ÜBUNGEN FÜR 19.03.2014

Exercise 1. *Let $\langle \kappa_i : i \in \alpha \rangle$ be a sequence of cardinals. We define the infinite sum of cardinals to be:*

$$\sum_{i \in \alpha} \kappa_i = |\bigcup_{i \in \alpha} X_i|,$$

where $\{X_i : i \in \alpha\}$ is a disjoint family of sets such that $|X_i| = \kappa_i$ for each $i \in \alpha$. Show that this definition makes sense (using AC) and show that for an infinite cardinal λ the following equation holds: $\sum_{i \in \lambda} \kappa_i = \lambda \cdot \sup_{i \in \lambda} \kappa_i$.

Exercise 2. *Let $\langle \kappa_i : i \in \alpha \rangle$ be as above. We define the infinite product of cardinals as follows:*

$$\prod_{i \in \alpha} \kappa_i = |\prod_{i \in \alpha} X_i|,$$

where the X_i's are such that $|X_i| = \kappa_i$ for each $i \in \alpha$ and $\prod_{i \in \alpha} X_i := \{f : f$ is a function, $\mathrm{dom}(f) = \alpha$, and $\forall i \in \alpha (f(i) \in X_i)\}$. Show that this is a well-defined notion and show that for an infinite cardinal λ and a non-decreasing sequence $\langle \kappa_i : i \in \lambda \rangle$ of cardinals the following equation holds: $\prod_{i \in \lambda} \kappa_i = (\sup_{i \in \lambda} \kappa_i)^\lambda$.

Exercise 3. *Prove that if $\langle \kappa_i : i \in \alpha \rangle$ and $\langle \lambda_i : i \in \alpha \rangle$ are two sequences of cardinals such that for each $i \in \alpha$, $\kappa_i < \lambda_i$, then $\sum_{i \in \alpha} \kappa_i < \prod_{i \in \alpha} \lambda_i$. Use this to prove Koenig's Theorem (Kunen Theorem 1.13.12.).*

Exercise 4. *Prove that $|[\beth_\omega]^\omega| = \prod_{n \in \omega} \beth_n = \beth_{\omega+1}$.*

Exercise 5. *Let κ be an infinite cardinal and $\alpha < \kappa^+$. Prove that there exist $X_n \subset \alpha$, $n \in \omega$, such that o.t.$(X_n) < \kappa^n$ (here we condider ordinal exponentiation) and $\alpha = \bigcup_{n \in \omega} X_n$.*

The last fact is known as "Milner-Rado Paradox".

Exercise 6. *Let κ be an infinite cardinal and $\prec$ be a well-order on κ. Prove that there exists $X \in [\kappa]^\kappa$ such that $\prec \cap X^2 = \in \cap X^2$, i.e., $\prec$ and $\in$ coincide on X.*

Übungen für 26.03.2014

Exercise 1 (Kunen I.13.34). *Let W be a vector space over some field F, and let $W^* = Hom(W, F)$ be the dual vector space. Consider W as a subspace of W^{**} as usually ($x \in W$ is identified with the map $\phi \mapsto \phi(x)$ in W^{**}). Let $W_0 = W$ and $W_{n+1} = W_n^{**}$, so that $W_n \subset W_{n+1}$. Let $W_\omega = \bigcup_{n\in\omega} W_n$. Prove that if $|F| < \beth_\omega$ and $\omega \leq \dim(W) < \beth_\omega$, then $|W_\omega| = \dim(W_\omega) = \beth_\omega$.*

Exercise 2 (Kunen I.13.36). *Assume CH. Prove that $\omega_n^\omega = \omega_n$ for all $n < \omega$.*

Exercise 3 (Kunen I.13.39). *Suppose that κ is an infinite cardinal, $\alpha = \bigcup_{n<c} X_n$ for some $c < \omega$, and the order type of each X_n is less than κ^ω (ordinal exponentiation!). Show that $\alpha < \kappa^\omega$.*

Exercise 4 (Kunen 1.15.10). *Let $\mathfrak{B}$ be any structure for $\mathcal{L}$ such that $\max\{|\mathcal{L}|, \omega\} \leq \kappa \leq |B|$ for some infinite cardinal κ. Suppose that $S \subset B$ has size $|S| \leq \kappa$. Show that there exists an elementary submodel $\mathfrak{A}$ of $\mathfrak{B}$ such that $S \subset A$ and $|A| = \kappa$.*
***Hint**: Use I.13.22 and I.13.21, or just look it up in some model theory book.*

Exercise 5. *Prove that $(\mathbb{Q}, <)$ is an elementary substructure of $(\mathbb{R}, <)$.*
***Hint**: Use the previous exercise to find countable $X \supset \mathbb{Q}$ such that $(X, <)$ is an elementary substructure of $(\mathbb{R}, <)$. Then construct a monotone bijection $\phi : \mathbb{Q} \to X$ (this is the famous Cantor's back and forth argument which you may find in many books or just reinvent!), and argue that it may be extended to a monotone bijection $\bar{\phi} : \mathbb{R} \to \mathbb{R}$ by the completeness of $\mathbb{R}$.*
There are of course other approaches.

Übungen für 02.04.2014

Exercise 1 (Kunen I.16.6)**.** *(ZF^-). Let pow(x, y) be $\forall z(z \subset x \to z \in y)$. Let γ be a limit ordinal and $a, b \in R(\gamma)$. Prove that $R(\gamma) \models pow(a, b)$ iff $b = \mathcal{P}(a)$, i.e., $R(\gamma) \prec_{pow} V$.*

Exercise 2 (Kunen I.16.8)**.** *(ZFC^-). Assume that $0 < \gamma < \delta$ are ordinals and $R(\gamma) \prec R(\delta)$. Prove that $R(\gamma) \models ZFC$, and hence also $R(\delta) \models ZFC$. You may use the fact that $R(\gamma) \models ZC$ for any limit γ.*

Exercise 3 (Kunen I.16.9)**.** *Assume that $ZFC \vdash \exists\gamma[R(\gamma) \models ZFC]$. Show that ZFC is inconsistent.*

Exercise 4 (Kunen I.16.10)**.** *Show how to modify Definition I.15.5 to give a correct definition of $(V, \in) \models \varphi(\delta)$ in the case of Δ_0 formulas.*

Exercise 5 (Kunen I.16.17)**.** *Describe a two-element non-transitive M that is isomorphic to $\{0, 1\}$, such that $\cap^M$ is defined but $\cap$ is not absolute for M, and such that $\subset$ is not absolute for M.*

Übungen für 09.04.2014

Work in ZF unless otherwise indicated.

Exercise 1. *Which axioms of ZF are true in ON?*

Exercise 2. *(AC). For* $\kappa > \omega$*, show that* $|H(\kappa)| = 2^{<\kappa}$*.*

Exercise 3. *(AC). For* $\kappa > \omega$*, show that* $H(\kappa) = R(\kappa)$ *iff* $\kappa = \beth_\kappa$*.*

Exercise 4. *Show that in* $R(\omega + \omega)$*, it is not true that every well-ordering is isomorphic to an ordinal.*
Hint. Consider $2 \times \omega$*, ordered lexicographically. Track down the specific instance of Replacement which fails in* $R(\omega + \omega)$*.*

Exercise 5. *(AC.) Recall that Zermelo set theory, Z, is ZF without Replacement. Show that for all* $\kappa > \omega$*,* $H(\kappa)$ *is a model for* $Z - P$*. Show that the Power Set Axiom is true in* $H(\kappa)$ *iff* $\kappa = \beth_\gamma$ *for some limit* γ*. Show that Replacement fails in* $H(\beth_\omega)$*.*

Übungen für 30.04.2014

Exercise 1 (II.4.8)**.** *Prove that the notions "R well-orders A" and "R is well-founded on A" are absolute for $R(\gamma)$ for any limit γ.*

Why can't we use here II.4.7 directly?

Exercise 2 (II.4.6)**.** *Let γ be a limit ordinal such that $\forall\alpha < \gamma[\alpha^2 < \gamma]$. Show that ordinal sum and product are defined in $R(\gamma)$ and are absolute for $R(\gamma)$.*

Exercise 3 (II.4.9)**.** *(ZFC). Prove that $R(\gamma) \Vdash AC^+$ and $H(\kappa) \Vdash AC^+$ for any limit γ and regular κ.*

Exercise 4 (II.4.21)**.** *Let AI be "our standard" Axiom of Infinity, and let AU denote the* ***Axiom des Undendlichen*** *of Zermello: $\exists x(\emptyset \in x \wedge \forall y \in x(\{y\} \in x))$. Work in ZFC and produce transitive models for ZC+$\neg AU$ and for ZC–Inf+AU+$\neg AI$.*

Exercise 5 (II.4.22)**.** *Find a transitive $M \vDash ZC - P$ in which $\omega \times \omega$ and $\omega^* := \{\{n\} : n \in \omega\}$ do not exist.*

There are hints to almost all of these exercises in the book. Feel free to use them!

Übungen für 7.05.2014

Exercise 1 (II.4.26). *Let M be a transitive class, and assume that the axioms of Extensionality, Comprehension, Pairing, Union, and Infinity hold in M. Prove that $\omega \in M$.*

Exercise 2 (II.4.29). *Let M be a transitive model for ZF–P. Let $\star, * \in M$ be two group operations on ω. Prove that the statement $(\omega, *) \cong (\omega, \star)$ is absolute for M.*

Exercise 3 (II.5.6). *Assume AC. Find a formula ϕ such that every transitive M satisfying $M \prec_\phi V$ is of the form $R(\gamma)$ for some ordinal $\gamma = \beth_\gamma$.*

Exercise 4 (II.5.12). *Work in ZFC plus the assumption that $R(\gamma) \models$ZFC for some γ. Prove that the minimal such γ has cofinality ω.*

Exercise 5 (II.5.13). *Show that there is a finite set Λ of instances of the Comprehension axiom such that Λ together with the axioms of ZF other than Comprehension, proves all instances of Comprehension.*

There are hints in the book simplifying these exercises greatly!

Übungen für 14.05.2014. Mengenlehre 1

Exercise 1 (II.6.30)**.** *Convince yourself*[1] *that the class $L[A]$ defined in II.6.29 is a transitive model of ZFC if A consists of ordinals. Find the place in the argument where the fact that $A \subset ON$ is used! Prove that $L[A] \vDash GCH$ for $A \subset \omega$.*

Exercise 2 (II.6.31)**.** *Suppose that $V = L[A]$ for some $A \subset \omega_1$. Prove that GCH holds in $L[A]$.*

Later we shall show that $V = L[A]$ is essential in the above exercise.

Exercise 3 (II.6.33)**.** *Assume $V = L$ and prove that $L(\alpha) = R(\alpha)$ iff $\alpha = \aleph_\alpha$.*

Exercise 4 (III.2.7)**.** *Let κ be singular. Show that there is a family $\mathcal{A}$ of κ two-element subsets of κ such that no $\mathcal{B} \in [\mathcal{A}]^\kappa$ forms a delta system.*

Exercise 5 (Folklore)**.** *Let $\mathcal{A}$ be an uncountable collection of finite subsets of ω_1 and M an elementary submodel of $H(\omega_2)$ containing $\mathcal{A}$ as an element. Let $A \in \mathcal{A} \setminus M$ and $D = A \cap M$. Prove that there exists an uncountable delta system $\mathcal{B} \subset \mathcal{A}$, $\mathcal{B} \in M$, with kernel D.*

Hint: pick in M a maximal delta subsystem of $\mathcal{A}$ with the kernel D and show that it is uncountable. Use the fact that if $|X| = \omega$ and $X \in M$ then $X \subset M$.

The same ideas as in the above exercise allow to prove also more general instances of the delta system lemma.

[1] I will not ask you to present this near blackboard because this is analogous to the case of L and lengthy.

Übungen für 21.05.2014. Mengenlehre 1

Recall from III.3.23 that a subset C of a poset $\mathbb{P}$ is *centered*, if for any $n \in \omega$ and all $p_1, \dots, p_n \in C$ there exists $q \in \mathbb{P}$ such that $q \leq p_i$ for all $i \leq n$. If, moreover, q may be found in C, then C is called a *filter*. A poset $\mathbb{P}$ is called *σ-centered* if it can be written as a countable union of its centered subsets.

Exercise 1 (III.3.27(part 1))**.** *If X is a compact Hausdorff space, then X is separable iff O_X is σ-centered iff O_X is a countable union of filters. Here O_X is ordered by inclusion, i.e., $U \leq V$ means $U \subset V$.*

The standard base for the topology on 2^A consists of sets $[s]$, $s \in Fn(A, 2)$, where $[s] = \{x \in 2^A : x \restriction \mathrm{dom}(s) = s\}$. Thus $U \subset 2^A$ is open iff it is a union of a collection of sets of the form $[s]$.

Exercise 2 (III.3.27(part 2))**.** *Let κ be a cardinal and $X = 2^\kappa$. Show that O_X is ccc. Show that O_X is σ-centered iff $\kappa \leq 2^\omega$.*

Hint: If $\kappa \leq 2^\omega$, then take any metrizable separable topology on κ (e.g., via some bijection with a subset of $\mathbb{R}$), fix a countable base $\mathcal{B}$ for this topology, and look at characteristic functions of finite unions of elements of $\mathcal{B}$. For the case $\kappa > 2^\omega$ show that a separable space cannot have more than 2^ω mutually different clopen subsets.

Exercise 3 (IV.2.8)**.** *Let $\tau = \{\langle \emptyset, p\rangle, \langle \{\langle \emptyset, q\rangle\}, r\rangle\}$. Compute τ_G for each of the 8 possibilities for p, q, r being $\in$ or $\notin G$.*

Exercise 4 (IV.2.16)**.** *Using the notation of Lemma IV.2.15, replace the definition of π by: $\pi = \{\langle \upsilon, p\rangle : \exists \langle \sigma, q\rangle \in \tau \exists r[\langle \upsilon, r\rangle \in \sigma \wedge p \leq r \wedge p \leq q]\}$. Let $b = \pi_G$ and show that $\cup a = b$.*

Exercise 5 (IV.2.28)**.** *Let M be a ctm for ZFC. Find a poset $\mathbb{P}$ and a sentence $\psi \in \mathcal{FL}_{\mathbb{P}} \cap M$ and two different generic filters G, H with $M[G] = M[H]$ and $M[G] \vDash \psi$ and $M[H] \nvDash \psi$ because some τ_G differs from τ_H.*

Übungen für 28.05.2014. Mengenlehre 1

Exercise 1. *Let M be a ctm for ZFC and $\mathbb{P} \in M$ be a poset. Let also $G \subset \mathbb{P}$ be a filter. Show that the following conditions are equivalent:*

(1) $G \cap D \neq \emptyset$, *whenever* $D \in M$ *and* D *is dense in* $\mathbb{P}$;
(2) $G \cap A \neq \emptyset$, *whenever* $A \in M$ *and* A *is a maximal antichain in* $\mathbb{P}$;
(3) $G \cap E \neq \emptyset$, *whenever* $E \in M$ *and for every* $p \in \mathbb{P}$ *there exists* $q \in E$ *such that* p *and* q *are compatible.*

Furthermore, show that in all these items, if we assume that G is just a centered subset of $\mathbb{P}$, then it is automatically a filter.

Exercise 2 (IV.2.46)**.** *Assume that M is a ctm for ZFC, and let $\mathbb{P} = Fn(\omega, 2)$. Then there is a filter G on $\mathbb{P}$ such that there is no transitive $N \supset M$ such that $G \in N$, $N \vDash ZF - P$, and $o(N) = o(M)$.*

Exercise 3. *Let M be a ctm for ZFC and $\mathbb{P} \in M$ be a poset. Suppose that $\tau \in M^{\mathbb{P}}$ and $\mathrm{dom}(\tau) \subset \{\check{n} : n \in \omega\}$. Let*

$$\sigma = \{\langle \check{n}, p \rangle : \forall q \in \mathbb{P}(\langle \check{n}, q \rangle \in \tau \to p \perp q)\}.$$

Show that $\sigma_G = \omega \setminus \tau_G$, where G is a $\mathbb{P}$-generic over M.

Exercise 4. *Let M be a ctm for ZFC and $\mathbb{P} = (2^{<\omega_1})^M$, where $p \leq q$ means that p is an extension of q. Let G be a $\mathbb{P}$-generic over M. Show that in $M[G]$ there exists a bijection between $(\omega_1)^M$ and $(2^\omega)^M$.*

Hint: Look at the restrictions of $\bigcup G$ to intervals $[\alpha, \alpha + \omega)$ for $\alpha < (\omega_1)^M$.

Exercise 5 (IV.2.47)**.** *Assume that M is a ctm for ZFC. Give an example of $\mathbb{P} \in M$ and a (non-generic) filter G on $\mathbb{P}$ for which $\mathbb{P} \setminus G \notin M[G]$.*

Übungen für 4.06.2014. Mengenlehre 1

In the following, unless we state otherwise: M represents a c.t.m. for ZFC, $\mathbb{P} \in M$ is a p.o., and G is a filter which is $\mathbb{P}$-generic over M.

Exercise 1. *Assume that $\mathbb{P}$ doesn't have the largest element. For an element $x \in M$ redefine the name $\check{x}$ so that $\check{x}_G = x$.*

Exercise 2. *Suppose $\langle \mathbb{P}, \leq \rangle$ is a partial order in M which may or may not have a largest element. In M, fix $1 \notin \mathbb{P}$, and define the p.o. $\langle \mathbb{Q}, \leq, 1 \rangle$ by: $\mathbb{Q} = \mathbb{P} \cup \{1\}$ where $\mathbb{P}$ retains the same order and $\forall p \in \mathbb{P}(p < 1)$. Show that if $G \subset \mathbb{P}$ is a filter, G is $\mathbb{P}$-generic over M iff $G \cup \{1\}$ is $\mathbb{Q}$-generic over M, and $M[G]$ (defined as a $\mathbb{P}$-extension) is the same as $M[G \cup \{1\}]$ (defined as a $\mathbb{Q}$-extension).*

Exercise 3. *Assume $f : A \to M$ and $f \in M[G]$. Show that there is a set $B \in M$ such that $f : A \to B$.*

Hint. Let $B = \{b : \exists p \in \mathbb{P}(p \Vdash \check{b} \in \mathrm{ran}(\tau))\}$, *where* $f = \tau_G$.

Exercise 4. *Assume α is a cardinal of M. Show that the following are equivalent.*

(1) *Whenever $B \in M$, ${}^{\alpha}B \cap M = {}^{\alpha}B \cap M[G]$;*
(2) *${}^{\alpha}M \cap M = {}^{\alpha}M \cap M[G]$;*
(3) *In M: The intersection of α many dense open subsets of $\mathbb{P}$ is dense.*

Recall that a subset O of $\mathbb{P}$ is open if for every $p \in O$ and $q \leq p$ we have $q \in O$ (i.e., O is downwards closed).

A p.o. satisfying (3) is called α^+-*Baire*. κ-*Baire* means that the intersection of less than κ dense open sets is dense.

Exercise 5. *Let $\mathbb{P} \in M$ be non-atomic. Let*

$$M = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots \quad (n \in \omega)$$

be such that $M_{n+1} = M_n[G_n]$ for some G_n which is $\mathbb{P}$-generic over M_n. Show that $\bigcup_{n\in\omega} M_n$ cannot satisfy the Power Set Axiom. Furthermore, show that the G_n may be chosen so that there is no c.t.m. N for ZFC with $\langle G_n : n \in \omega \rangle \in N$ and $o(N) = o(M)$.

Hint, $\{n : p \in G_n\}$ *can code* $o(M)$.

Übungen für 11.06.2014.

In the following, unless we state otherwise: M represents a c.t.m. for ZFC, $\mathbb{P} \in M$ is a p.o., and G is a filter which is $\mathbb{P}$-generic over M.

A poset $\mathbb{P}$ is called λ-*closed*, where λ is a cardinal, if every decreasing sequence $\langle p_\xi : \xi < \alpha \rangle$ of elements of $\mathbb{P}$ of length $\alpha < \lambda$ has a lower bound.

Exercise 1. *Prove that every λ-closed poset is λ-Baire (see the definition on the previous exercise sheet). Show that if $\mathbb{P}$ is λ-closed and λ is singular then $\mathbb{P}$ is λ^+-closed.*

Exercise 2. *Suppose that $\mathbb{P}$ is countable and non-atomic. Show that there is a dense embedding from $\{p \in Fn(\omega, \omega) : \mathrm{dom}(p) \in \omega\}$ into $\mathbb{P}$.*

Hint. Map $\{p : \mathrm{dom}(p) = 1\}$ onto an infinite antichain in $\mathbb{P}$, now handle $\{p : \mathrm{dom}(p) = 2\}$, etc.

It follows from the exercise above that all countable non-atomic posets yield the same generic extensions.

Observe that for every forcing poset $\mathbb{P}$, each map $i : \mathbb{P} \to \mathbb{P}$ gives rise to a natural map $i^* : M^{\mathbb{P}} \to M^{\mathbb{P}}$ defined as follows: $i^*(\tau) = \{\langle i^*(\sigma), i(p)\rangle : \langle \sigma, p\rangle \in \tau\}$.

Exercise 3. *If $\mathbb{P}$ (i.e., $\langle \mathbb{P}, \leq, 1_{\mathbb{P}}\rangle$) is a p.o., an automorphism of $\mathbb{P}$ is a 1-1 map i from $\mathbb{P}$ onto $\mathbb{P}$ which preserves $\leq$ and satisfies $i(1_{\mathbb{P}}) = 1_{\mathbb{P}}$; thus also $i^*(\check{x}) = \check{x}$ for each x. $\mathbb{P}$ is called* almost homogeneous *iff for all $p, q \in \mathbb{P}$, there is an automorphism i of $\mathbb{P}$ such that $i(p)$ and q are compatible. Suppose that $\mathbb{P} \in M$ and $\mathbb{P}$ is almost homogeneous in M. Show that if $p \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$, then $1_{\mathbb{P}} \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$; thus, either $1_{\mathbb{P}} \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$ or $1_{\mathbb{P}} \Vdash \neg\phi(\check{x}_1, \ldots, \check{x}_n)$.*

Exercise 4. *Show that any $Fn(I, J, \kappa)$ is almost homogeneous.*

For $\mathbb{P} = Fn(\omega, 2)$ give an example showing that the conclusion of the previous exercise is not any more true for arbitrary names.

Exercise 5. *Let κ be a cardinal of uncountable cofinality and $f : \kappa \to \kappa$. Show that there exists a closed and unbounded $C \subset \kappa$ such that for all $\alpha \in C$ and $\beta \in \alpha$ we have that $f(\beta) \in \alpha$ (i.e., $\mathrm{range}(f \restriction \alpha) \subset \alpha$).*

Übungen für 18.06.2014.

In the following, unless we state otherwise: M represents a c.t.m. for ZFC, $\mathbb{P} \in M$ is a p.o., and G is a filter which is $\mathbb{P}$-generic over M.

Exercise 1. *κ is called* strongly Mahlo *iff κ is strongly inaccessible and*

$\{\alpha < \kappa : \alpha \textit{ is regular}\}$

is stationary in κ. Show that for such κ,

$\{\alpha < \kappa : \alpha \textit{ is strongly inaccessible}\}$

is stationary in κ.

Exercise 2. *Let $(\mathbb{P} = Fn(I, 2, \omega_1))^M$, where $(|I| \geq \omega_1)^M$. Show that $M[G]$ satisfies CH, regardless of whether M does.*

Exercise 3. *Suppose, in M, $\omega = \mathrm{cf}(\lambda) < \lambda$. Show that $Fn(\lambda, 2, \lambda)^M$ adds a map from ω onto λ^+.*

Exercise 4. *Assume in M that $\kappa > \omega$, κ is regular, and $\mathbb{P}$ has the κ-c.c. In $M[G]$, let $C \subset \kappa$ be c.u.b. Show that there exists $C' \subset C$ such that $C' \in M$ and C' is c.u.b. in κ.*

Exercise 5. *Suppose that in M: $S \subset \omega_1$ is stationary and $\mathbb{P}$ is either c.c.c. or ω_1-closed. Show that S remains stationary in $M[G]$.*