

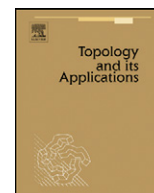


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Sequential properties of function spaces with the compact-open topology[☆]

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ABSTRACT

The main results of the paper are:

- (1) If X is metrizable but not locally compact topological space, then $C_k(X)$ contains a closed copy of S_2 , and hence does not have the property AP;
- (2) For any zero-dimensional Polish X , the space $C_k(X, 2)$ is sequential if and only if X is either locally compact or the derived set X' is compact; and
- (3) All spaces of the form $C_k(X, 2)$, where X is a non-locally compact Polish space whose derived set is compact, are homeomorphic, and have the topology determined by an increasing sequence of Cantor subspaces, the n th one nowhere dense in the $(n + 1)$ st.

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1. Introduction

Let $C_k(X)$ be the space of continuous real-valued functions on X with the compact-open topology. $C_k(X)$ for metrizable X is typically not a k -space, in particular not sequential. Indeed, by a theorem of R. Pol [8], for X paracompact first countable (in particular, metrizable), $C_k(X)$ is a k -space if and only if X is locally compact, in which case X is a topological sum of locally compact σ -compact spaces and $C_k(X)$ is a product of completely metrizable spaces. A similar result holds for $C_k(X, [0, 1])$: it is a k -space if and only if X is the topological sum of a discrete space and a locally compact σ -compact space, in which case $C_k(X)$ is the product of a compact space and a completely metrizable space. It follows that, for separable metric X , the following are equivalent:

- (1) $C_k(X)$ is a k -space;
- (2) $C_k(X)$ is first countable;
- (3) $C_k(X)$ is a complete separable metrizable space, i.e., a Polish space;
- (4) X is a locally compact space.

The same equivalences hold for $C_k(X, [0, 1])$. On the other hand, for Polish X , $C_k(X)$ always has the (strong) Pytkeev property [9].

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A space X has the property AP if whenever $x \in \bar{A} \setminus A$, there is some $B \subseteq A$ such that $x \in \bar{B} \subseteq A \cup \{x\}$. The space X has the property WAP when a subset A of X is closed if and only if there is no $B \subseteq A$ such that $|\bar{B} \setminus A| = 1$. Thus, every Fréchet space is AP and every sequential space is WAP . It was asked in [5] whether $C_k(\omega^\omega)$ is WAP .

In this note, we first show that if X is metrizable but not locally compact, then $C_k(X)$ contains a closed copy of Arens space S_2 , and hence it is not AP . In fact, such a closed copy of S_2 is contained in $C_k(M, 2)$, where M is the countable metric fan. We then show that $C_k(M, 2)$ is sequential, in contrast to the full function space $C_k(M)$. Next we show that for a zero-dimensional Polish space X , if $C_k(X, 2)$ is not metrizable (which is the case if and only if X is not locally compact), then $C_k(X, 2)$ is sequential if and only if the derived set X' is compact. We obtain a complete description of $C_k(X, 2)$ for a non-locally compact Polish X such that X' is compact: any such $C_k(X, 2)$ is homeomorphic to the space $(2^\omega)^\infty$, which is the space with the topology determined by an increasing sequence of Cantor sets, the n th one nowhere dense in the $(n + 1)$ st.

2. When $C_k(X)$ contains S_2

Arens's space S_2 is the set

$$\{(0, 0), (1/n, 0), (1/n, 1/nm) : n, m \in \omega \setminus \{0\}\} \subseteq \mathbb{R}^2$$

carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0 = \{(0, 0), (\frac{1}{n}, 0) : n > 0\}$ and $C_n = \{(\frac{1}{n}, 0), (\frac{1}{n}, \frac{1}{nm}) : m > 0, n > 0\}$. The *sequential fan* is the quotient space $S_\omega = S_2/C_0$ obtained from the Arens space by identifying the points of the sequence C_0 [6]. It is easy to check that S_ω is a non-metrizable Fréchet–Urysohn space, and S_2 is sequential and not Fréchet–Urysohn. In fact, any space which is sequential but not Fréchet–Urysohn contains S_2 as a subspace.

The *countably infinite metric fan* is the space $M = (\omega \times \omega) \cup \{\infty\}$, where points of $\omega \times \omega$ are isolated, and the basic neighborhoods of ∞ are $U(n) = \{\infty\} \cup ((\omega \setminus n) \times \omega)$, $n \in \omega$. The space M is not locally compact at its non-isolated point ∞ .

Lemma 2.1. *The space $C_k(M, 2)$ contains a closed copy of S_2 .*

Proof. For each $n > 1$ and each k , let

$$U(n, k) = (\{0\} \times n) \cup ((n \setminus \{0\}) \times k) \cup U(n),$$

and let $f_{n,k}$ be the member of $C_k(M)$ which is 0 on $U(n, k)$ and 1 otherwise (i.e., the characteristic function of $M \setminus U(n, k)$).

Let $f_n \in C_k(M)$ be the function which is 1 on $\{0\} \times (\omega \setminus n)$ and 0 otherwise, and let c_0 be the constant 0 function. For each n , we have $\lim_k f_{n,k} = f_n$, and $\lim_n f_n = c_0$. Thus, c_0 is a limit point of the set $A = \{f_{n,k} : n > 1, k \in \omega\}$. Let $S = \{f_n : n > 1\}$, and $X = \{c_0\} \cup S \cup A$.

We claim that X is homeomorphic to the Arens space S_2 . It suffices to show that for each sequence $(k_n)_{n>1}$, the point c_0 is not in the closure of the set $\{f_{n,k} : k < k_n, n > 1\}$. Given $(k_n)_{n>1}$, set $K = \{(n - 1, k_n) : n > 1\} \cup \{\infty\}$. Then K is a sequence convergent to ∞ , and for each $f \in \{f_{n,k} : k < k_n, n > 1\}$ there exists $x \in K$ (namely $x = (n - 1, k_n)$, where $f = f_{n,k}$) such that $f(x) = 1$. Therefore $\{f_{n,k} : k < k_n, n > 1\}$ does not intersect the neighborhood $\{f \in C_k(M, 2) : f \upharpoonright K \equiv 0\}$ of c_0 , and hence does not contain c_0 in its closure.

By [6, Corollary 2.6], if every point z in a topological space Z is regular G_δ (i.e., $\{z\}$ is equal to $\bigcap_n \bar{U}_n$ for some open neighborhoods U_n of z), and Z contains a copy of S_2 , then Z contains a closed copy of S_2 . Since every point of $C_k(M, 2)$ is regular G_δ , the space $C_k(M, 2)$ contains a closed copy of S_2 . In fact, the space X constructed above is closed, even in $C_p(M, 2)$. \square

Theorem 2.2. *If X is metrizable and not locally compact, then $C_k(X)$ contains closed copies of S_2 and S_ω .*

Proof. By Lemma 8.3 of [4], a first countable space X contains a closed topological copy of the space M if and only if X is not locally compact. E.A. Michael [7, Theorem 7.1] observed that, for a closed subspace Y of a metrizable space X , the linear extender $e : C(Y) \rightarrow C(X)$ given by the Dugundji extension theorem is a homeomorphic embedding when both $C(Y)$ and $C(X)$ are given the compact-open topology (or the topology of uniform convergence, or pointwise convergence). Thus for each metrizable space X which is not locally compact, $C_k(M, 2)$ is embedded in $C_k(X)$ as a closed subspace, and hence $C_k(X)$ contains a closed copy of S_2 . Finally, $C_k(X)$ also contains a closed copy of S_ω because any topological group G contains a (closed) copy of S_2 if and only if it contains a (closed) copy of S_ω . \square

Remark 2.3. C.J.R. Borges [2] showed that the Dugundji extension theorem holds for the class of stratifiable spaces, and hence Theorem 2.2 holds more generally for first countable stratifiable spaces.

3. Sequentiality of $C_k(X, 2)$

A topological space X carries the inductive topology with respect to a closed cover \mathcal{C} of X , if for each $F \subseteq X$, the set F is closed whenever $F \cap C$ is closed in C for each $C \in \mathcal{C}$. A topological space is a k -space (respectively, sequential space) if it carries the inductive topology with respect to its cover by compact (respectively, compact metrizable) subspaces. X is sequential if and only if for every non-closed $A \subseteq X$, there exists a sequence in A converging to a point in $X \setminus A$. For a topological space X , the set of all non-isolated points of X is denoted by X' .

The following theorem is the main result of this section.

Theorem 3.1. *Let X be a zero-dimensional Polish space which is not locally compact. Then $C_k(X, 2)$ is sequential if and only if the set X' is compact.*

We divide the proof of Theorem 3.1 into a sequence of lemmata.

Lemma 3.2. *The space $C_k(M, 2)$ is sequential.*

Proof. Suppose not. Then there is $A \subseteq C_k(M, 2)$ which is not closed and yet contains all limit points of convergent sequences of its elements. As M is zero-dimensional, $C_k(M, 2)$ is homogeneous. Thus, without loss of generality, we may assume that $c_0 \in \bar{A} \setminus A$, where c_0 is the constant 0 function. We may additionally assume that $f(\infty) = 0$ for all $f \in A$. Let

$$A_n = \{f \in A : f(U(n)) = \{0\}\}.$$

Note that the sets A_n are increasing with n , and their union is A .

Claim 3.3. *There exists a sequence $(k_n)_{n \in \omega}$ such that $1 \in f(\bigcup_{i \leq n} \{i\} \times k_i)$ for all n and $f \in A_{n+1}$.*

Proof. By induction. Assume that for all $i < n$, there are k_i such that $f \in A_{i+1}$ implies $1 \in f(\bigcup_{j \leq i} \{j\} \times k_j)$, but that for each m , there is $f_m \in A_{n+1}$ such that $f_m((\bigcup_{i < n} \{i\} \times k_i) \cup (\{n\} \times m)) = \{0\}$. Let $f'_m = f_m \upharpoonright (n+1) \times \omega$. As $2^{(n+1) \times \omega}$ is homeomorphic to the Cantor space, there is a subsequence $\{f'_{m_i}\}$ of $\{f'_m\}$, converging to an element $f' \in 2^{(n+1) \times \omega}$. It follows from $f_m \in A_{n+1}$ that $f_m(U(n+1)) = \{0\}$. Define $g \in C_k(M)$ by $g(U(n+1)) = \{0\}$ and $g \upharpoonright (n+1) \times \omega = f'$. Then in $C_k(M)$, $g = \lim_i f_{m_i}$, and therefore $g \in A$. As $f'_m(\{n\} \times m) = \{0\}$ for all m , $g(\{n\} \times \omega) = \{0\}$. We have $g(U(n+1)) = \{0\}$ so $g(U(n)) = \{0\}$, and thus $g \in A_n$. But $g(\bigcup_{i \leq n-1} \{i\} \times k_i) = \{0\}$ (indeed, this holds for all f_m 's), contradicting the induction hypothesis. \square

Let

$$K = \left(\bigcup_{i \in \omega} \{i\} \times k_i \right) \cup \{\infty\}.$$

Let V be the set of all functions which map K into the interval $(-1/2, 1/2)$. Then V is a neighborhood of c_0 which misses A , a contradiction. \square

It is worth mentioning that $C_k(M)$ and $C_k(M, [0, 1])$ are not k -spaces [8] since the metric fan M is not locally compact.

A topological space Y has the strong Pytkeev property [9] (respectively, countable cs^* -character) if for each $y \in Y$, there is a countable family \mathcal{N} of subsets of Y , such that for each neighborhood U of y and each $A \subseteq Y$ with $y \in \bar{A} \setminus A$ (respectively, each sequence A in $Y \setminus \{y\}$ converging to y), there is $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite.

For every Polish space X the space $C_k(X)$ has the strong Pytkeev property [9, Corollary 8]. Thus, any subspace of $C_k(X)$ has the strong Pytkeev property, and therefore has countable cs^* -character.

An mk_ω -space is a topological space which carries the inductive topology with respect to a countable cover of compact metrizable subspaces. A topological group G is an mk_ω -group if G is an mk_ω -space.

The space $C_k(X, 2)$ has a natural structure of a topological group.

Theorem 3.4. ([3]) *Let G be a sequential non-metrizable topological group of countable cs^* -character. Then G contains an open mk_ω -subgroup H and thus is homeomorphic to the product $H \times D$ for some discrete space D .*

Corollary 3.5. *Let G be a sequential separable topological group of countable cs^* -character. If G is not metrizable, then G is σ -compact.*

The following statement is well known and easy to prove.

Lemma 3.6. *Let X be a zero-dimensional first countable space. Then $C_k(X, 2)$ is metrizable if and only if X is locally compact and σ -compact.*

Claim 3.7. Let Z be a non-discrete metrizable separable zero-dimensional space. Then $C_k(Z, 2)$ is not compact.

Proof. If Z is locally compact, then it contains a clopen infinite compact subset C . Then $C_k(C, 2)$ is a closed subset of $C_k(Z, 2)$ homeomorphic to ω , and hence $C_k(Z, 2)$ is not compact.

If Z is not locally compact, then Z contains a closed copy Y of M . By Lemma 2.1, the space $C_k(Y, 2)$ contains a closed copy of S_2 , and is thus not compact. As restriction to Y is a continuous map from $C_k(Z, 2)$ onto $C_k(Y, 2)$, the space $C_k(Z, 2)$ is not compact. \square

Claim 3.8. If X_i is not compact for any $i \in \omega$, then the product $\prod_{i \in \omega} X_i$ is not σ -compact.

Proof. A simple diagonalization argument. \square

Proof of Theorem 3.1. Assume that X' is compact and consider the subgroup $H = \{f \in C_k(X, 2) : f \upharpoonright X' \equiv 0\}$. It is easy to see that H is an open subgroup of $C_k(X, 2)$, and thus it suffices to prove that H is sequential. Since X is not locally compact, there is a clopen outer base $\{U_n : n \in \omega\}$ of X' such that $U_0 = X$, $U_{n+1} \subseteq U_n$, and $U_n \setminus U_{n+1}$ is infinite for all $n \in \omega$. Let $f : X \rightarrow M$ be a map such that $f(X') = \{\infty\}$ and $f \upharpoonright (U_n \setminus U_{n+1})$ is an injective map onto $\{n\} \times \omega$. Then the map

$$f^* : \{g \in C_k(M, 2) : g(\infty) = 0\} \rightarrow H$$

assigning to g the composition $g \circ f$ is easily seen to be a homeomorphism, and hence H is sequential.

Now, assume that X' is not compact. Then there exists a countable closed discrete subspace $T \subseteq X'$, and hence there exists a discrete family $\{U_t : t \in T\}$ of clopen subsets of X such that $t \in U_t$ for all $t \in T$. Observe that $C_k(X, 2)$ contains a closed copy of the product $\prod_{t \in T} C_k(U_t, 2)$.

Since $T \subseteq X'$, the set U_t is not discrete for all $t \in T$. By Claims 3.8 and 3.7, the product $\prod_{t \in T} C_k(U_t, 2)$ is not σ -compact. Thus, $C_k(X, 2)$ is not σ -compact.

As X is Polish, $C_k(X)$ has the strong Pytkeev property [9], and thus has countable cs^* -character. Consequently, so does its subspace $C_k(X, 2)$. As $C_k(X, 2)$ is separable and X is not locally compact, $C_k(X, 2)$ is not first countable, and hence it is not metrizable. To finish the proof apply Corollary 3.5. \square

Next, we prove two corollaries of Theorem 3.1. The first one is straightforward.

Corollary 3.9. The space $C_k(\omega \times M, 2)$ is not sequential. \square

Let $(0) \in 2^\omega$ be the constant zero sequence. Following [1], let $(2^\omega)^\infty$ be the space $\bigcup_{n \in \omega} (2^\omega)^n$, where $(2^\omega)^n$ is identified with the subspace $(2^\omega)^n \times \{(0)\}$ of $(2^\omega)^{n+1}$, with the inductive topology with respect to the cover $\{(2^\omega)^n : n \in \omega\}$.

Theorem 3.10. (Banach [1]) Every non-metrizable uncountable zero-dimensional mk_ω -group is homeomorphic to $(2^\omega)^\infty$.

Corollary 3.11. For zero-dimensional Polish spaces X , the following are equivalent:

- (1) $C_k(X, 2)$ is sequential but not metrizable;
- (2) $C_k(X, 2)$ is homeomorphic to $(2^\omega)^\infty$;
- (3) X is not locally compact but X' is compact.

Proof. (1) \Rightarrow (3): Since $C_k(X, 2)$ is not metrizable, X is not locally compact (Lemma 3.6). By Theorem 3.1, X' is compact.

(3) \Rightarrow (1): Since X is not locally compact, $C_k(X, 2)$ is not metrizable (Lemma 3.6). By Theorem 3.1, the compactness of X' implies that $C_k(X, 2)$ is sequential.

(1) \Rightarrow (2): By [9, Corollary 8], $C_k(X, 2)$ has countable cs^* -character. Applying Theorem 3.4, we conclude that $C_k(X, 2)$ contains an open mk_ω -subgroup. Since $C_k(X, 2)$ is separable, it is an mk_ω -group. To finish the proof apply Theorem 3.10. \square

The space $C_k(\omega \times M, 2)$ is homeomorphic to $C_k(M, 2)^\omega$, and hence to $((2^\omega)^\infty)^\omega$. Thus, a negative answer to the following question would imply that $C_k(P)$ is not WAP for “most” Polish spaces, including ω^ω and some σ -compact ones.

Question 3.12. Does the space $((2^\omega)^\infty)^\omega$ have the property WAP? What about S_2^ω ?

References

- [1] T. Banach, Topological classification of zero-dimensional \mathcal{M}_ω -groups, *Mat. Stud.* 15 (2001) 109–112.
- [2] C. Borges, On stratifiable spaces, *Pacific J. Math.* 17 (1966) 1–16.
- [3] T. Banach, L. Zdomskyy, The topological structure of (homogeneous) spaces and groups with countable cs^* -character, *Appl. Gen. Topol.* 5 (2004) 25–48.

- [4] E. van Douwen, The integers and topology, in: K. Kunen, J.E. Vaughan (Eds.), *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 111–167.
- [5] G. Gruenhage, K. Tamano, If X is σ -compact Polish, then $C_k(X)$ has a σ -closure-preserving base, *Topology Appl.* 151 (2005) 99–106.
- [6] S. Lin, A note on Arens space and sequential fan, *Topology Appl.* 81 (1997) 185–196.
- [7] E. Michael, Some extension theorems for continuous functions, *Pacific J. Math.* 3 (1953) 789–806.
- [8] R. Pol, Normality in function spaces, *Fund. Math.* 84 (1974) 145–155.
- [9] B. Tsaban, L. Zdomskyy, On the Pytkeev property in spaces of continuous functions, II, *Houston J. Math.* 35 (2009) 563–571.