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CLOSED LOCALLY PATH-CONNECTED SUBSPACES OF FINITE-DIMENSIONAL GROUPS ARE LOCALLY COMPACT

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Abstract. We prove that each closed locally continuum-connected subspace of a finite dimensional topological group is locally compact. This allows us to construct many 1-dimensional metrizable separable spaces that are not homeomorphic to closed subsets of finite-dimensional topological groups, which answers in negative a question of D.Shakhmatov. Another corollary is a characterization of Lie groups as finite-dimensional locally continuum-connected topological groups. For locally path connected topological groups this characterization was proved by Gleason and Palais in 1957.

1. Introduction

It follows from the classical Menger-Nöbeling-Pontryagin Theorem that each separable metrizable space X of dimension $n = \dim X < \infty$ admits a topological embedding $e : X \to G$ into a metrizable separable group G of dimension $\dim(G) = 2n + 1$ (for such a group G we can take the $(2n + 1)$-dimensional Euclidean space $\mathbb{R}^{2n+1}$). In [15] (see also [4, Question 7]) Dmitri Shakhmatov asked if we can additionally require of $e : X \to G$ to be a **closed** embedding? In this paper we shall give a strongly negative answer to this question constructing simple 1-dimensional spaces that are

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not homeomorphic to closed subspaces of finite-dimensional topological groups.

First we state a Key Lemma treating locally continuum-connected subspaces of finite-dimensional topological groups. By a *continuum* we understand a connected compact Hausdorff space. We shall say that two points x, y of a topological space X are connected by a *subcontinuum* $K \subset X$ if $x, y \in K$.

Following [3] we define a topological space X to be *locally continuum-connected* at a point $x \in X$ if for every neighborhood $U \subset X$ of x there is another neighborhood $V \subset U$ of x such that each point $y \in V$ can be connected with x by a subcontinuum $K \subset U$. A space X is *locally continuum-connected* if it is locally continuum-connected at each point. It is clear that X is locally continuum-connected at $x \in X$ if X is locally path-connected at x.

The "locally path-connected" version of the following lemma was proved by A.Gleason [6] and D.Montgomery [13] and then was used by A.Gleason and R.Palais [7] to prove that locally path-connected finite-dimensional topological groups are Lie groups. We shall say that a topological group G is *compactly finite-dimensional* if

$$\text{co-dim}(G) = \sup\{\dim(K) : K \text{ is a compact subspace of } G\}$$

is finite. Observe that a subgroup of a compactly finite- dimensional group also is compactly finite-dimensional.

Lemma 1.1 (Key Lemma). *If a subspace X of a compactly finite-dimensional topological group G is locally continuum-connected at a point $x \in X$, then some neighborhood $U \subset X$ of x has compact closure in G.*

Because of its technical character, we postpone the proof of this lemma till the end of the paper. Now we state several corollaries of this lemma. We recall that a regular topological space X is *cosmic* if it is a continuous image of a separable metrizable space. This is equivalent to the countability of the network weight of X.

Theorem 1.2. *Each (closed) locally continuum-connected cosmic subspace X of a compactly finite-dimensional topological group G is metrizable (and locally compact).*

Proof. Replacing G by the group hull of X, if necessary, we may assume that X algebraically generates G. Then G is a cosmic space and hence all compact subsets of G are metrizable. By Key Lemma, each point $x \in X$ has a neighborhood $U \subset X$ with compact (and thus metrizable) closure $\overline{U}$ in G. If X is closed in G, then $\overline{U} \subset X$ is compact, witnessing that X is locally compact. Being locally metrizable and paracompact (because of Lindelöf), the space X is metrizable. □

Corollary 1.3. *For a finite-dimensional locally continuum-connected cosmic space X the following conditions are equivalent:*

(1) *X admits a closed embedding into a compactly finite-dimensional topological group;*
(2) *X admits a closed embedding into $\mathbb{R}^{2n+1}$ where $n = \dim(X)$;*
(3) *X is metrizable and locally compact.*

This corollary supplies us with many examples of finite-dimensional second countable spaces admitting no closed embedding into (compactly) finite-dimensional topological groups.

Probably the simplest one is the hedgehog

$$H_\omega = \bigcup_{n\in\omega} [0,1] \cdot \vec{e}_n \subset l_2$$

where $(\vec{e}_n)$ is the standard orthonormal basis in the Hilbert space l_2.

Corollary 1.4. *No compactly finite-dimensional topological group contains a closed subspace homeomorphic to the hedgehog H_ω.*

Besides the metrizable topology the hedgehog carries also a natural non-metrizable topology, namely the strongest topology inducing the original Euclidean topology on each needle $[0,1] \cdot \vec{e}_n$, $n \in \omega$. The hedgehog endowed with this non-metrizable topology will be denoted by V_ω and will be referred to as the *Fréchet-Urysohn hedgehog* (by analogy with the Fréchet-Urysohn fan).

The metrizable hedgehog H_ω admits no closed embedding into a compactly finite-dimensional group, but can be embedded into the 2-dimensional group $\mathbb{R}^2$. In contrast with this, by Theorem 1.2, nothing similar cannot be done for the Fréchet- Urysohn hedgehog V_ω.

Corollary 1.5. *No compactly finite-dimensional topological group contains a subspace homeomorphic to the Fréchet-Urysohn hedgehog V_ω.*

Remark 1.6. By Corollaries 1.4 and 1.5, a topological group G containing a topological copy of V_ω **or** a closed topological copy of H_ω is not finite-dimensional. By its form, this results resembles a result of [2] or [1]: a topological group with countable pseudocharacter containing a topological copy of V_ω **and** a closed topological copy of H_ω is not sequential. We do not know if this resemblance is occasional.

It is also interesting to compare Corollaries 1.4 and 1.5 with a result of J.Kulesza [11] who proved that the metric hedgehog H_{ω_1} with ω_1 spines cannot be embedded into a finite-dimensional topological group. The other his result says that the hengehog H_3 with three spines cannot be embedded into a 1-dimensional topological group.

Key Lemma has another interesting corollary related to the famous fifth problem of Hilbert.

Corollary 1.7. *A topological group G is a Lie group if and only if G is compactly finite-dimensional and locally continuum-connected.*

Proof. The "only if" part is trivial. To prove the "if" part, take any locally continuum-connected compactly finite-dimensional topological group G. Key Lemma implies that G is locally compact and hence finite-dimensional. By [14, p.185], the group G is a Lie group, being locally compact, locally connected, and finite-dimensional. □

This corollary generalizes Gleason-Palais Theorem [7] stating that each locally path-connected finite-dimensional topological group is a Lie group. The Gleason-Palais Theorem was applied by S. N. Hudson [9] to show that each path-connected locally connected finite-dimensional topological group is a Lie group.

Problem 1.8. *Let G be a locally connected finite-dimensional topological group. Is G a Lie group if G is continuum-connected? (The latter means that any two points of G can be connected by a subcontinuum of G).*

It should be mentioned that there exists a connected locally connected subgroup of $\mathbb{R}^2$ that contains no arc, see [10].

2. A Dimension Lemma

The proof of Key Lemma relies on the following (probably known) fact from Dimension Theory.

Lemma 2.1. *Let $K_1, \dots, K_n$ be continua and for every $i \le n$ let $a_i, b_i \in K_i$ be two distinct points. Let $K = \prod_{i=1}^n K_i$ and $A_i = \mathrm{pr}_i^{-1}(a_i)$, $B_i = \mathrm{pr}_i^{-1}(b_i)$ where $\mathrm{pr}_i : K \to K_i$ is the projection. Let $f : K \to X$ be a continuous map to a Hausdorff topological space such that $f(A_i) \cap f(B_i) = \emptyset$ for all $i \le n$. Then $\dim f(K) \ge n$.*

Proof. Assume conversely that $\dim f(K) < n$ and apply Theorem on Partitions [5, 3.2.6] to find closed subsets $P_1, \dots, P_n$ of $f(K)$ such that $\bigcap_{i=1}^n P_i = \emptyset$ and each P_i is a partition between $f(A_i)$ and $f(B_i)$ in $f(K)$ (the latter means that A_i and B_i lie in different connected components of $f(K) \setminus P_i$).

Then for every $i \le n$ the set $L_i = f^{-1}(P_i)$ is a partition between A_i and B_i in K and $\bigcap_{i=1}^n L_i = \emptyset$. This means that the sequence (A_1, B_1), $\dots$, (A_n, B_n) is inessential in $K = \prod_{i=1}^n K_i$, which contradicts the results of Holsztyński [8] or Lifanov [12], see also [5, 1.8.K]. □

3. Proof of Key Lemma

Let G be a topological group with $n = \text{co-dim}(G) < \infty$ and X be a subspace of G that is locally continuum-connected at a point $e \in X$. We have to show that e has a neighborhood $U \subset X$ with compact closure in G.

Assuming the converse we shall derive a contradiction. Without loss of generality, the point e is the neutral element of the group G. By finite induction, we shall construct a sequence $(C_i)_{i=0}^n$ of subcontinua of $X \subset G$ connecting e with some other point $a_i \in X \setminus \{e\}$ such that for the "cube" $K_i = \prod_{j \le i} C_j$ and the projections $\mathrm{pr}_{k,i} : \prod_{j \le i} C_j \to C_k$, $k \le i$, the "faces" $A_{k,i} = \mathrm{pr}_{k,i}^{-1}(e)$, $B_{k,i} = \mathrm{pr}_{k,i}^{-1}(a_k)$ have disjoint images under the map

$$f_i : \prod_{j \le i} C_j \to G, \quad f_i : (x_0, \dots, x_i) \mapsto x_0 \cdot \dots \cdot x_i.$$

By Dimension Lemma 2.1 this will imply

$$n = \text{co-dim}\, G \ge \dim f_i(K_i) \ge i + 1,$$

which is not possible for $i = n$.

Assume that for some $i \le n$ the pointed continua (C_j, a_j), $j < i$, have been constructed, so that for every $k < i$ the "faces" $A_{k,i-1}$, $B_{k,i-1}$ have disjoint images $f_{i-1}(A_{k,i-1})$ and $f_{i-1}(B_{k,i-1})$ (we start the induction from the trivial case $i = 0$). The compactness argument yields us a neighborhood $U_i \subset G$ of e such that the sets $f_{i-1}(A_{k,i-1}) \cdot U_i$ and $f_{i-1}(B_{k,i-1}) \cdot U_i$ do not intersect for all $k < i$. By the local continuum-connectedness of X at e, there is a neighborhood $V_i \subset U_i \cap X$ of e in X such that any point $x \in V_i$ can be connected with e by a subcontinuum $C \subset U_i$. Consider the compact subset $K = \{x^{-1}y : x, y \in f_{i-1}(K_{i-1})\}$ of G. By our assumption, no neighborhood of e in X has compact closure in G. Consequently, $V_i \not\subset K$ and we can find a point $a_i \in V_i \setminus K$. By the choice of V_i, there is a compact connected subset $C_i \in U_i$ containing the points e and a_i.

To complete the inductive step it now suffices to check that

$$f_i(A_{k,i}) \cap f_i(B_{k,i}) = \emptyset$$

for all $k \le i$.

If $k < i$, then $A_{k,i} = A_{k,i-1} \times C_i$ and $f_i(A_{k,i}) \subset f_{i-1}(A_{k,i-1}) \cdot C_i \subset f_{i-1}(A_{k,i-1}) \cdot U_i$. By analogy, $f_i(B_{k,i}) \subset f_{i-1}(B_{k,i-1}) \cdot U_i$. Consequently, the sets $f_i(A_{k,i})$ and $f_i(B_{k,i})$ do not intersect by the choice of the neighborhood U_i.

If $k = i$, then $A_{k,i} = K_{i-1} \times \{e\}$ and $B_{k,i} = K_{i-1} \times \{a_i\}$. Consequently, $f_i(A_{k,i}) = f_{i-1}(K_{i-1})$ and $f_i(B_{k,i}) = f_{i-1}(K_{i-1}) \cdot a_i$. Now the choice of the point $a_i \notin K$, implies that $f_i(A_{k,i}) \cap f_i(B_{k,i}) = \emptyset$. This finishes the inductive construction as well as the proof of Key Lemma.

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