any interval I, we define $\bar{\mu}_{\mathbf{x}}(I)$ by

$$\bar{\mu}_{\mathbf{x}}(I) := \sup\{\, \mu(I) : \mu \in M(\mathbf{x}) \,\}.$$

Remark 2.3. Note that $\bar{\mu}_{\mathbf{x}}(I) \geq \limsup_{n\to\infty} \mu_{\mathbf{x},n}(I)$ while equality does not hold in general: Take $x_n = \frac{1}{n}$ and $I = \{0\}$, then $M(\mathbf{x}) = \{\delta_0\}$, $\delta_0(I) = 1$ but $\mu_{\mathbf{x},n}(I) = 0$ for all n.

Theorem 2.4. *Let $\mathbf{n} = (n_0, n_1, \ldots)$ be a sequence of natural numbers, and assume $q := \liminf_k (n_{k+1}/n_k) > 1$. Then the set*

$$\mathscr{U} := \{\, \alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is uniformly distributed w.r.t. } \lambda \,\}$$

is meager.

Moreover: There is a number $Q > 0$ such that for all intervals I the set

$$\{\, \alpha : \bar{\mu}_{\mathbf{n}\alpha}(I) > \frac{Q}{-\log \lambda(I)} \,\}$$

is residual.

Equivalently, the set $\{\alpha : \forall I\ \bar{\mu}_{\mathbf{n}\alpha}(I) > \frac{Q}{-\log\lambda(I)}\}$ is residual.

Remark 2.5.
(1) The sentence "Equivalently …" follows from the previous sentence because it is enough to prove this for intervals with rational end points.
(2) Note that for short intervals I we have $\frac{Q}{-\log\lambda(I)} \gg \lambda(I)$.
(3) Choosing Q small enough, the inequality $\bar{\mu}_{\mathbf{n}\alpha}(I) > \frac{Q}{-\log\lambda(I)}$ will be trivially true for large intervals I, say for all I with $\lambda(I) \geq \frac{1}{q}$. So it is enough to consider only intervals I with $\lambda(I) < \frac{1}{q}$.

(4) Results from [AHK 83] or [B 83] show that the growth condition in Theorem 2.4 cannot be weakened. Boshernitzan for instance shows that for every sequence of integers m_k with $\lim_{k\to\infty} \sqrt[k]{m_k} = 1$ there are $n_k \geq m_k$ such that $(n_k\alpha)_{k\in\mathbb{N}}$ is uniformly distributed mod 1 for all irrational α.

Idea of the proof of Theorem 2.4: Fix a short interval I. Let c be large with respect to q and I (see below for details). If we consider only every c-th term in the sequence $\mathbf{n}$, i.e., the sequence $\mathbf{n}' = (n'_k)_{k\in\mathbb{N}}$ with $n'_k = n_{ck}$, then the n'_k will increase so fast that

(1) $\quad R := \{\alpha : \{n'_k\alpha, \ldots, n'_{2k-1}\alpha\} \subseteq I$ for infinitely many k $\}$ is residual.

So for $\alpha \in R$, $\bar{\mu}_{\mathbf{n}'\alpha}(I) \geq \frac{1}{2}$, and $\bar{\mu}_{\mathbf{n}\alpha}(I) \geq \frac{1}{2c}$.
Upon closer inspection we see that $c \approx \frac{1}{-\log_q \lambda(I)}$ is sufficient for (1).
Similar methods will be used in the proof of Theorem 2.6.

Theorem 2.6. *Let $\mathbf{n} = (n_0, n_1, \ldots)$ be a sequence of natural numbers, and assume $\lim_k(n_{k+1}/n_k) = \infty$. Then the set*

$$\{\, \alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is maldistributed}\}$$

is residual.

Weaker versions of irregular distribution also occur for certain classes of slowly increasing $n_1 < n_2 < \ldots$. The following two theorems elaborate on remarks of the referee of a previous version of this paper, for which we are grateful:

Theorem 2.7. *Let X be the set of all increasing $\mathbf{n} = (n_k)_{k\in\mathbb{N}}$ of integers with $n_{k+1} - n_k \in \{1, 2\}$. Then the set $\{(\alpha, \mathbf{n}) \in \mathbb{R} \times X : \mathbf{n}\alpha$ is not u.d.$\}$ is residual in $\mathbb{R} \times X$.*

More refined investigations in this spirit will be the content of Section 3.

The last result of this section indicates that for individual sequences of $\mathbf{n}$ the situation can be very diverse and hence complicated:

Theorem 2.8. *For $\mathbf{n} = (2^k)_{k\in\mathbb{N}}$ there is a set $M \subseteq \mathscr{M}(X)$ such that $M = M(\mathbf{n}\alpha)$ for generic $\alpha \in X$. M contains exactly all measures which are invariant under $x \mapsto 2x$. In contrast, for $\mathbf{n}' = (n'_k)_{k\in\mathbb{N}}$ with $n'_k = 2^k + 1$, there is no $M' \subseteq \mathscr{M}(X)$ such that $M' = M(\mathbf{n}'\alpha)$ for generic $\alpha \in X$.*

For related constructions yielding results in terms of Hausdorff dimension we refer to [P 79].

2.2. Notation. For notational convenience we sometimes identify $\alpha + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ with the unique representative $\alpha \in [0, 1) \subseteq \mathbb{R}$. Very often we are in the situation that an intersection $I \cap B$ of an interval I with a Borel set B is residual in I. Note that this can be interpreted as a generalized implication of the type:

Except for a meager set, $x \in I$ implies $x \in B$.

Therefore we introduce the following notation.

Definition 2.9. For an open interval I and a Borel set B we write

$$I \Vdash B$$

as an abbreviation for "$B \cap I$ is residual in I" or equivalently, "$I \setminus B$ is meager". We read this also as "the typical element of I is in B".

The following fact is a folklore consequence of Baire's theorem:

Fact 2.10. Let I be an open interval.

(1) If B_n is a Borel set for every $n \in \{0, 1, 2, \ldots\}$, and $I \cap \bigcup_n B_n$ is residual in I, then there is some open nonempty $J \subseteq I$ and some n such that B_n is residual in J, or abbreviated:
$$I \Vdash \bigcup_{n \in \mathbb{N}} B_n \;\Rightarrow\; \exists J \subseteq I \,\exists n \in \mathbb{N} : \; J \Vdash B_n$$

(2) If B_n is a Borel set for every $n \in \{0, 1, 2, \ldots\}$, then $I \cap \bigcap_n B_n$ is residual in I iff each $I \cap B_n$ is residual in B_n:
$$I \Vdash \bigcap_{n \in \mathbb{N}} B_n \;\Leftrightarrow\; \forall n \in \mathbb{N} : \; I \Vdash B_n$$

(3) If B is a Borel set then $B \cap I$ is *not* residual in I iff there is some open interval $J \subseteq I$ such that B is meager in J:
$$I \nVdash B \;\Leftrightarrow\; \exists J \subseteq I : J \Vdash (-B).$$
(Here we write $-B$ for the complement of B.)

Proof. Let $\mathbf{B}$ be the family of all sets with the Baire property, i.e., all sets which can be written as $A \,\Delta\, M$, where A is an open set, M is meager, and Δ denotes the symmetric difference of two sets. Then clearly

- $\mathbf{B}$ contains all open sets
- $\mathbf{B}$ is closed under countable unions
- $\mathbf{B}$ contains all closed sets, as each closed set C can be written as $A \,\Delta\, M$, where A is the open kernel of C and $M = \partial C = C \setminus A$ is nowhere dense
- $\mathbf{B}$ is closed under complements: If $X = A \,\Delta\, M$, then $(-X) = (-A) \,\Delta\, M$; write $-A$ as $A' \,\Delta\, M'$ with A' open, M' meager, then $(-X) = (A' \,\Delta\, M') \,\Delta\, M = A' \,\Delta\, (M' \,\Delta\, M)$, where $M' \,\Delta\, M$ is meager.

Hence $\mathbf{B}$ contains all Borel sets.

To prove (1), write each B_n as $A_n \Delta M_n$ with A_n open and M_n meager. Not all A_n can be empty (otherwise the set $\bigcup_n B_n = \bigcup_n M_n$ would be meager); let J be an interval contained in any nonempty A_n.

(2) is easy.

To prove (3), assume that $B \cap I$ is not residual, and write $I \setminus (B \cap I)$ as $A \,\Delta\, M$ for some open A and meager M; as $I \setminus (B \cap I)$ is not meager, A is not empty; let J be any nonempty open interval with $J \subseteq A$. $\square$

Definition 2.11. We say that a family $(f_1, f_2, \ldots)$ of functions $f_i : [0,1] \to [0,1)$ is ε-mixing if: whenever $J_1, J_2, \ldots$ are intervals of length ε, then for all $k \in \mathbb{N}$:

$$\bigcap_{n=1}^{k} f_n^{-1}(J_n) \text{ contains an inner point.}$$

More generally we say that $(f_1, f_2, \ldots)$ is ε-mixing in δ if: for all sequences $J_1, J_2, \ldots$ of intervals of length ε, and all $k \in N$, and all intervals J' of length δ:

$$J' \cap \bigcap_{n=1}^{k} f_n^{-1}(J_n) \text{ contains an inner point.}$$

Remark 2.12. Although we work in $I = [0,1)$, we identify elements in I with their equivalence classes in $\mathbb{R}/\mathbb{Z}$, so an (open) interval can either be of the form (a,b) or of the form $[0,a) \cup (b,1)$ (for $0 \le a < b \le 1$). However, since we are mainly concerned with very short intervals, it is no loss of generality to only consider intervals of the first form.

2.3. Proof of Theorem 2.4.

Lemma 2.13. *Let $f_k : [0,1) \to [0,1)$ be the function mapping α to $n_k\alpha$ modulo 1, where $\mathbf{n} = (n_k)_{k\in\mathbb{N}}$ is a sequence of natural numbers satisfying*

(1) *$n_{k+1} > \frac{2}{\varepsilon} n_k$ for all k.*
(2) *$n_1 > \frac{\varepsilon}{\delta}$.*

Then $(f_1, f_2, \ldots)$ is ε-mixing in δ.

Proof. Let $J_1, J_2, \ldots$ be intervals of length ε, J' an interval of length δ.

We will show (by induction on k) that each set

$$J' \cap \bigcap_{i=1}^{k} f_i^{-1}(J_i)$$

contains in fact an interval I_k of length ε/n_k. This is clear for $k = 0$, as the length of J' is $\delta > \varepsilon/n_0$.

Consider $k > 0$. Note that $f_k^{-1}(J_k) = \bigcup_{j=1}^{n_k}\{\alpha \in [0,1) : n_k\alpha - j \in J_k\}$ is a union of n_k many disjoint intervals, each of length ε/n_k.

By inductive assumption, the set $J' \cap \bigcap_{i=1}^{k-1} f_i^{-1}(J_i)$ contains an interval I_{k-1} of length ε/n_{k-1}:

$$I_{k-1} \subseteq J' \cap \bigcap_{i=1}^{k-1} f_i^{-1}(J), \qquad \lambda(I_{k-1}) = \frac{\varepsilon}{n_{k-1}}$$

Since $\varepsilon/n_{k-1} > 2/n_k$, we can find a natural number $j < n_k$ such that the interval

$$[\frac{j}{n_k}, \frac{j+1}{n_k}] = \{\, \alpha : n_k\alpha - j \in [0,1] \,\}$$

is contained in I_{k-1}. Hence the set

$$I_k := \{\, \alpha : n_k\alpha - j \in J_k \,\},$$

an interval of length ε/n_k, is also contained in I_{k-1}. $\square$

Proof of Theorem 2.4. Choose $Q > 0$ so small that $(\frac{1}{4Q} - 1) - \log 2 > 1$.

Without loss of generality we may assume $\forall k : \frac{n_{k+1}}{n_k} > q$.

Let $\varepsilon := \lambda(I)$. By 2.5(3), we may assume $\varepsilon < \frac{1}{q}$, so $(-\log \varepsilon) > 1$. (In this proof, log denotes the logarithm with base q.)

So we have $(\frac{1}{4Q} - 1) \cdot (-\log \varepsilon) - \log 2 > 1$, hence the interval

$$(\log 2 - \log \varepsilon, -\frac{1}{4Q} \log \varepsilon)$$

has length > 1. Let c be an integer in this interval. Thus,

- $q^c > \frac{2}{\varepsilon}$
- $\dfrac{1}{2c} > \dfrac{2Q}{-\log \varepsilon}$

Now assume that the theorem is false. Since the set $\{\alpha : \bar{\mu}_{\mathbf{n}\alpha}(I) > \frac{Q}{-\log \varepsilon}\}$ is a Borel set and not residual, by 2.10(3) we know that its complement will be residual in I, for some open interval I:

$$I \Vdash \left\{ \alpha : \bar{\mu}_{\mathbf{n}\alpha}(I) \le \frac{Q}{-\log \varepsilon} \right\}$$

Now, by Remark 2.3, the set $\left\{ \alpha : \bar{\mu}_{\mathbf{n}\alpha}(I) \le \frac{Q}{-\log \varepsilon} \right\}$ is contained in the set

$$\left\{ \alpha : \exists m \, \forall N \ge m : \mu_{\mathbf{n}\alpha,N}(I) < \frac{2Q}{-\log \varepsilon} \right\}.$$

We will write $Z_N(\alpha)$ for the set $\{j < N : n_j\alpha \in I\}$. So $\mu_{\mathbf{n}\alpha,N}(I) = \frac{\# Z_N(\alpha)}{N}$ and we have

$$I \Vdash \bigcup_m \bigcap_{N \ge m} \left\{ \alpha : \frac{\# Z_N(\alpha)}{N} < \frac{2Q}{-\log \varepsilon} \right\}.$$

By 2.10(1), we can find an open interval $J \subseteq I$ and a k^* such that

$$J \Vdash \bigcap_{N \ge k^*} \left\{ \alpha : \frac{\# Z_N(\alpha)}{N} < \frac{2Q}{-\log \varepsilon} \right\}.$$

In other words: for all $N \ge k^*$:

$$J \Vdash \left\{ \alpha : \frac{\# Z_N(\alpha)}{N} < \frac{2Q}{-\log \varepsilon} \right\}. \tag{2}$$

Let $\delta := \lambda(J)$. Wlog we have $n_{k^*c} > \frac{\varepsilon}{\delta}$ (otherwise we just increase k^*).

Now we consider the functions $f_{k^*c}, f_{(k^*+1)c}, f_{(k^*+2)c}, \ldots, f_{(2k^*-1)c}$. Since

$$\frac{n_{(k^*+i+1)c}}{n_{(k^*+i)c}} \geq q^c > \frac{2}{\varepsilon},$$

and $n_{k^*c} > \frac{\varepsilon}{\delta}$, these functions are ε-mixing in δ (Lemma 2.13).

So there is an open interval $K \subseteq J$ such that for all $\alpha \in K$, and all $i \in \{0, \ldots, k^*\}$:

$$\alpha \in f^{-1}_{n_{(k^*+i)c}}(I) \qquad \text{i.e.,} \quad n_{(k^*+i)c}\alpha \in I.$$

Thus $\forall \alpha \in K$:

$$\#Z_{2k^*c}(\alpha) = \#\{i < 2k^*c : n_i\alpha \in I\} \geq \#\{\, k^*c, (k^*+1)c, \ldots, (2k^*-1)c \,\} = k^*.$$

Hence for $\alpha \in K$:

$$\frac{\#Z_{2k^*c}(\alpha)}{2k^*c} > \frac{1}{2c}. \tag{3}$$

However, $\frac{1}{2c} > \frac{2Q}{-\log \varepsilon}$ and $K \subseteq J$, so we get from (2) for $N := 2k^*c$:

$$K \Vdash \left\{ \alpha : \frac{\#Z_{2k^*c}(\alpha)}{2k^*c} \leq \frac{1}{2c} \right\}. \tag{4}$$

Now consider the set $\left\{ \alpha : \frac{\#Z_{2k^*c}(\alpha)}{2k^*nc} \leq \frac{1}{2c} \right\} \cap K$. By (3), this set is empty, but by (4) it is residual in K; this is a contradiction. $\square$

2.4. Proof of Theorem 2.6.

Fact 2.14. For any sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$, the set $M(\mathbf{x})$ is closed.

Fact 2.15. For any sequence $\mathbf{n} = (n_k)_{k \in \mathbb{N}}$, the set

$$M := \{\mu \in \mathscr{P} : \text{ for typical } \alpha,\ \mu \in M(\mathbf{n}\alpha)\,\}$$

is closed in $\mathscr{P} = \mathscr{M}([0,1])$.

Proof. We show that M is closed under limits of sequences. So let $\mu_n \to \mu$, with all $\mu_n \in M$. Let

$$A_n := \{\alpha : \mu_n \in M(\mathbf{n}\alpha)\}$$

Now $\mu_n \in M$ just means that A_n is residual; so $A := \bigcap_n A_n$ is also residual, and by 2.14 we have $\mu \in M(\mathbf{n}\alpha)$ for all $\alpha \in A$. $\square$

Definition 2.16. For any list $\vec{e} = (e_0, \ldots, e_{\ell-1})$ of natural numbers, and any $\eta > 0$ we let

$$M_{\vec{e},\eta} := \left\{ \mu \in \mathscr{P} : \ \forall i \in \{0, \ldots, \ell-1\}\ \left|\mu([\frac{i}{\ell}, \frac{i+1}{\ell})) - \frac{e_i}{e}\right| < \eta \right\}$$

and $e := \sum e_i$.

By 2.15, the following are equivalent for any $\mathbf{n}$:

(i) The set $\{\alpha : M(\mathbf{n}\alpha) = \mathscr{P}\}$ is residual.

(ii) For each $\vec{e}$ and each η, the set $\{\alpha : M(\mathbf{n}\alpha) \cap M_{\vec{e},\eta} \neq \emptyset\}$ is residual.
(iii) For each $\vec{e}$ and each η, the set $\{\alpha : \exists^\infty N\, \mu_{\mathbf{n}\alpha,N} \in M_{\vec{e},\eta}\}$ is residual.

Proof of Theorem 2.6. Assume that property (iii) above fails. As in the proof of 2.4, this means that we can find a nonempty interval I, a natural number N_0, a sequence $\vec{e} = (e_0, \dots, e_{\ell-1})$ of natural numbers, and a real number η such that

$$I \Vdash \{\alpha : \forall N \geq N_0 : \mu_{\mathbf{n}\alpha,N} \notin M_{\vec{e},\eta}\}. \tag{5}$$

Clearly we may assume $N_0 > \frac{1}{\eta}$, that $e := \sum e_i$ divides N_0, and that

$$\frac{n_{k+1}}{n_k} > 2\ell, \quad n_{N_0} > \frac{1}{\lambda(I)}. \tag{6}$$

Choose a sequence $(I_j : j = 1, \dots, N_0^2)$ of intervals such that for each $i \in \{0, \dots, \ell - 1\}$ the set

$$\{j \in \{1, \dots, N_0^2\} : \ I_j = [\frac{i}{\ell}, \frac{i+1}{\ell})\}$$

has cardinality $\frac{e_i}{e} N_0^2$. So each I_j has length $\frac{1}{\ell}$.

Let $f_j(x) = n_j x$ for $j \in \{N_0 + 1, \dots, N_0^2\}$. By (6) and Lemma 2.13 these functions are $\frac{1}{\ell}$-mixing in $\lambda(I)$, so we can find an interval

$$J \subseteq I \cap \bigcap_{j=N_0+1}^{N_0^2} f_j^{-1}(I_j).$$

We now claim that

$$\forall \alpha \in J : \mu_{\mathbf{n}\alpha,N_0^2} \in M_{\vec{e},\eta}, \tag{7}$$

which clearly contradicts (5).

Indeed, let $\alpha \in J$. Then for any $j \in \{N_0 + 1, \dots, N_0^2\}$ we have $f_j(\alpha) \in I_j$, hence (writing $O(1)$ for a quantity that lies between -1 and 1) we get

$$\mu_{\mathbf{n}\alpha,N_0^2}([\frac{i}{\ell}, \frac{i+1}{\ell})) = \frac{1}{N_0^2}\big(\frac{e_i}{e} N_0^2 + O(1)N_0\big) = \frac{e_i}{e} + \frac{O(1)}{N_0},$$

so $\mu_{\mathbf{n}\alpha,N_0^2} \in M_{\vec{e},\eta}$. $\square$

2.5. Proof of Theorem 2.7. The set in question has the Baire property, and we will show that each vertical section is residual.

So fix $\alpha \in (0, 1)$. We can find a number $\varepsilon > 0$ such that the intervals $(0, \varepsilon)$ and $(\alpha, \alpha + \varepsilon)$ (computed modulo 1) are disjoint.

We claim

> Whenever $(n_1, \dots, n_k)$ is a finite sequence in which $n_{j+1} - n_j \in \{1, 2\}$ holds for all $j < k$,
> there is an infinite extension $(n_1, \dots, n_k, n_{k+1}, n_{k+2}, \dots) \in X$ such that
> $$\forall j > k : \ n_j \alpha \notin (0, \varepsilon).$$

This claim implies that for each k the closed set

$$\bigcap_{j>k}\{\vec{n}:\frac{|\{i<j:n_i\alpha\in(0,\varepsilon)\}|}{j}\geq\frac{\varepsilon}{2}\}$$

is nowhere dense, so the set $\{\vec{n}:\vec{n}\alpha \text{ is u.d.}\}$ is meager.

Proof of the claim: We can construct the numbers $n_{k+1}, n_{k+2}, \ldots$ by induction. Given n_j, we either have $(n_j+1)\alpha\notin(0,\varepsilon)$ — in that case we may choose $n_{j+1}:=n_j+1$. Or we have $(n_j+1)\alpha\in(0,\varepsilon)$ — in that case we have $(n_j+2)\alpha\in(\alpha,\alpha+\varepsilon)$, hence $(n_j+2)\alpha\notin(0,\varepsilon)$, so we may choose $n_{j+1}:=n_j+2$.

2.6. Proof of Theorem 2.8. For Theorem 2.8 it suffices to prove the four statements of the following lemma. Recall that we focus on the sequences $\mathbf{n}=(n_k)_{k\in\mathbb{N}}$ with $n_k=2^k$ and $\mathbf{n}'=(n'_k)_{k\in\mathbb{N}}$ with $n'_k=2^k+1$.

Lemma 2.17.
(1) *Let X be any compact metric space, $T:X\to X$ continuous, $x\in X$, $\mathbf{x}=(T^n x)_{n\in\mathbb{N}}$ and $\mu\in M(\mathbf{x})$. Then μ is T-invariant.*
(2) *Let $\mu\in\mathscr{M}(X)$ be T-invariant for $T:X\to X$, $x\mapsto 2x$, on $X=\mathbb{R}/\mathbb{Z}$. Then $\mu\in M(\mathbf{n}\alpha)$ for generic $\alpha\in X$.*
(3) *If $\alpha\in(0,\frac{1}{16})$ then $\mu\in M(\mathbf{n}'\alpha)$ implies $\mu(I)\leq\frac{5}{6}$ for $I:=(\frac{1}{2},\frac{3}{4})$.*
(4) *For generic $\alpha\in(\frac{1}{2},\frac{3}{4})$ there is a $\mu\in M(\mathbf{n}'\alpha)$ with $\mu(I)=1$ for $I:=(\frac{1}{2},\frac{3}{4})$.*

Proof of Theorem 2.8. Assume that Lemma 2.17 holds. Let M denote the set of all T-invariant measures $\mu\in\mathscr{M}(\mathbb{R}/\mathbb{Z})$ for $T:x\mapsto 2x$. Then the first statement of the lemma tells us that $M(\mathbf{n}\alpha)\subseteq M$ for all $\alpha\in X$. Conversely, the second statement guarantees that for each $\mu\in M$ the set $R_\mu=\{\alpha:\ \mu\in M(\mathbf{n}\alpha)\}$ is residual. There is an at most countable set $M_0=\{\mu_n:\ n\in\mathbb{N}\}$ with $\overline{M_0}=M$. Let $R=\bigcap_{n\in\mathbb{N}}R_{\mu_n}$. Then R is residual and $M_0\subseteq M(\mathbf{n}\alpha)$ for all $\alpha\in R$. Since every set of the form $M(\mathbf{x})$ is closed we have $M=\overline{M_0}\subseteq M(\mathbf{n}\alpha)$ for all such α, hence $M(\mathbf{n}\alpha)=M$ for residual $\alpha\in X$, establishing the first two sentences in Theorem 2.8, while the third sentence follows by combining the third and the fourth statement of the Lemma. Thus Theorem 2.8 indeed follows from Lemma 2.17. □

We are now going to prove the four statements of Lemma 2.17.

Proof of statement (1) of Lemma 2.17. All we have to prove is $\int f\,d\mu=\int f\circ T\,d\mu$ for any continuous $f:X\to\mathbb{R}$. $\mu\in M(\mathbf{x})$ means that $\lim_{k\to\infty}\mu_{\mathbf{x},n_k}=\mu$ for some $n_1<n_2<\ldots\in\mathbb{N}$. Thus we easily obtain

$$\begin{aligned}\int f\,d\mu &= \lim_{k\to\infty}\tfrac{1}{n_k}\textstyle\sum_{j=1}^{n_k}f(T^jx)=\lim_{k\to\infty}\tfrac{1}{n_k}\sum_{j=2}^{n_k+1}f(T^jx)=\\ &= \lim_{k\to\infty}\tfrac{1}{n_k}\textstyle\sum_{j=1}^{n_k}f\circ T(T^jx)=\int f\circ T\,d\mu.\end{aligned}$$

□

Proof of the statement (2) of Lemma 2.17. Birkhoff's ergodic theorem guarantees that, for each continuous $f : X \to \mathbb{R}$, the set R_f of all $\alpha \in X$ with

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} f(2^j \alpha) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} f(T^j \alpha) = \int f \, d\mu$$

has full measure $\mu(R_f) = 1$. Let $f_1, f_2, \dots$ be any sequence of continuous $f : X \to \mathbb{R}$, then $\mu(R_\mu) = 1$ for $R_\mu := \bigcap_{n\in\mathbb{N}} R_{f_n}$, in particular $R_\mu \neq \emptyset$. We may assume that the f_i, $i \in \mathbb{N}$, are $\|\cdot\|_\infty$-dense in the space of all continuous $f : X \to \mathbb{R}$. Take any $\alpha_0 \in R_\mu$. Then $M(\mathbf{n}\alpha_0) = M((2^k\alpha_0)_{k\in\mathbb{N}}) = \{\mu\}$. In order to show $\mu \in M(\mathbf{n}\alpha)$ for generic α assume that $I \subseteq X = [0, 1)$ is any nonempty open subset of X. Then I contains an interval $I_0 = (\frac{k-1}{2^{k_0}}, \frac{k}{2^{k_0}})$ for some $k_0 \in \mathbb{N}$ and $k \in \{1, 2, \dots, k_0\}$. X is a compact metrizable space, hence so is $\mathscr{M}(X)$ and there is a sequence $U_1 \supset U_2 \supset \dots$ of open $U_i \subseteq \mathscr{M}(X)$ forming a neighbourhood base for μ. Note that in X we have $2^{k_0+k}(\alpha') = 2^k\alpha_0$ for $\alpha' := \frac{k-1}{2^{k_0}} + \frac{\alpha_0}{2^{k_0}}$. It follows that for each j there is an open neighbourhood $V = (\alpha' - \varepsilon_j, \alpha' + \varepsilon_j) \subseteq I_0$, $\varepsilon_j > 0$, such that $\mu_{\mathbf{n}\alpha,k_j} \in U_j$ for some $k_j \geq j$ and all $\alpha \in V$. In particular each $R_j = \{\alpha : \mu_{\mathbf{n}\alpha,k} \in U_j \text{ for some } k \geq j\}$ contains an open dense set. Thus $R = \bigcap_{j\in\mathbb{N}} R_j$ is residual with $\mu \in M(\mathbf{n}\alpha)$ for all $\alpha \in R$. □

Proof of statement (3) of Lemma 2.17. Assume $0 < \alpha < \frac{1}{16}$. Note that for every $J \subseteq X$ we have $n'_k\alpha = 2^k\alpha + \alpha \in J$ if and only if $n_k\alpha = 2^k\alpha \in J - \alpha$. In order to obtain the desired estimate we take for J instead of $I = (\frac{1}{2}, \frac{3}{4})$ the interval $I' := (\frac{1}{2} - \frac{\alpha}{3}, \frac{3}{4} + \frac{\alpha}{3})$ and, accordingly $I_\alpha := I' - \alpha = (\frac{1}{2} - \frac{4}{3}\alpha, \frac{3}{4} - \frac{2}{3}\alpha) = I^- \cup I^+$ with $I^- := (\frac{1}{2} - \frac{4}{3}\alpha, \frac{1}{2}]$ and $I^+ := (\frac{1}{2}, \frac{3}{4} - \frac{2}{3}\alpha)$. Observe that for $T : x \mapsto 2x$ each of the sets $T(I^-), T^2(I^-)$ and $T(I^+)$ has empty intersection with I_α. If D^- and D^+ denote the sets of all $k \in \mathbb{N}$ such that $2^k\alpha \in I^-$ resp. $2^k\alpha \in I^+$, this shows that the upper densities of D^- and D^+ are at most $\frac{1}{3}$ resp. $\frac{1}{2}$. It follows that the set of all $k \in \mathbb{N}$ with $n_k\alpha = 2^k\alpha \in I_\alpha$ or, equivalently, $n'_k\alpha \in I'$ has upper density at most $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Since the interval I' is open and contains the closure of I this yields that $\mu(I) \leq \frac{5}{6}$ for every $\mu \in M(\mathbf{n}'\alpha)$. □

Proof of statement (4) of Lemma 2.17. Similar arguments as several times before show that a generic α contains extremely long blocks of 0's in its binary representation $\alpha = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$. To be more precise, the set R of all α such that $(a_j, \dots, a_{j^2}) = (0, \dots, 0)$ for infinitely many $j \in \mathbb{N}$ is residual. For $\alpha \in (\frac{1}{2}, \frac{3}{4}) \cap R$ this implies that the upper density of the set $D = \{k \in \mathbb{N} : 2^{k+1}\alpha = 2^k\alpha + \alpha \in (\frac{1}{2}, \frac{3}{4})\}$ is 1, implying that $\mu(I) = 1$ for some $\mu \in M(\mathbf{n}'\alpha)$. □

3. Baire spaces of subsequences

In this section λ denotes an arbitrary but fixed Borel probability measure on a compact metric space X.

3.1. Notation and statement of the main results of this section. We assume $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \approx \lambda$ (see below). Typical examples of this type are $x_n = n\alpha$, $\alpha \in \mathbb{R}/\mathbb{Z}$, or, more generally, sequences induced by uniquely ergodic dynamical systems as ergodic group rotations, i.e. $x_n = ng$ where g is a topological generator of a monothetic compact group, λ the Haar measure. To state our results we need a lot of notation. Therefore the following list might be for the reader's convenience.

- We fix a measure $\lambda \in \mathscr{M}(X)$ (e.g. the Lebesgue measure).
- $\mathscr{C}(\lambda)$ denotes the system of all λ-continuity sets C, i.e. of those $C \subseteq X$ with $\lambda(\partial C) = 0$, where ∂C is the topological boundary of C. Similarly $\mathscr{C}(\lambda, \mu) = \mathscr{C}(\lambda) \cap \mathscr{C}(\mu)$ etc.
- $\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$, the shift acting on arbitrary infinite sequences.
- We write $\mathbf{x} \approx \lambda$ if $\mathbf{x} \sim \lambda$ in fact is well distributed. This, by definition, means that $\lim_{N\to\infty} \mu_{\sigma^k(\mathbf{x}),N} = \lambda$ uniformly in $k \in \mathbb{N}$. (Since $\mathscr{M}(X)$ is compact there is a unique uniform structure and this notion is well defined.) Clearly $\mathbf{x} \approx \lambda$ implies $\mathbf{x} \sim \lambda$ but not conversely. $\mathbf{x} \approx \lambda$ is equivalent to the condition that for all $A \in \mathscr{C}(\lambda)$ the limit
$$\lim_{N\to\infty} \frac{1}{N} |\{k < n \leq N + k : \ x_n \in A\}| = \lambda(A)$$
is uniform in $k \in \mathbb{N}$.
- $\mathbf{I} = (I_j)_{j\in\mathbb{N}}$ denotes a partition of $\mathbb{N}$ into intervals: $I_j = \{n \in \mathbb{N} : a_{j-1} < n \leq a_j\}$, $0 = a_0 < a_1 < \cdots \in \mathbb{N}$. The $b_j = a_j - a_{j-1}$ are called block lengths.
- For a sequence $\mathbf{m} = (m_j)_{j\in\mathbb{N}}$ of nonnegative integers we define $M_k = \sum_{i=1}^{k} m_j$.
- $S_0 = \{\mathbf{n} = (n_k)_{k\in\mathbb{N}} : 0 < n_1 < n_2 < n_2 < \cdots\}$ denotes the set of all strictly increasing sequences of natural numbers. S_0 is a Baire space, i.e. nonempty open sets are not meager. A topological base of open sets is given by all cylinder sets $[n'_1, n'_2, \ldots, n'_k]$ containing those $\mathbf{n} \in S_0$ with $n_i = n'_i$ for $i = 1, \ldots, k$.
- $S(\mathbf{I}, \mathbf{m})$ defines the closed (and hence Baire) subspace of S_0 containing those $\mathbf{n}$ having with each I_j exactly m_j members in common.
- Admissible $(\mathbf{I}, \mathbf{m})$ and $\mathbf{q}$: Given $\mathbf{I} = (I_j)_{j\in\mathbb{N}}$ and $\mathbf{m} = (m_j)_{j\in\mathbb{N}}$ with $m_j \leq b_j = |I_j|$, we consider the sequence $\mathbf{q}$ of ratios $q_j = \frac{m_j}{b_j} \in [0, 1]$ and, for each $N \in \mathbb{N}$, the measure
$$\pi_{\mathbf{I},\mathbf{m},N} = M_N^{-1} \sum_{j=1}^{N} m_j \delta_{q_j}$$
on $[0, 1]$. $(\mathbf{I}, \mathbf{m})$ is called admissible if the further conditions $\lim_{j\to\infty} b_j = \infty$ and $\lim_{n\to\infty} \frac{m_n}{M_n} = 0$ are satisfied. We only consider admissible $(\mathbf{I}, \mathbf{m})$.

- $P(\mathbf{I}, \mathbf{m}) = A((\pi_{\mathbf{I},\mathbf{m},N})_{N\in\mathbb{N}})$, the set of accumulation measures of the $\pi_{\mathbf{I},\mathbf{m},N}$, $N \in \mathbb{N}$.
- For $\pi \in \mathscr{P}$ let $F_\pi : [0,1] \to [0,1]$ be defined by
$$F_\pi(t_0) = \pi([0,t_0]) + t_0 \int_{(t_0,1]} \frac{d\pi(t)}{t}$$
for $0 \le t_0 \le 1$, in particular $F_\pi(0) = \pi(\{0\})$.
- For each $\pi \in \mathscr{P}$ we define the set
$$M(\pi, \lambda) = \{\mu \in \mathscr{M}(X) : \ \mu \le F_\pi \circ \lambda\}.$$
- $M(\lambda, \mathbf{I}, \mathbf{m}) = \bigcup_{\pi \in P(\mathbf{I},\mathbf{m})} M(\pi, \lambda)$.

Given $\mathbf{I}$, $\mathbf{m}$ and $\mathbf{x}$, we are interested in the distribution behavior of a subsequence $\mathbf{xn} = (x_{n_k})_{k\in\mathbb{N}}$ for typical $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$. Theorem 3.1 shows that limit measures of such sequences cannot be too far from λ, where the precise statement, of course, depends on the parameters $\mathbf{I}$ and $\mathbf{m}$. Theorem 3.2 shows that everything which might happen, happens typically in the Baire sense, i.e. all measures not excluded by Theorem 3.1 are limit measures of a generic subsequence $\mathbf{xn}$, $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$.

Now we are ready to state our results:

Theorem 3.1. *Let $\mathbf{x} \approx \lambda$. Then $M(\mathbf{xn}) \subseteq M(\lambda, \mathbf{I}, \mathbf{m})$ for all $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$.*

Theorem 3.2. *Suppose $\mathbf{x} \approx \lambda$ with supp$(\lambda) = X$. Then $M(\mathbf{xn}) = M(\lambda, \mathbf{I}, \mathbf{m})$ for most $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$, i.e. the exceptional set of those $\mathbf{n}$ with $M(\mathbf{xn}) \ne M(\lambda, \mathbf{I}, \mathbf{m})$ is meager in $S(\mathbf{I}, \mathbf{m})$.*

3.2. Preliminaries. Not surprisingly, a rigorous proof of Theorems 3.1 and 3.2 requires a lot of lemmata and combinatorial technicalities. Several of these auxiliary results are collected in:

Lemma 3.3.
1. *Given at most countably many $\mu_i \in \mathscr{M}(X)$, there is an open basis of X contained in $\bigcap_{n\in\mathbb{N}} \mathscr{C}(\mu_n)$.*
2. *Let $\mu, \mu_1, \mu_2, \ldots \in \mathscr{M}(X)$ and V a neighborhood of μ. Then there is an $\varepsilon > 0$ and a finite partition $X = A_1 \cup \cdots \cup A_s$, $A_i \in \bigcap_{n\in\mathbb{N}} \mathscr{C}(\mu_n)$ such that, for all $\nu \in \mathscr{M}(X)$, $|\nu(A_i) - \mu(A_i)| < \varepsilon$ for $i = 1, \ldots, s$, implies $\nu \in V$.*
3. *For any $\pi \in \mathscr{P}$,*
$$F_\pi(t_0) = \pi([0,t_0]) + t_0 \int_{(t_0,1]} \frac{d\pi(t)}{t}$$
defines a function $F_\pi : [0,1] \to [0,1]$ which is monotonically nondecreasing, continuous and concave.
4. *Assume $\lim_n \pi_n = \pi$. Then $\lim_{n\to\infty} F_{\pi_n} = F_\pi$ uniformly.*
5. *If $\mu(A) \le F_\pi(\lambda(A))$ for all $A \in \mathscr{C}(\lambda, \mu)$, then the same inequality holds for all Borel sets A, i.e. $\mu \le F_\pi \circ \lambda$.*

Proof. (1) Standard.

(2) Standard.

(3) Note that, for any $\pi \in \mathscr{P}$ and fixed $t_0 \in [0,1]$, $F_\pi(t_0) = \int_0^1 f_{t_0}(t)d\pi(t)$ with the continuous function f_{t_0} defined by $f_{t_0}(t) = 1$ for $0 \le t \le t_0$ and $f_{t_0}(t) = \frac{t_0}{t}$ for $t_0 < t \le 1$. This shows $0 \le F_\pi(t_0) \le 1$ and that $t_0 \le t_1$ implies $f_{t_0} \le f_{t_1}$ and hence $F_\pi(t_0) \le F_\pi(t_1)$. Thus $F_\pi : [0,1] \to [0,1]$ is monotonic. Continuity at 0 follows from monotonic convergence:

$$\lim_{t_0 \to 0} F_\pi(t_0) = \lim_{t_0 \to 0} \int_{[0,1]} f_{t_0}(t)d\pi(t) = \int_{[0,1]} f_0(t)d\pi(t) = F_\pi(0).$$

For other points $0 < t_0 < t_1$ observe that

$$F_\pi(t_1) - F_\pi(t_0) = A - B + C$$

with $A = \pi((t_0, t_1])$, $B = t_0 \int_{(t_0,t_1]} \frac{d\pi(t)}{t}$ and $C = (t_1 - t_0) \int_{(t_1,1]} \frac{d\pi(t)}{t}$. For fixed t_0 and $t_1 \to t_0$ all three values tend to 0, while for fixed t_1 and $t_0 \to t_1$ both A and B tend to $\pi(\{t_1\})$ and C tends to 0. This shows that F_π is continuous on the whole interval $[0,1]$.

In order to see that F_π is concave we introduce for $0 < t_0 < t_1 < t_2 \le 1$ the abbreviations $A_0 = (0, t_0]$, $A_1 = (t_0, t_1]$, $A_2 = (t_1, t_2]$, $A_3 = (t_2, 1]$, $p_i = \pi(A_i)$, and $c_i = \int_{A_i} \frac{d\pi(t)}{t}$. It suffices to show that $2F_\pi(t_1) \ge F_\pi(t_0) + F_\pi(t_2)$ for $t_0 = t_1 - \varepsilon$ and $t_2 = t_1 + \varepsilon$. In this case we have
$F_\pi(t_0) = p_0 + (t_1 - \varepsilon)(c_1 + c_2 + c_3)$,
$F_\pi(t_1) = p_0 + p_1 + t_1(c_2 + c_3)$, and
$F_\pi(t_2) = p_0 + p_1 + p_2 + (t_1 + \varepsilon)c_3$.
Thus the above inequality reduces to $p_1 + t_1c_2 + \varepsilon(c_1 + c_2) \ge p_2 + t_1c_1$, which follows from $p_1 \ge (t_1 - \varepsilon)c_1$ and $p_2 \le (t_1 + \varepsilon)c_2$.

(4) Using the function f_{t_0} from part (3) one gets pointwise convergence

$$\lim_{n \to \infty} F_{\pi_n}(t_0) = \lim_{n \to \infty} \int_{[0,1]} f_{t_0} d\pi_n(t) = F_\pi(t_0)$$

immediately from the definition of the convergence $\lim_{n\to\infty} \pi_n = \pi$ of measures. The uniformity in $t_0 \in [0,1]$ finally follows from a standard argument on the convergence of monotonic functions.

(5) Standard.

□

3.3. Proof of Theorem 3.1.

Lemma 3.4. *Suppose that X is finite, $A \subseteq X$, $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$, $(\mathbf{I}, \mathbf{m})$ admissible, $\lim_{k\to\infty} \mu_{\mathbf{xn},M_{N_k}} = \mu$, $\lim_{k\to\infty} \pi_{\mathbf{I},\mathbf{m},N_k} = \pi$ for some $N_1 < N_2 < \cdots \in \mathbb{N}$, and $\lambda(A) \le t_0$. Then $\mu(A) \le F_\pi(t_0)$.*

Proof. By continuity of F_π (Lemma 3.3(3)) it suffices to prove the statement for $\lambda(A) = t_0 > 0$. Fix any $\varepsilon > 0$. Since $\mathbf{x} \approx \lambda$ and $\lim_{j\to\infty} b_j = \infty$ there is a

$j(\varepsilon) \in \mathbb{N}$ such that, letting $c_j := |\{n_k \in I_j : x_{n_k} \in A\}|$, we have

$$b_j(t_0 - \varepsilon) \leq c_j \leq b_j(t_0 + \varepsilon)$$

for all $j > j(\varepsilon)$. For fixed $k \in \mathbb{N}$ define

$$\begin{array}{rcl} J_1 & = & \{1, \ldots, j(\varepsilon)\}, \\ J_2 & = & \{j :\ j(\varepsilon) < j \leq M_{N_k} :\ q_j = \frac{m_j}{b_j} \leq \lambda(A) = t_0\}, \text{ and} \\ J_3 & = & \{j :\ j(\varepsilon) < j \leq M_{N_k} :\ q_j = \frac{m_j}{b_j} > \lambda(A) = t_0\}. \end{array}$$

We are going to estimate $C = C_1 + C_2 + C_3$, $C_i = \sum_{j \in J_i} c_j$, $i = 1, 2, 3$. Abbreviate $\pi_{\mathbf{I},\mathbf{m},N_k}$ by π'_k. Now $C_1 \leq M_{j(\varepsilon)}$ is a constant not depending on k,

$$C_2 \leq \sum_{j \in J_2} m_j \leq M_{n_k} \pi'_k([0, t_0]),$$

and, using $b_j \leq \frac{m_j}{t_0}$ for $j \in C_3$,

$$\begin{array}{rcl} C_3 & \leq & \sum_{j \in J_3} b_j(t_0 + \varepsilon) \leq t_0 \sum_{j \in J_3} \frac{m_j}{q_j} + \frac{\varepsilon}{t_0} \sum_{j \in J_3} m_j \leq \\ & \leq & t_0 M_{N_k} \int_{(t_0, 1]} \frac{d\pi'_k(t)}{t} + \frac{\varepsilon}{t_0} M_{N_k}. \end{array}$$

Hence

$$\lim_{k \to \infty} \mu_{\mathbf{x}, M_{n_k}}(A) \leq \lim_{k \to \infty} \frac{M_{j(\varepsilon)}}{M_{n_k}} + \pi'_k([0, t_0]) + t_0 \int_{(t_0, t]} \frac{d\pi'_k(t)}{t} + \frac{\varepsilon}{t_0} = F_{\pi'_k}(t_0) + \frac{\varepsilon}{t_0}.$$

Since this holds for all $\varepsilon > 0$, Lemma 3.3(4) proves the assertion. □

Lemma 3.5. *Theorem 3.1 holds whenever X is finite.*

Proof. Let $\mu \in M(\mathbf{xn})$, $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$. This means that some subsequence of the $\mu_{\mathbf{xn},N}$, $N \in \mathbb{N}$, converges to μ. Since $m_k = o(M_k)$ $(k \to \infty)$ and since $\mathscr{P}$ is compact we may assume $\lim_{k \to \infty} \mu_{\mathbf{xn}, M_{N_k}} = \mu$ and, if necessary again by taking an appropriate subsequence, $\lim_{k \to \infty} \pi_{\mathbf{I},\mathbf{m},N_k} = \pi$ for some $\pi \in P(\mathbf{I}, \mathbf{m})$. Since X is finite, this implies $\mu(A) \leq F_\pi \lambda(A)$ for all $A \subseteq X$. Thus $\mu \in M(\pi, \lambda) \subseteq M(\lambda, \mathbf{I}, \mathbf{m})$. □

Proof of Theorem 3.1. Let $\mu \in M(\mathbf{xn})$, $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$ and $A \in \mathscr{C}(\lambda, \mu)$. Consider the finite space $X' = \{0, 1\}$ with measures $\mu'(\{1\}) = \mu(A)$, $\lambda'(\{1\}) = \lambda(A)$ and the sequence $\mathbf{x}' = (x'_n)_{n \in \mathbb{N}}$ with $x'_n = 1$ iff $x_n \in A$. Then we are in the situation of Lemma 3.5 to conclude $\mu' \in M(\pi, \lambda')$ for some $\pi \in P(\mathbf{I}, \mathbf{n})$, i.e.

$$\mu(A) = \mu'(\{1\}) \leq F_\pi \lambda'(\{1\}) = F_\pi \lambda(A).$$

Since this holds for arbitrary $A \in \mathscr{C}(\lambda, \mu)$, Lemma 3.3(5) yields $\mu \in M(\pi, \lambda) \subseteq M(\lambda, \mathbf{I}, \mathbf{m})$. □

3.4. Proof of Theorem 3.2.

Lemma 3.6. *Assume that* $X = \{1, 2, \dots, s\}$ *is finite,* $\lim_{k\to\infty} \pi_{\mathbf{I},\mathbf{m},N_k} = \pi$, $\varepsilon > 0$, $\mu \leq F_\pi \lambda$, $\mathbf{n}' \in S(\mathbf{I}, \mathbf{m})$, $\mathbf{x} \approx \lambda$, *supp*$(\lambda) = X$ *and* $j_0 \in \mathbb{N}$. *Then there exist* $j_1 \geq j_0$, $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$, *with* $n'_k = n_k$ *for all* $n_k \in \bigcup_{j=1}^{j_0} I_j$ *and* $|\mu(i) - \mu_{\mathbf{xn},M}(i)| < \varepsilon$ *for* $M = \sum_{j=1}^{j_1} m_j$ *and all* $i \in X$.

Proof. Let $n_1 < n_2 < \cdots < n_{M_0}$ be given such that $M_0 = \sum_{j=1}^{j_0} m_j$ and $|I'_j| = m_j$ for $j = 1, \dots, j_0$ and $I'_j = I_j \cap \{n_k : 1 \leq k \leq M_0\}$. We assume $\lim_{k\to\infty} \pi_{\mathbf{I},\mathbf{m},N_k} = \pi$ for $N_1 < N_2 < \cdots$, furthermore $\mu \leq F_\pi \lambda$ and $\varepsilon > 0$. We have to find a number $j_1 = N_k \geq j_0$ and an extension $n_1 < n_2 < \cdots < n_{M_0} < \cdots < n_M$, $M = \sum_{j=1}^{j_1} m_j$ such that, putting $I'_j = I_j \cap \{n_k : k \in \mathbb{N}\}$, we have $|I'_j| = m_j$ for all $j = 1, \dots, j_1$ (we call such an $\mathbf{n} = (n_1, \dots, n_M)$ admissible) and $|d_i| < \varepsilon$ for all $d_i = \mu(i) - \mu_{\mathbf{xn},M}(i)$, $i \in X$.

To do this let, w.l.o.g., $\lambda(1) = \min_{i\in X} \lambda(i)$ which is positive since X is finite and supp$(\lambda) = X$. Define $c_j^{(i)} = |\{n \in I_j : x_n = i\}|$. Since $\mathbf{x} \approx \lambda$ and $\lim_{j\to\infty} b_j = \infty$ there is some $j' \geq j_0$ such that $|\frac{c_j^{(i)}}{b_j} - \lambda(i)| < \frac{\lambda(1)\varepsilon}{3s^2}$ for all $j \geq j'$. Choose $j_1 = N_k > j'$ such that $M'/M < \varepsilon/3s$ ($M = \sum_{j=1}^{j_1} m_j$, $M' = \sum_{j=1}^{j'} m_j$), $\frac{1}{M} < \frac{\varepsilon}{2s^2}$ and $F_{\pi_{\mathbf{I},\mathbf{m},N_k}} > F_\pi - \frac{\varepsilon}{3}$ (Lemma 3.3(4)). Let now, for our given admissible $\mathbf{n}$, $A(\mathbf{n}, i) = \mu_{\mathbf{xn},M}(i) = |\{k \leq M : x_{n_k} = i\}|$. Rearrange the d_i in such a way that $d_{i_1} \geq d_{i_2} \geq \cdots \geq d_{i_s}$. Since the set of admissible $\mathbf{n}$ is finite, there is a nonempty set of admissible $\mathbf{n}$ for which $D(\mathbf{n}) = \sum_{i=1}^{s} |d_i|$ takes a minimal value, say D_0. Among these $\mathbf{n}$ choose one which leads to the s-tuple $(d_{i_1}, \dots, d_{i_s})$ which is minimal with respect to the lexicographic ordering. Everything we have to show is $d_{i_1} < \varepsilon/s$, since then $\sum_{i\in X} d_i = 0$ implies $\max_{i\in X} |d_i| = \max\{d_{i_1}, -d_{i_s}\} \leq \varepsilon$.

Assume therefore, by contradiction, $|d_{i_1}| \geq \varepsilon/s$. We treat only the case $d_{i_1} \geq \varepsilon/s > 0$, since $d_{i_1} \leq -\varepsilon/s < 0$ is similar. $\sum_{i=1}^{s} d_i = 0$ and $d_{i_1} \geq \cdots \geq d_{i_s}$ implies $d_{i_s} \leq 0$. It follows that there is some r such that $d_r - d_{r+1} > \frac{\varepsilon}{s^2}$ and $d_r > \frac{\varepsilon}{s^2}$. Let $Y = \{i_1, \dots, i_r\}$. For $j = j_0 + 1, \dots, j_1$ we claim that
(i) $x_n \in Y$ for all $n \in I'_j$ whenever $\sum_{i\in Y} c_j^{(i)} \geq m_j$;
(ii) $x_n \notin Y$ for all $n \in I_j \setminus I'_j$ whenever $\sum_{i\in Y} c_j^{(i)} < m_j$.
To see this, note that, if (i) failed by some $x_n \notin Y$, $n \in I'_j$, $\sum_{i\in Y} c_j^{(i)} \geq m_j$, we could replace n in I'_j by some $n' \in I_j$ with $x_{n'} \in Y$ to get a contradiction to the extremal choice of $\mathbf{n}$. A similar argument shows (ii).

Let now $A_j = |\{n_k \in I_j : x_{n_k} \in Y\}|$. Then, for $j' \leq j \leq j_1$, (i) and (ii) together with the extremal choice of $\mathbf{n}$ guarantee the following two implications:
$q_j \leq \lambda(Y)$ implies $A_j \geq m_j(1 - \frac{\lambda_1\varepsilon}{3s}) \geq m_j(1 - \frac{\varepsilon}{3s})$ and
$q_j > \lambda(Y)$ implies $A_j \geq (\lambda(Y) - \frac{\lambda_1\varepsilon}{3s})b_j$.
Since $d_i \geq 0$ for all $i \in Y$ and $i_1 \in Y$, $d_{i_1} \geq \varepsilon/s$ implies

$$\mu(Y) \geq \mu_{\mathbf{xn},M}(Y) + \frac{\varepsilon}{s}.$$

In

$$\mu_{\mathbf{xn},M}(Y) = \frac{1}{M}\sum_{j\le j_1} A_j$$

we split the sum $\sum_{j\le j_1} A_j$ into three sums S_i, $i=0,1,2$, where the summation runs over all $j\in J_i$. Here J_0 contains all $j<j'$, J_1 all j with $j'\le j\le j_1$ and $q_j\le\lambda(Y)$, J_2 all j with $j'\le j\le j_1$ and $q_j>\lambda(Y)$. By using our lower bounds for A_j if $j\in J_1$ resp. if $j\in J_2$ we get

$$\mu(Y) \ge \frac{1}{M}\left(\sum_{j\in J_1} m_j(1-\frac{\varepsilon}{3s}) + \sum_{j\in J_2}(\lambda(Y)-\frac{\lambda_1\varepsilon}{3s})b_j\right)+\frac{\varepsilon}{s}.$$

Write now J_i' for the set of all $j\le j_1$ (including those $<j'$) with $q_j\le\lambda(Y)$ (if $i=1$) resp. $q_j>\lambda(Y)$ (if $i=2$), $S_i'=\sum_{j\in J_i} m_j$, $i=1,2$. By separating positive and negative terms in the above inequality and by using $M'/M<\frac{\varepsilon}{3s}$ and, if $q_j\le\lambda(Y)$, $\lambda(Y)b_j\ge q_jb_j=m_j$ and $b_j<\frac{m_j}{\lambda(Y)}\le\frac{m_j}{\lambda_1}$, we can continue our estimation with

$$\mu(Y)\ge\frac{1}{M}(S_1'+S_2')-\frac{\varepsilon}{3sM}(S_1'+S_2')+\frac{\varepsilon}{s}-\frac{\varepsilon}{3s}.$$

Note that $\frac{1}{M}(S_1'+S_2')=F_{\pi_{\mathbf{I},\mathbf{m},N_k}}(\lambda(Y))>F_\pi(\lambda(Y))-\frac{\varepsilon}{3s}$ and $S_1'+S_2'\le\sum_j m_j=M$ to finally obtain

$$\mu(Y)>F_\pi(\lambda(Y)),$$

contradicting $\mu\le F_\pi\lambda$. $\square$

Lemma 3.7. *Theorem 3.2 holds whenever X is finite.*

Proof. Recall that for $n_1<n_2<\cdots<n_l\in\mathbb{N}$ the symbol $[n_1,\ldots,n_l]$ denotes the set of all $\mathbf{n}$ with this initial part. The family of these cylinder sets forms an open base for the topology on S_0 and their intersections with $S(\mathbf{I},\mathbf{m})$ form an open base for the subspace $S(\mathbf{I},\mathbf{m})$. Thus the following statement is just a reformulation of Lemma 3.6:

> For all nonempty open sets $O\subseteq S(\mathbf{I},\mathbf{m})$, $\mu\le F_\pi\lambda$, $\pi\in P(\mathbf{I},\mathbf{m})$, neighborhoods V of μ and $l_0\in\mathbb{N}$ there is a nonempty open set $U\subseteq O$ and some $l\ge l_0$ such that $\mathbf{n}\in U$ implies $\mu_{\mathbf{xn},l}\in V$.

This implies that the set $S(V,l_0,\mu)$ of all $\mathbf{n}\in S(\mathbf{I},\mathbf{m})$ such that $\mu_{\mathbf{xn},l}\in V$ for some $l\ge l_0$ is a residual subset of $S(\mathbf{I},\mathbf{m})$. Since μ has a countable neighborhood base consisting of some $V_1,V_2,\ldots$, the intersection $S(\mu)$ of all $S(V_n,l_0,\mu)$, $n,l_0\in\mathbb{N}$, is residual in $S(\mathbf{I},\mathbf{m})$. Furthermore $\mu\in M(\mathbf{xn})$ for all $\mathbf{n}\in S(\mu)$.

Finally take a sequence $\mu_1,\mu_2,\ldots$ which is dense in $M(\lambda,\mathbf{I},\mathbf{m})$. Then $S'=\bigcap_{i=1}^\infty S(\mu_i)$ is residual and $\mathbf{n}\in S'$ implies $\mu_i\in M(\mathbf{xn})$ for all i, hence

$$M(\lambda,\mathbf{I},\mathbf{m})=\overline{\{\mu_i:\ i\in\mathbb{N}\}}\subseteq M(\mathbf{xn}).$$

This together with Theorem 3.1 proves the lemma. $\square$

Proof of Theorem 3.2. First fix any $\mu \in M(\lambda, \mathbf{I}, \mathbf{m})$, say $\mu \in M(\pi, \lambda)$ for some fixed $\pi \in P(\mathbf{I}, \mathbf{m})$. For any given neighborhood V of μ there is a partition $X = A_1 \cup \cdots \cup A_s$, $A_i \in \mathscr{C}(\lambda, \mu)$ (cf. Lemma 3.3, parts(1) and (2)), such that $\nu(A_i) = \mu(A_i)$ implies $\nu \in V$. Similar to the proof of Theorem 3.1 we can here apply the finite case (Lemma 3.7) to the induced structure and see that the set $S(V)$ of all $\mathbf{n} \in S(\mathbf{I}, \mathbf{m})$ with $M(\mathbf{xn}) \cap V \neq \emptyset$ is residual. By considering the residual set $S(\mu) = \bigcap_{n=1}^{\infty} S(V_n)$, where $V_1, V_2, \ldots$ form an neighborhood base of μ, and using that all $M(\mathbf{xn})$ are closed, one sees that $\mu \in M(\mathbf{xn})$ for all $\mathbf{n} \in S(\mu)$. Now the same argument as in the proof of Lemma 3.7 implies the theorem. $\square$

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