THE TOPOLOGICAL STRUCTURE OF (HOMOGENEOUS) SPACES AND GROUPS WITH COUNTABLE cs*-CHARACTER

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Abstract. In this paper we introduce and study three new cardinal topological invariants called the cs*-, cs-, and sb-characters. The class of topological spaces with countable cs*-character is closed under many topological operations and contains all $\aleph$-spaces and all spaces with point-countable cs*-network. Our principal result states that each non-metrizable sequential topological group with countable cs*-character has countable pseudo-character and contains an open k_ω-subgroup. This result is specific for topological groups: under Martin Axiom there exists a sequential topologically homogeneous k_ω-space X with $\aleph_0 = \mathrm{cs}^*_\chi(X) < \psi(X)$.

Introduction

In this paper we introduce and study three new local cardinal invariants of topological spaces called the sb-character, the cs-character and cs*-character, and describe the structure of sequential topological groups with countable cs*-character. All these characters are based on the notion of a *network* at a point x of a topological space X, under which we understand a collection $\mathcal{N}$ of subsets of X such that for any neighborhood $U \subset X$ of x there is an element $N \in \mathcal{N}$ with $x \in N \subset U$, see [Lin].

A subset B of a topological space X is called a *sequential barrier* at a point $x \in X$ if for any sequence $(x_n)_{n\in\omega} \subset X$ convergent to x, there is $m \in \omega$ such that $x_n \in B$ for all $n \geq m$, see [Lin]. It is clear that each neighborhood of a point $x \in X$ is a sequential barrier for x while the converse in true for Fréchet-Urysohn spaces.

Under a sb-*network* at a point x of a topological space X we shall understand a network at x consisting of sequential barriers at x. In other words, a collection $\mathcal{N}$ of subsets of X is a sb-network at x if for any neighborhood U of x there is an element N of $\mathcal{N}$ such that for any sequence $(x_n) \subset X$ convergent to x the set N contains almost all elements of (x_n). Changing two quantifiers in this definition by their places we get a definition of a cs-network at x.

Namely, we define a family $\mathcal{N}$ of subsets of a topological space X to be a cs-*network* (resp. a cs*-*network*) at a point $x \in X$ if for any neighborhood $U \subset X$ of x and any sequence $(x_n) \subset X$ convergent to x there is an element $N \in \mathcal{N}$ such that $N \subset U$ and N contains almost all (resp. infinitely many) members of the sequence (x_n). A family $\mathcal{N}$ of subsets of a topological space X is called a cs-*network* (resp. cs*-*network*) if it is a cs-network (resp. cs*-network) at each point $x \in X$, see [Na].

The smallest size $|\mathcal{N}|$ of an sb-network (resp. cs-network, cs*-network) $\mathcal{N}$ at a point $x \in X$ is called the sb-*character* (resp. cs-*character,* cs*-*character*) of X at the point x and is denoted by $\mathrm{sb}_\chi(X,x)$ (resp. $\mathrm{cs}_\chi(X,x)$, $\mathrm{cs}^*_\chi(X,x)$). The cardinals $\mathrm{sb}_\chi(X) = \sup_{x\in X} \mathrm{sb}_\chi(X,x)$, $\mathrm{cs}_\chi(X) = \sup_{x\in X} \mathrm{cs}_\chi(X,x)$ and $\mathrm{cs}^*_\chi(X) = \sup_{x\in X} \mathrm{cs}^*_\chi(X,x)$ are called the sb-*character*, cs-*character* and cs*-*character* of the topological space X, respectively. For the empty topological space $X=\emptyset$ we put $\mathrm{sb}_\chi(X)=\mathrm{cs}_\chi(X)=\mathrm{cs}^*_\chi(X)=1$.

In the sequel we shall say that a topological space X has *countable* sb-*character (resp.* cs-, cs*-*character)* if $\mathrm{sb}_\chi(X)\le\aleph_0$ (resp. $\mathrm{cs}_\chi(X)\le\aleph_0$, $\mathrm{cs}^*_\chi(X)\le\aleph_0$). In should be mentioned that under different names topological spaces with countable sb- or cs-character have already occured in topological literature. In particular, a topological space has countable cs-character if and only if it is *csf*-countable in the sense of [Lin]; a (sequential) space X has countable sb-character if and only if it is universally *csf*-countable in the sense of [Lin] (if and only if it is weakly first-countable in the sense of [Ar$_1$] if and only if it is 0-metrizable in the sense of Nedev [Ne]). From now on, all the topological spaces considered in the paper are T_1-spaces. At first we consider the interplay between the characters introduced above.

Proposition 1. *Let X be a topological space. Then*

1. $\mathrm{cs}^*_\chi(X)\le\mathrm{cs}_\chi(X)\le\mathrm{sb}_\chi(X)\le\chi(X)$*;*
2. $\chi(X)=\mathrm{sb}_\chi(X)$ *if X is Fréchet-Urysohn;*
3. $\mathrm{cs}^*_\chi(X)<\aleph_0$ *iff* $\mathrm{cs}_\chi(X)<\aleph_0$ *iff* $\mathrm{sb}_\chi(X)<\aleph_0$ *iff* $\mathrm{cs}^*_\chi(X)=1$ *iff* $\mathrm{cs}_\chi(X)=1$ *iff* $\mathrm{sb}_\chi(X)=1$ *iff each convergent sequence in X is trivial;*
4. $\mathrm{sb}_\chi(X)\le 2^{\mathrm{cs}^*_\chi(X)}$*;*
5. $\mathrm{cs}_\chi(X)\le\mathrm{cs}^*_\chi(X)\cdot\sup\{\big|[\kappa]^{\le\omega}\big|:\kappa<\mathrm{cs}^*_\chi(X)\}\le\big(\mathrm{cs}^*_\chi(X)\big)^{\aleph_0}$ *where* $[\kappa]^{\le\omega}=\{A\subset\kappa:|A|\le\aleph_0\}$.

Here "iff" is an abbreviation for "if and only if". The Arens' space S_2 and the sequential fan S_ω give us simple examples distinguishing between some of the characters considered above. We recall that the *Arens' space* S_2 is the set $\{(0,0),(\frac1n,0),(\frac1n,\frac1{nm}):n,m\in\mathbb N\}\subset\mathbb R^2$ carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0=\{(0,0),(\frac1n,0):n\in\mathbb N\}$ and $C_n=\{(\frac1n,0),(\frac1n,\frac1{nm}):m\in\mathbb N\}$, $n\in\mathbb N$. The quotient space $S_\omega=S_2/C_0$ obtained from the Arens' space S_2 by identifying the points of the sequence C_0 is called the *sequential fan*, see [Lin]. The sequential fan S_ω is the simplest example of a non-metrizable Fréchet-Urysohn space while S_2 is the simplest example of a sequential space which is not Fréchet-Urysohn.

We recall that a topological space X is *sequential* if a subset $A\subset X$ if closed if and only if A is *sequentially closed* in the sense that A contain the limit point of any sequence $(a_n)\subset A$, convergent in X. A topological space X is *Fréchet-Urysohn* if for any cluster point $a\in X$ of a subset $A\subset X$ there is a sequence $(a_n)\subset A$, convergent to a.

Observe that $\aleph_0=\mathrm{cs}^*_\chi(S_2)=\mathrm{cs}_\chi(S_2)=\mathrm{sb}_\chi(S_2)<\chi(S_2)=\mathfrak d$ while $\aleph_0=\mathrm{cs}^*_\chi(S_\omega)=\mathrm{cs}_\chi(S_\omega)<\mathrm{sb}_\chi(S_\omega)=\chi(S_\omega)=\mathfrak d$. Here $\mathfrak d$ is the well-known in Set Theory small uncountable cardinal equal to the cofinality of the partially ordered set $\mathbb N^\omega$ endowed with the natural partial order: $(x_n)\le(y_n)$ iff $x_n\le y_n$ for all n, see [Va]. Besides $\mathfrak d$, we will need two other small cardinals: $\mathfrak b$ defined as the smallest size of a subset of uncountable cofinality in $(\mathbb N^\omega,\le)$, and $\mathfrak p$ equal to the smallest size $|\mathcal F|$ of a family of infinite subsets of ω closed under finite intersections and having no infinite pseudo-intersection in the sense that there is no infinite subset $I\subset\omega$ such that the complement $I\setminus F$ is finite for any $F\in\mathcal F$, see [Va], [vD]. It is known that $\aleph_1\le\mathfrak p\le\mathfrak b\le\mathfrak d\le\mathfrak c$ where $\mathfrak c$ stands for the size of continuum. Martin Axiom implies $\mathfrak p=\mathfrak b=\mathfrak d=\mathfrak c$, [MS]. On the other hand, for any uncountable regular cardinals $\lambda \le \kappa$ there is a model of ZFC with $\mathfrak{p} = \mathfrak{b} = \mathfrak{d} = \lambda$ and $\mathfrak{c} = \kappa$, see [vD, 5.1].

Unlike to the cardinal invariants cs_χ, sb_χ and χ which can be distinguished on simple spaces, the difference between the cardinal invariants cs_χ and cs^*_χ is more subtle: they cannot be distinguished in some models of Set Theory!

Proposition 2. *Let X be a topological space. Then $\mathrm{cs}^*_\chi(X) = \mathrm{cs}_\chi(X)$ provided one of the following conditions is satisfied:*

1. $\mathrm{cs}^*_\chi(X) < \mathfrak{p}$;
2. $\kappa^{\aleph_0} \le \mathrm{cs}^*_\chi(X)$ *for any cardinal* $\kappa < \mathrm{cs}^*_\chi(X)$;
3. $\mathfrak{p} = \mathfrak{c}$ *and* $\lambda^\omega \le \kappa$ *for any cardinals* $\lambda < \kappa \ge \mathfrak{c}$;
4. $\mathfrak{p} = \mathfrak{c}$ *(this is so under MA) and X is countable;*
5. *the Generalized Continuum Hypothesis holds.*

Unlike to the usual character, the cs*-, cs-, and sb-characters behave nicely with respect to many countable topological operations.

Among such operation there are: the Tychonov product, the box-product, producing a sequentially homeomorphic copy, taking image under a sequentially open map, and forming inductive topologies.

As usual, under *the box-product* $\square_{i\in\mathcal{I}} X_i$ of topological spaces X_i, $i \in \mathcal{I}$, we understand the Cartesian product $\prod_{i\in\mathcal{I}} X_i$ endowed with the box-product topology generated by the base consisting of products $\prod_{i\in\mathcal{I}} U_i$ where each U_i is open in X_i. In contrast, by $\prod_{i\in\mathcal{I}} X_i$ we denote the usual Cartesian product of the spaces X_i, endowed with the Tychonov product topology.

We say that a topological space X carries *the inductive topology* with respect to a cover $\mathcal{C}$ of X if a subset $F \subset X$ is closed in X if and only if the intersection $F \cap C$ is closed in C for each element $C \in \mathcal{C}$. For a cover $\mathcal{C}$ of X let $\mathrm{ord}(\mathcal{C}) = \sup_{x\in X} \mathrm{ord}(\mathcal{C}, x)$ where $\mathrm{ord}(\mathcal{C}, x) = |\{C \in \mathcal{C} : x \in C\}|$. A topological space X carrying the inductive topology with respect to a countable cover by closed metrizable (resp. compact, compact metrizable) subspaces is called an *$\mathcal{M}_\omega$-space* (resp. a *k_ω-space*, *$\mathcal{MK}_\omega$-space*).

A function $f : X \to Y$ between topological spaces is called *sequentially continuous* if for any convergent sequence (x_n) in X the sequence $(f(x_n))$ is convergent in Y to $f(\lim x_n)$; f is called a *sequential homeomorphism* if f is bijective and both f and f^{-1} are sequentially continuous. Topological spaces X, Y are defined to be *sequentially homeomorphic* if there is a sequential homeomorphism $h : X \to Y$. Observe that two spaces are sequentially homeomorphic if and only if their sequential coreflexions are homeomorphic. Under *the sequential coreflexion* σX of a topological space X we understand X endowed with the topology consisting of all sequentially open subsets of X (a subset U of X is *sequentially open* if its complement is sequentially closed in X; equivalently U is a sequential barrier at each point $x \in U$). Note that the identity map $\mathrm{id} : \sigma X \to X$ is continuous while its inverse is sequentially continuous, see [Lin].

A map $f : X \to Y$ is *sequentially open* if for any point $x_0 \in X$ and a sequence $S \subset Y$ convergent to $f(x_0)$ there is a sequence $T \subset X$ convergent to x_0 and such that $f(T) \subset S$. Observe that a bijective map f is sequentially open if its inverse f^{-1} is sequentially continuous.

The following technical Proposition is an easy consequence of the corresponding definitions.

Proposition 3. (1) *If X is a subspace of a topological space Y, then* $\mathrm{cs}^*_\chi(X) \le \mathrm{cs}^*_\chi(Y)$, $\mathrm{cs}_\chi(X) \le \mathrm{cs}_\chi(Y)$ *and* $\mathrm{sb}_\chi(X) \le \mathrm{sb}(Y)$.

(2) *If $f : X \to Y$ is a surjective continuous sequentially open map between topological spaces, then* $\mathrm{cs}^*_\chi(Y) \le \mathrm{cs}^*_\chi(X)$ *and* $\mathrm{sb}_\chi(Y) \le \mathrm{sb}_\chi(X)$.

(3) *If $f : X \to Y$ is a surjective sequentially continuous sequentially open map between topological spaces, then* $\min\{\mathrm{cs}^*_\chi(Y), \aleph_1\} \le \min\{\mathrm{cs}^*_\chi(X), \aleph_1\}$, $\min\{\mathrm{cs}_\chi(Y), \aleph_1\} \le \min\{\mathrm{cs}_\chi(X), \aleph_1\}$, *and* $\min\{\mathrm{sb}_\chi(Y), \aleph_1\} \le \min\{\mathrm{sb}_\chi(X), \aleph_1\}$.

(4) *If X, Y are sequentially homeomorphic topological spaces, then* $\min\{\mathrm{cs}^*_\chi(X), \aleph_1\} = \min\{\mathrm{cs}_\chi(X), \aleph_1\} = \min\{\mathrm{cs}_\chi(Y), \aleph_1\} = \min\{\mathrm{cs}^*_\chi(Y), \aleph_1\}$, *and* $\min\{\mathrm{sb}_\chi(Y), \aleph_1\} = \min\{\mathrm{sb}_\chi(X), \aleph_1\}$.

(5) $\min\{\mathrm{sb}_\chi(X), \aleph_1\} = \min\{\mathrm{sb}_\chi(\sigma X), \aleph_1\} \le \mathrm{sb}_\chi(\sigma X) \ge \mathrm{sb}_\chi(X)$ *and* $\mathrm{cs}_\chi(X) \le \mathrm{cs}_\chi(\sigma X) \ge \min\{\mathrm{cs}_\chi(\sigma X), \aleph_1\} = \min\{\mathrm{cs}_\chi(X), \aleph_1\} = \min\{\mathrm{cs}^*_\chi(X), \aleph_1\} = \min\{\mathrm{cs}^*_\chi(\sigma X), \aleph_1\} \le \mathrm{cs}^*_\chi(\sigma X) \ge \mathrm{cs}^*_\chi(X)$ *for any topological space X.*

(6) *If $X = \prod_{i\in\mathcal{I}} X_i$ is the Tychonov product of topological spaces X_i, $i \in \mathcal{I}$, then* $\mathrm{cs}^*_\chi(X) \le \sum_{i\in\mathcal{I}} \mathrm{cs}^*_\chi(X_i)$, $\mathrm{cs}_\chi(X) \le \sum_{i\in\mathcal{I}} \mathrm{cs}_\chi(X_i)$ *and* $\mathrm{sb}_\chi(X) \le \sum_{i\in\mathcal{I}} \mathrm{sb}_\chi(X_i)$.

(7) *If $X = \square_{i\in\mathcal{I}} X_i$ is the box-product of topological spaces X_i, $i \in \mathcal{I}$, then* $\mathrm{cs}^*_\chi(X) \le \sum_{i\in\mathcal{I}} \mathrm{cs}^*_\chi(X_i)$ *and* $\mathrm{cs}_\chi(X) \le \sum_{i\in\mathcal{I}} \mathrm{cs}_\chi(X_i)$.

(8) *If a topological space X carries the inductive topology with respect to a cover $\mathcal{C}$ of X, then* $\mathrm{cs}^*_\chi(X) \le \mathrm{ord}(\mathcal{C}) \cdot \sup_{C\in\mathcal{C}} \mathrm{cs}^*_\chi(C)$.

(9) *If a topological space X carries the inductive topology with respect to a point-countable cover $\mathcal{C}$ of X, then* $\mathrm{cs}_\chi(X) \le \sup_{C\in\mathcal{C}} \mathrm{cs}_\chi(C)$.

(10) *If a topological space X carries the inductive topology with respect to a point-finite cover $\mathcal{C}$ of X, then* $\mathrm{sb}_\chi(X) \le \sup_{C\in\mathcal{C}} \mathrm{sb}_\chi(C)$.

Since each first-countable space has countable cs^*-character, it is natural to consider the class of topological spaces with countable cs^*-character as a class of generalized metric spaces. However this class contains very non-metrizable spaces like $\beta\mathbb{N}$, the Stone-Čech compactification of the discrete space of positive integers. The reason is that $\beta\mathbb{N}$ contains no non-trivial convergent sequence. To avoid such pathologies we shall restrict ourselves by sequential spaces. Observe that a topological space is sequential provided X carries the inductive topology with respect to a cover by sequential subspaces. In particular, each $\mathcal{M}_\omega$-space is sequential and has countable cs^*-character. Our principal result states that for topological groups the converse is also true. Under an *$\mathcal{M}_\omega$-group* (resp. *$\mathcal{MK}_\omega$-group*) we understand a topological group whose underlying topological space is an $\mathcal{M}_\omega$-space (resp. $\mathcal{MK}_\omega$-space).

Theorem 1. *Each sequential topological group G with countable* cs^**-character is an $\mathcal{M}_\omega$-group. More precisely, either G is metrizable or else G contains an open $\mathcal{MK}_\omega$-subgroup H and is homeomorphic to the product $H \times D$ for some discrete space D.*

For $\mathcal{M}_\omega$-groups the second part of this theorem was proven in [Ba$_1$]. Theorem 1 has many interesting corollaries.

At first we show that for sequential topological groups with countable cs^*-character many important cardinal invariants are countable, coincide or take some fixed values. Let us remind some definitions, see [En$_1$]. For a topological space X recall that

- the *pseudocharacter* $\psi(X)$ is the smallest cardinal κ such that each one-point set $\{x\} \subset X$ can be written as the intersection $\{x\} = \cap\mathcal{U}$ of some family $\mathcal{U}$ of open subsets of X with $|\mathcal{U}| \le \kappa$;
- the *cellularity* $c(X)$ is the smallest cardinal κ such that X contains no family $\mathcal{U}$ of size $|\mathcal{U}| > \kappa$ consisting of non-empty pairwise disjoint open subsets;
- the *Lindelöf number* $l(X)$ is the smallest cardinal κ such that each open cover of X contains a subcover of size $\le \kappa$;
- the *density* $d(X)$ is the smallest size of a dense subset of X;
- the *tightness* $t(X)$ is the smallest cardinal κ such that for any subset $A \subset X$ and a point $a \in \bar{A}$ from its closure there is a subset $B \subset A$ of size $|B| \le \kappa$ with $a \in \bar{B}$;
- the *extent* $e(X)$ is the smallest cardinal κ such that X contains no closed discrete subspace of size $> \kappa$;
- the *compact covering number* $kc(X)$ is the smallest size of a cover of X by compact subsets;
- the *weight* $w(X)$ is the smallest size of a base of the topology of X;
- the *network weight* $nw(X)$ is the smallest size $|\mathcal{N}|$ of a topological network for X (a family $\mathcal{N}$ of subsets of X is a *topological network* if for any open set $U \subset X$ and any point $x \in U$ there is $N \in \mathcal{N}$ with $x \in N \subset U$);
- the *k-network weight* $knw(X)$ is the smallest size $|\mathcal{N}|$ of a k-network for X (a family $\mathcal{N}$ of subsets of X is a *k-network* if for any open set $U \subset X$ and any compact subset $K \subset U$ there is a finite subfamily $\mathcal{M} \subset \mathcal{N}$ with $K \subset \cup\mathcal{M} \subset U$).

For each topological space X these cardinal invariants relate as follows:

$$\max\{c(X), l(X), e(X)\} \le nw(X) \le knw(X) \le w(X).$$

For metrizable spaces all of them are equal, see [En$_1$, 4.1.15].

In the class of k-spaces there is another cardinal invariant, the k-ness introduced by E. van Douwen, see [vD, §8]. We remind that a topological space X is called a *k-space* if it carries the inductive topology with respect to the cover of X by all compact subsets. It is clear that each sequential space is a k-space. The *k-ness* $k(X)$ of a k-space is the smallest size $|\mathcal{K}|$ of a cover $\mathcal{K}$ of X by compact subsets such that X carries the inductive topology with respect to the cover $\mathcal{K}$. It is interesting to notice that $k(\mathbb{N}^\omega) = \mathfrak{d}$ while $k(\mathbb{Q}) = \mathfrak{b}$, see [vD]. Proposition 3(8) implies that $\mathrm{cs}^*_\chi(X) \le k(X) \cdot \psi(X) \ge kc(X)$ for each k-space X. Observe also that a topological space X is a k_ω-space if and only if X is a k-space with $k(X) \le \aleph_0$.

Besides cardinal invariants we shall consider an ordinal invariant, called the sequential order. Under *the sequential closure* $A^{(1)}$ of a subset A of a topological space X we understand the set of all limit point of sequences $(a_n) \subset A$, convergent in X. Given an ordinal α define the α-th sequential closure $A^{(\alpha)}$ of A by transfinite induction: $A^{(\alpha)} = \bigcup_{\beta<\alpha}(A^{(\beta)})^{(1)}$. Under the *sequential order* $\mathrm{so}(X)$ of a topological space X we understand the smallest ordinal α such that $A^{(\alpha+1)} = A^{(\alpha)}$ for any subset $A \subset X$. Observe that a topological space X is Fréchet-Urysohn if and only if $\mathrm{so}(X) \le 1$; X is sequential if and only if $\mathrm{cl}_X(A) = A^{(\mathrm{so}(X))}$ for any subset $A \subset X$.

Besides purely topological invariants we shall also consider a cardinal invariant, specific for topological groups. For a topological group G let $ib(G)$, the *boundedness index* of G be the smallest cardinal κ such that for any nonempty open set $U \subset G$ there is a subset $F \subset G$ of size $|F| \le \kappa$ such that $G = F \cdot U$. It is known that

$ib(G) \le \min\{c(G), l(G), e(G)\}$ and $w(G) = ib(G) \cdot \chi(G)$ for each topological group, see [Tk].

Theorem 2. *Each sequential topological group G with countable* cs^**-character has the following properties:* $\psi(G) \le \aleph_0$, $\mathrm{sb}_\chi(G) = \chi(G) \in \{1, \aleph_0, \mathfrak{d}\}$, $ib(G) = c(G) = d(G) = l(G) = e(G) = nw(G) = knw(G)$, *and* $\mathrm{so}(G) \in \{1, \omega_1\}$.

We shall derive from Theorems 1 and 2 an unexpected metrization theorem for topological groups. But first we need to remind the definitions of some of α_i-spaces, $i = 1, \dots, 6$ introduced by A.V. Arkhangelski in [Ar_2], [Ar_4]. We also define a wider class of α_7-spaces.

A topological space X is called

- an α_1*-space* if for any sequences $S_n \subset X$, $n \in \omega$, convergent to a point $x \in X$ there is a sequence $S \subset X$ convergent to x and such that $S_n \setminus S$ is finite for all n;
- an α_4*-space* if for any sequences $S_n \subset X$, $n \in \omega$, convergent to a point $x \in X$ there is a sequence $S \subset X$ convergent to x and such that $S_n \cap S \neq \emptyset$ for infinitely many sequences S_n;
- an α_7*-space* if for any sequences $S_n \subset X$, $n \in \omega$, convergent to a point $x \in X$ there is a sequence $S \subset X$ convergent to some point y of X and such that $S_n \cap S \neq \emptyset$ for infinitely many sequences S_n;

Under a sequence converging to a point x of a topological space X we understand any countable infinite subset S of X such that $S \setminus U$ if finite for any neighborhood U of x. Each α_1-space is α_4 and each α_4-space is α_7. Quite often α_7-spaces are α_4, see Lemma 7. Observe also that each sequentially compact space is α_7. It can be shown that a topological space X is an α_7-space if and only if it contains no closed copy of the sequential fan S_ω in its sequential coreflexion σX. If X is an α_4-space, then σX contains no topological copy of S_ω.

We remind that a topological group G is *Weil complete* if it is complete in its left (equivalently, right) uniformity. According to [PZ, 4.1.6], each k_ω-group is Weil complete. The following metrization theorem can be easily derived from Theorems 1, 2 and elementary properties of $\mathcal{MK}_\omega$-groups.

Theorem 3. *A sequential topological group G with countable* cs^**-character is metrizable if one of the following conditions is satisfied:*

1. $\mathrm{so}(G) < \omega_1$;
2. $\mathrm{sb}_\chi(G) < \mathfrak{d}$;
3. $ib(G) < k(G)$;
4. *G is Fréchet-Urysohn;*
5. *G is an α_7-space;*
6. *G contains no closed copy of S_ω or S_2;*
7. *G is not Weil complete;*
8. *G is Baire;*
9. $ib(G) < |G| < 2^{\aleph_0}$.

According to Theorem 1, each sequential topological group with countable cs^*-character is an $\mathcal{M}_\omega$-group. The first author has proved in [Ba_3] that the topological structure of a non-metrizable punctiform $\mathcal{M}_\omega$-group is completely determined by its density and the compact scatteredness rank.

Recall that a topological space X is *punctiform* if X contains no compact connected subspace containing more than one point, see [En$_2$, 1.4.3]. In particular, each zero-dimensional space is punctiform.

Next, we remind the definition of the scatteredness height. Given a topological space X let $X_{(1)} \subset X$ denote the set of all non-isolated points of X. For each ordinal α define the α-th derived set $X_{(\alpha)}$ of X by transfinite induction: $X_{(\alpha)} = \bigcap_{\beta<\alpha}(X_{(\beta)})_{(1)}$. Under the *scatteredness height* $\mathrm{sch}(X)$ of X we understand the smallest ordinal α such that $X_{(\alpha+1)} = X_{(\alpha)}$. A topological space X is *scattered* if $X_{(\alpha)} = \emptyset$ for some ordinal α. Under the *compact scatteredness rank* of a topological space X we understand the ordinal $\mathrm{scr}(X) = \sup\{\mathrm{sch}(K) : K$ is a scattered compact subspace of $X\}$.

Theorem 4. *Two non-metrizable sequential punctiform topological groups G, H with countable* cs**-character are homeomorphic if and only if $d(G) = d(H)$ and $\mathrm{scr}(G) = \mathrm{scr}(H)$.*

This theorem follows from Theorem 1 and "Main Theorem" of [Ba$_3$] asserting that two non-metrizable punctiform $\mathcal{M}_\omega$-groups G, H are homeomorphic if and only if $d(G) = d(H)$ and $\mathrm{scr}(G) = \mathrm{scr}(H)$. For countable k_ω-groups this fact was proven by E.Zelenyuk [Ze$_1$].

The topological classification of non-metrizable sequential locally convex spaces with countable cs*-character is even more simple. Any such a space is homeomorphic either to $\mathbb{R}^\infty$ or to $\mathbb{R}^\infty \times Q$ where $Q = [0,1]^\omega$ is the Hilbert cube and $\mathbb{R}^\infty$ is a linear space of countable algebraic dimension, carrying the strongest locally convex topology. It is well-known that this topology is inductive with respect to the cover of $\mathbb{R}^\infty$ by finite-dimensional linear subspaces. The topological characterization of the spaces $\mathbb{R}^\infty$ and $\mathbb{R}^\infty \times Q$ was given in [Sa]. In [Ba$_2$] it was shown that each infinite-dimensional locally convex $\mathcal{MK}_\omega$-space is homeomorphic to $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times Q$. This result together with Theorem 1 implies the following classification

Corollary 1. *Each non-metrizable sequential locally convex space with countable* cs**-character is homeomorphic to $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times Q$.*

As we saw in Theorem 2, each sequential topological group with countable cs*-character has countable pseudocharacter. The proof of this result is based on the fact that compact subsets of sequential topological groups with countable cs*-character are first countable. This naturally leads to a conjecture that compact spaces with countable cs*-character are first countable. Surprisingly, but this conjecture is false: assuming the Continuum Hypothesis N. Yakovlev [Ya] has constructed a scattered sequential compactum which has countable sb-character but fails to be first countable. In [Ny$_2$] P.Nyikos pointed out that the Yakovlev construction still can be carried under the assumption $\mathfrak{b} = \mathfrak{c}$. More precisely, we have

Proposition 4. *Under $\mathfrak{b} = \mathfrak{c}$ there is a regular locally compact locally countable space Y whose one-point compactification αY is sequential and satisfies $\aleph_0 = \mathrm{sb}_\chi(\alpha Y) < \psi(\alpha Y) = \mathfrak{c}$.*

We shall use the above proposition to construct examples of topologically homogeneous spaces with countable cs-character and uncountable pseudocharacter. This shows that Theorem 2 is specific for topological groups and cannot be generalized to topologically homogeneous spaces. We remind that a topological space X

is *topologically homogeneous* if for any points $x, y \in X$ there is a homeomorphism $h : X \to X$ with $h(x) = y$.

Theorem 5.

(1) *There is a topologically homogeneous countable regular k_ω-space X_1 with $\aleph_0 = \mathrm{sb}_\chi(X_1) < \chi(X_1) = \mathfrak{d}$ and $\mathrm{so}(X_1) = \omega$;*
(2) *Under $\mathfrak{b} = \mathfrak{c}$ there is a sequential topologically homogeneous zero-dimensional k_ω-space X_2 with $\aleph_0 = \mathrm{cs}_\chi(X_2) < \psi(X_2) = \mathfrak{c}$;*
(3) *Under $\mathfrak{b} = \mathfrak{c}$ there is a sequential topologically homogeneous totally disconnected space X_3 with $\aleph_0 = \mathrm{sb}_\chi(X_3) < \psi(X_3) = \mathfrak{c}$.*

We remind that a space X is *totally disconnected* if for any distinct points $x, y \in X$ there is a continuous function $f : X \to \{0, 1\}$ such that $f(x) \neq f(y)$, see [En$_2$].

Remark 1. The space X_1 from Theorem 5(1) is the well-known Arkhangelski-Franklin example [AF] (see also [Co, 10.1]) of a countable topologically homogeneous k_ω-space, homeomorphic to no topological group (this also follows from Theorem 2). On the other hand, according to [Ze$_2$], each topologically homogeneous countable regular space (in particular, X_1) is homeomorphic to a *quasitopological group*, that is a topological space endowed with a separately continuous group operation with continuous inversion. This shows that Theorem 2 cannot be generalized onto quasitopological groups (see however [Zd] for generalizations of Theorems 1 and 2 to some other topologo-algebraic structures).

Next, we find conditions under which a space with countable cs*-character is first-countable or has countable sb-character. Following [Ar$_3$] we define a topological space X to be *c-sequential* if for each closed subspace $Y \subset X$ and each non-isolated point y of Y there is a sequence $(y_n) \subset Y \setminus \{y\}$ convergent to y. It is clear that each sequential space is c-sequential. A point x of a topological space X is called *regular G_δ* if $\{x\} = \cap\mathcal{B}$ for some countable family $\mathcal{B}$ of closed neighborhood of x in X, see [Lin].

First we characterize spaces with countable sb-character (the first three items of this characterization were proved by Lin [Lin, 3.13] in terms of (universally) csf-countable spaces).

Proposition 5. *For a Hausdorff space X the following conditions are equivalent:*

(1) *X has countable* sb*-character;*
(2) *X is an α_1-space with countable* cs**-character;*
(3) *X is an α_4-space with countable* cs**-character;*
(4) $\mathrm{cs}^*_\chi(X) \leq \aleph_0$ *and* $\mathrm{sb}_\chi(X) < \mathfrak{p}$.

Moreover, if X is c-sequential and each point of X is regular G_δ, then the conditions (1)–(4) are equivalent to:

(5) $\mathrm{cs}^*_\chi(X) \leq \aleph_0$ *and* $\mathrm{sb}_\chi(X) < \mathfrak{d}$.

Next, we give a characterization of first-countable spaces in the same spirit (the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5) were proved by Lin [Lin, 2.8]).

Proposition 6. *For a Hausdorff space X with countable* cs**-character the following conditions are equivalent:*

(1) *X is first-countable;*
(2) *X is Fréchet-Urysohn and has countable* sb*-character;*
(3) *X is Fréchet-Urysohn α_7-space;*

(4) *$\chi(X) < \mathfrak{p}$ and X has countable tightness.*

Moreover, if each point of X is regular G_δ, then the conditions (1)–(4) are equivalent to:

(5) *X is a sequential space containing no closed copy of S_2 or S_ω;*

(6) *X is a sequential space with $\chi(X) < \mathfrak{d}$.*

For Fréchet-Urysohn (resp. dyadic) compacta the countability of the cs*-character is equivalent to the first countability (resp. the metrizability). We remind that a compact Hausdorff space X is called *dyadic* if X is a continuous image of the Cantor discontinuum $\{0,1\}^\kappa$ for some cardinal κ.

Proposition 7.

(1) *A Fréchet-Urysohn countably compact space is first-countable if and only if it has countable* cs*-*character.*

(2) *A dyadic compactum is metrizable if and only if its has countable* cs*-*character.*

In light of Proposition 7(1) one can suggest that $\mathrm{cs}^*_\chi(X) = \chi(X)$ for any compact Fréchet-Urysohn space X. However that is not true: under CH, $\mathrm{cs}_\chi(\alpha D) \neq \chi(\alpha D)$ for the one-point compactification αD of a discrete space D of size $|D| = \aleph_2$. Surprisingly, but the problem of calculating the cs*- and cs-characters of the spaces αD is not trivial and the definitive answer is known only under the Generalized Continuum Hypothesis. First we note that the cardinals $\mathrm{cs}^*_\chi(\alpha D)$ and $\mathrm{cs}_\chi(\alpha D)$ admit an interesting interpretation which will be used for their calculation.

Proposition 8. *Let D be an infinite discrete space. Then*

(1) $\mathrm{cs}^*_\chi(\alpha D) = \min\{w(X) : X$ *is a (regular zero-dimensional) topological space of size $|X| = |D|$ containing non no-trivial convergent sequence*$\}$;

(2) $\mathrm{cs}_\chi(\alpha D) = \min\{w(X) : X$ *is a (regular zero-dimensional) topological space of size $|X| = |D|$ containing no countable non-discrete subspace*$\}$.

For a cardinal κ we put $\log\kappa = \min\{\lambda : \kappa \le 2^\lambda\}$ and $\mathrm{cof}([\kappa]^{\le\omega})$ be the smallest size of a collection $\mathcal{C} \subset [\kappa]^{\le\omega}$ such that each at most countable subset $S \subset \kappa$ lies in some element $C \in \mathcal{C}$. Observe that $\mathrm{cof}([\kappa]^{\le\omega}) \le \kappa^\omega$ but sometimes the inequality can be strict: $1 = \mathrm{cof}([\aleph_0]^{\le\omega}) < \aleph_0$ and $\aleph_1 = \mathrm{cof}([\aleph_1]^{\le\omega}) < \aleph_1^{\aleph_0}$. In the following proposition we collect all the information on the cardinals $\mathrm{cs}^*_\chi(\alpha D)$ and $\mathrm{cs}_\chi(\alpha D)$ we know.

Proposition 9. *Let D be an uncountable discrete space. Then*

(1) $\aleph_1 \cdot \log|D| \le \mathrm{cs}^*_\chi(\alpha D) \le \mathrm{cs}_\chi(\alpha D) \le \min\{|D|, 2^{\aleph_0} \cdot \mathrm{cof}([\log|D|]^{\le\omega})\}$ *while* $\mathrm{sb}_\chi(\alpha D) = \chi(\alpha D) = |D|$;

(2) $\mathrm{cs}^*_\chi(\alpha D) = \mathrm{cs}_\chi(\alpha D) = \aleph_1 \cdot \log|D|$ *under GCH.*

In spite of numerous efforts some annoying problems concerning cs*- and cs-characters still rest open.

Problem 1. *Is there a (necessarily consistent) example of a space X with $\mathrm{cs}^*_\chi(X) \neq \mathrm{cs}_\chi(X)$? In particular, is $\mathrm{cs}^*_\chi(\alpha\mathfrak{c}) \neq \mathrm{cs}_\chi(\alpha\mathfrak{c})$ in some model of ZFC?*

In light of Proposition 8 it is natural to consider the following three cardinal characteristics of the continuum which seem to be new:

$$\mathfrak{w}_1 = \min\{w(X) : X \text{ is a topological space of size } |X| = \mathfrak{c} \text{ containing no non-trivial convergent sequence}\};$$

$$\mathfrak{w}_2 = \min\{w(X) : X \text{ is a topological space of size } |X| = \mathfrak{c} \text{ containing no non-discrete countable subspace}\};$$

$$\mathfrak{w}_3 = \min\{w(X) : X \text{ is a } P\text{-space of size } |X| = \mathfrak{c}\}.$$

As expected, a *P-space* is a T_1-space whose any G_δ-subset is open. Observe that $\mathfrak{w}_1 = \mathrm{cs}^*_\chi(\alpha\mathfrak{c})$ while $\mathfrak{w}_2 = \mathrm{cs}_\chi(\alpha\mathfrak{c})$. It is clear that $\aleph_1 \le \mathfrak{w}_1 \le \mathfrak{w}_2 \le \mathfrak{w}_3 \le \mathfrak{c}$ and hence the cardinals $\mathfrak{w}_i$, $i = 1,2,3$, fall into the category of small uncountable cardinals, see [Va].

Problem 2. *Are the cardinals $\mathfrak{w}_i$, $i = 1,2,3$, equal to (or can be estimated via) some known small uncountable cardinals considered in Set Theory? Is $\mathfrak{w}_1 < \mathfrak{w}_2 < \mathfrak{w}_3$ in some model of ZFC?*

Our next question concerns the assumption $\mathfrak{b} = \mathfrak{c}$ in Theorem 5.

Problem 3. *Is there a ZFC-example of a sequential space X with $\mathrm{sb}_\chi(X) < \psi(X)$ or at least $\mathrm{cs}^*_\chi(X) < \psi(X)$?*

Propositions 1 and 5 imply that $\mathrm{sb}_\chi(X) \in \{1, \aleph_0\} \cup [\mathfrak{d}, \mathfrak{c}]$ for any c-sequential topological space X with countable cs*-character. On the other hand, for a sequential topological group G with countable cs*-character we have a more precise estimate $\mathrm{sb}_\chi(G) \in \{1, \aleph_0, \mathfrak{d}\}$.

Problem 4. *Is $\mathrm{sb}_\chi(X) \in \{1, \aleph_0, \mathfrak{d}\}$ for any sequential space X with countable cs*-character?*

As we saw in Proposition 7, $\chi(X) \le \aleph_0$ for any Fréchet-Urysohn compactum X with $\mathrm{cs}_\chi(X) \le \aleph_0$.

Problem 5. *Is $\mathrm{sb}_\chi(X) \le \aleph_0$ for any sequential (scattered) compactum X with $\mathrm{cs}_\chi(X) \le \aleph_0$?*

Now we pass to proofs of our results.

On sequence trees in topological groups

Our basic instrument in proofs of main results is the concept of a sequence tree. As usual, under a *tree* we understand a partially ordered subset $(T, \le)$ such that for each $t \in T$ the set $\downarrow t = \{\tau \in T : \sigma \le t\}$ is well-ordered by the order $\le$. Given an element $t \in T$ let $\uparrow t = \{\tau \in T : \tau \ge t\}$ and $\mathrm{succ}(t) = \min(\uparrow t \setminus \{t\})$ be the set of successors of t in T. A maximal linearly ordered subset of a tree T is called a *branch* of T. By $\max T$ we denote the set of maximal elements of the tree T.

Definition 1. Under a *sequence tree* in a topological space X we understand a tree $(T, \le)$ such that

- $T \subset X$;
- T has no infinite branch;
- for each $t \notin \max T$ the set $\min(\uparrow t \setminus \{t\})$ of successors of t is countable and converges to t.

Saying that a subset S of a topological space X converges to a point $t \in X$ we mean that for each neighborhood $U \subset X$ of t the set $S \setminus U$ is finite.

The following lemma is well-known and can be easily proven by transfinite induction (on the ordinal $s(a, A) = \min\{\alpha : a \in A^{(\alpha)}\}$ for a subset A of a sequential space and a point $a \in \bar{A}$ from its closure)

Lemma 1. *A point $a \in X$ of a sequential topological space X belongs to the closure of a subset $A \subset X$ if and only if there is a sequence tree $T \subset X$ with $\min T = \{a\}$ and $\max T \subset A$.*

For subsets A, B of a group G let $A^{-1} = \{x^{-1} : x \in A\} \subset G$ be the inversion of A in G and $AB = \{xy : x \in A,\ y \in B\} \subset G$ be the product of A, B in G. The following two lemmas will be used in the proof of Theorem 1.

Lemma 2. *A sequential subspace $F \subset X$ of a topological group G is first countable if the subspace $F^{-1}F \subset G$ has countable* sb*-character at the unit e of the group G.*

Proof. Our proof starts with the observation that it is sufficient to consider the case $e \in F$ and prove that F has countable character at e.

Let $\{S_n : n \in \omega\}$ be a decreasing sb-network at e in $F^{-1}F$. First we show that for every $n \in \omega$ there exists $m > n$ such that $S_m^2 \cap (F^{-1}F) \subset S_n$. Otherwise, for every $m \in \omega$ there would exist $x_m, y_m \in S_m$ with $x_m y_m \in (F^{-1}F) \setminus S_n$. Taking into account that $\lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = e$, we get $\lim_{m\to\infty} x_m y_m = e$. Since S_n is a sequential barrier at e, there is a number m with $x_m y_m \in S_n$, which contradicts to the choice of the points x_m, y_m.

Now let us show that for all $n \in \omega$ the set $S_n \cap F$ is a neighborhood of e in F. Suppose, conversely, that $e \in \mathrm{cl}_F(F \setminus S_{n_0})$ for some $n_0 \in \omega$.

By Lemma 1 there exists a sequence tree $T \subset F$, $\min T = \{e\}$ and $\max T \subset F \setminus S_{n_0}$. To get a contradiction we shall construct an infinite branch of T. Put $x_0 = e$ and let m_0 be the smallest integer such that $S_{m_0}^2 \cap F^{-1}F \subset S_{n_0}$.

By induction, for every $i \geq 1$ find a number $m_i > m_{i-1}$ with $S_{m_i}^2 \cap F^{-1}F \subset S_{m_{i-1}}$ and a point $x_i \in \mathrm{succ}(x_{i-1}) \cap (x_{i-1} S_{m_i})$. To show that such a choice is always possible, it suffices to verify that $x_{i-1} \notin \max T$. It follows from the inductive construction that $x_{i-1} \in F \cap (S_{m_0} \cdots S_{m_{i-1}}) \subset F \cap S_{m_0}^2 \subset S_{n_0}$ and thus $x_{i-1} \notin \max T$ because $\max T \subset F \setminus S_{n_0}$.

Therefore we have constructed an infinite branch $\{x_i : i \in \omega\}$ of the sequence tree T which is not possible. This contradiction finishes the proof. $\square$

Lemma 3. *A sequential α_7-subspace F of a topological group G has countable* sb*-character provided the subspace $F^{-1}F \subset G$ has countable* cs*-character at the unit e of G.*

Proof. Suppose that $F \subset G$ is a sequential α_7-space with $\mathrm{cs}_\chi(F^{-1}F, e) \leq \aleph_0$. We have to prove that $\mathrm{sb}_\chi(F, x) \leq \aleph_0$ for any point $x \in F$. Replacing F by Fx^{-1}, if necessary, we can assume that $x = e$ is the unit of the group G. Fix a countable family $\mathcal{A}$ of subsets of G closed under group products in G, finite unions and finite intersections, and such that $F^{-1}F \in \mathcal{A}$ and $\mathcal{A}|F^{-1}F = \{A \cap (F^{-1}F) : A \in \mathcal{A}\}$ is a cs-network at e in $F^{-1}F$. We claim that the collection $\mathcal{A}|F = \{A \cap F : A \in \mathcal{A}\}$ is a sb-network at e in F.

Assuming the converse, we would find an open neighborhood $U \subset G$ of e such that for any element $A \in \mathcal{A}$ with $A \cap F \subset U$ the set $A \cap F$ fails to be a sequential barrier at e in F.

Let $\mathcal{A}' = \{A \in \mathcal{A} : A \subset F \cap U\} = \{A_n : n \in \omega\}$ and $B_n = \bigcup_{k\le n} A_k$. Let $m_{-1} = 0$ and $U_{-1} \subset U$ be any closed neighborhood of e in G. By induction, for every $k \in \omega$ find a number $m_k > m_{k-1}$, a closed neighborhood $U_k \subset U_{k-1}$ of e in G, and a sequence $(x_{k,i})_{i\in\omega}$ convergent to e so that the following conditions are satisfied:

(i) $\{x_{k,i} : i \in \omega\} \subset U_{k-1} \cap F \setminus B_{m_{k-1}}$;
(ii) the set $F_k = \{x_{n,i} : n \le k,\ i \in \omega\} \setminus B_{m_k}$ is finite;
(iii) $U_k \cap (F_k \cup \{x_{i,j} : i, j \le k\}) = \emptyset$ and $U_k^2 \subset U_{k-1}$.

The last condition implies that $U_0U_1 \cdots U_k \subset U$ for every $k \ge 0$.

Consider the subspace $X = \{x_{k,i} : k, i \in \omega\}$ of F and observe that it is discrete (in itself). Denote by $\bar{X}$ the closure of X in F and observe that $\bar{X} \setminus X$ is closed in F. We claim that e is an isolated point of $\bar{X} \setminus X$. Assuming the converse and applying Lemma 1 we would find a sequence tree $T \subset \bar{X}$ such that $\min T = \{e\}$, $\max T \subset X$, and $\mathrm{succ}(e) \subset \bar{X} \setminus X$.

By induction, construct a (finite) branch $(t_i)_{i\le n+1}$ of the tree T and a sequence $\{C_i : i \le n\}$ of elements of the family $\mathcal{A}$ such that $t_0 = e$, $|\mathrm{succ}(t_i) \setminus t_iC_i| < \aleph_0$ and $C_i \subset U_i \cap (F^{-1}F)$, $t_{i+1} \in \mathrm{succ}(t_i) \cap t_iC_i$, for each $i \le n$. Note that the infinite set $\sigma = \mathrm{succ}(t_n) \cap t_nC_n \subset X$ converges to the point $t_n \ne e$.

On the other hand, $\sigma \subset t_nC_n \subset t_{n-1}C_{n-1}C_n \subset \cdots \subset t_0C_0 \cdots C_n \subset U_0 \cdots U_n \subset U$. It follows from our assumption on $\mathcal{A}$ that $C_0 \cdots C_n \in \mathcal{A}$ and thus $(C_0 \cdots C_n) \cap F \subset B_{m_k}$ for some k. Consequently, $\sigma \subset X \cap B_{m_k}$ and $\sigma \subset \{x_{j,i} : j \le k,\ i \in \omega\}$ by the item (i) of the construction of X. Since e is a unique cluster point of the set $\{x_{j,i} : j \le k,\ i \in \omega\}$, the sequence σ cannot converge to $t_n \ne e$, which is a contradiction.

Thus e is an isolated point of $\bar{X} \setminus X$ and consequently, there is a closed neighborhood W of e in G such that the set $V = (\{e\} \cup X) \cap W$ is closed in F.

For every $n \in \omega$ consider the sequence $S_n = W \cap \{x_{n,i} : i \in \omega\}$ convergent to e. Since F is an α_7-space, there is a convergent sequence $S \subset F$ such that $S \cap S_n \ne \emptyset$ for infinitely many sequences S_n. Taking into account that V is a closed subspace of F with $|V \cap S| = \aleph_0$, we conclude that the limit point $\lim S$ of S belongs to the set V. Moreover, we can assume that $S \subset V$. Since the space X is discrete, $\lim S \in V \setminus X = \{e\}$. Thus the sequence S converges to e. Since $\mathcal{A}'$ is a cs-network at e in F, there is a number $n \in \omega$ such that A_n contains almost all members of the sequence S. Since $S_m \cap (S_k \cup A_n) = \emptyset$ for $m > k \ge n$, the sequence S cannot meet infinitely many sequences S_m. But this contradicts to the choice of S. □

Following [vD, §8] by $\mathbb{L}$ we denote the countable subspace of the plane $\mathbb{R}^2$:

$$\mathbb{L} = \{(0,0), (\tfrac{1}{n}, \tfrac{1}{nm}) : n, m \in \mathbb{N}\} \subset \mathbb{R}^2.$$

The space $\mathbb{L}$ is locally compact at each point except for $(0,0)$. Moreover, according to Lemma 8.3 of [vD], a first countable space X contains a closed topological copy of the space $\mathbb{L}$ if and only if X is not locally compact.

The following important lemma was proven in [Ba_1] for normal sequential groups.

Lemma 4. *If a sequential topological group G contains a closed copy of the space $\mathbb{L}$, then G is an α_7-space.*

Proof. Let $h : \mathbb{L} \to G$ be a closed embedding and let $x_0 = h(0,0)$, $x_{n,m} = h(\frac{1}{n}, \frac{1}{nm})$ for $n, m \in \mathbb{N}$. To show that G is an α_7-space, for every $n \in \mathbb{N}$ fix a sequence

$(y_{n,m})_{m\in\mathbb{N}} \subset G$, convergent to the unit e of G. Denote by $* : G \times G \to G$ the group operation on G.

It is easy to verify that for every n the subspace $D_n = \{x_{n,m} * y_{n,m} : m \in \mathbb{N}\}$ is closed and discrete in G. Hence there exists $k_n \in \mathbb{N}$ such that $x_0 \neq x_{n,m} * y_{n,m}$ for all $m > k_n$. Consider the subset

$$A = \{x_{n,m} * y_{n,m} : n > 0,\ m > k_n\}$$

and using the continuity of the group operation, show that $x_0 \notin A$ is a cluster point of A in G. Consequently, the set A is not closed and by the sequentiality of G, there is a sequence $S \subset A$ convergent to a point $a \notin A$. Since every space D_n is closed and discrete in G, we may replace S by a subsequence, and assume that $|S \cap D_n| \leq 1$ for every $n \in \mathbb{N}$. Consequently, S can be written as $S = \{x_{n_i,m_i} * y_{n_i,m_i} : i \in \omega\}$ for some number sequences (m_i) and (n_i) with $n_{i+1} > n_i$ for all i. It follows that the sequence $(x_{n_i,m_i})_{i\in\omega}$ converges to x_0 and consequently, the sequence $T = \{y_{n_i,m_i}\}_{i\in\omega}$ converges to $x_0^{-1} * a$. Since $T \cap \{y_{n_i,m}\}_{m\in\mathbb{N}} \neq \emptyset$ for every i, we conclude that G is an α_7-space. □

Lemma 4 allows us to prove the following unexpected

Lemma 5. *A non-metrizable sequential topological group G with countable* cs-*character has a countable* cs-*network at the unit, consisting of closed countably compact subsets of G.*

Proof. Given a non-metrizable sequential group G with countable cs-character we can apply Lemmas 2–4 to conclude that G contains no closed copy of the space $\mathbb{L}$. Fix a countable cs-network $\mathcal{N}$ at e, closed under finite intersections and consisting of closed subspaces of G. We claim that the collection $\mathcal{C} \subset \mathcal{N}$ of all countably compact subsets $N \in \mathcal{N}$ forms a cs-network at e in G.

To show this, fix a neighborhood $U \subset G$ of e and a sequence $(x_n) \subset G$ convergent to e. We must find a countably compact set $M \in \mathcal{N}$ with $M \subset U$, containing almost all points x_n. Let $\mathcal{A} = \{A_k : k \in \omega\}$ be the collection of all elements $N \subset U$ of $\mathcal{N}$ containing almost all points x_n. Now it suffices to find a number $n \in \omega$ such that the intersection $M = \bigcap_{k\leq n} A_k$ is countably compact. Suppose to the contrary, that for every $n \in \omega$ the set $\bigcap_{k\leq n} A_k$ is not countably compact. Then there exists a countable closed discrete subspace $K_0 \subset A_0$ with $K_0 \not\ni e$. Fix a neighborhood W_0 of e with $W_0 \cap K_0 = \emptyset$. Since $\mathcal{N}$ is a cs-network at e, there exists $k_1 \in \omega$ such that $A_{k_1} \subset W_0$.

It follows from our hypothesis that there is a countable closed discrete subspace $K_1 \subset \bigcap_{k\leq k_1} A_k$ with $K_1 \ni e$. Proceeding in this fashion we construct by induction an increasing number sequence $(k_n)_{n\in\omega} \subset \omega$, a sequence $(K_n)_{n\in\omega}$ of countable closed discrete subspaces of G, and a sequence $(W_n)_{n\in\omega}$ of open neighborhoods of e such that $K_n \subset \bigcap_{k\leq k_n} A_k$, $W_n \cap K_n = \emptyset$, and $A_{k_{n+1}} \subset W_n$ for all $n \in \omega$.

It follows from the above construction that $\{e\} \cup \bigcup_{n\in\omega} K_n$ is a closed copy of the space $\mathbb{L}$ which is impossible. □

Proofs of Main Results

Proof of Proposition 1. The first three items can be easily derived from the corresponding definitions. To prove the fourth item observe that for any cs*-network $\mathcal{N}$ at a point x of a topological space X, the family $\mathcal{N}' = \{\cup\mathcal{F} : \mathcal{F} \subset \mathcal{N}\}$ is an sb-network at x.

The proof of fifth item is more tricky. Fix any cs*-network $\mathcal{N}$ at a point $x \in X$ with $|\mathcal{N}| \le \mathrm{cs}^*_\chi(X)$. Let $\lambda = \mathrm{cof}(|\mathcal{N}|)$ be the cofinality of the cardinal $|\mathcal{N}|$ and write $\mathcal{N} = \bigcup_{\alpha<\lambda} \mathcal{N}_\alpha$ where $\mathcal{N}_\alpha \subset \mathcal{N}_\beta$ and $|\mathcal{N}_\alpha| < |\mathcal{N}|$ for any ordinals $\alpha \le \beta < \lambda$. Consider the family $\mathcal{M} = \{\cup \mathcal{C} : \mathcal{C} \in [\mathcal{N}_\alpha]^{\le\omega},\ \alpha < \lambda\}$ and observe that $|\mathcal{M}| \le \lambda \cdot \sup\{|[\kappa]^{\le\omega}| : \kappa < |\mathcal{N}|\}$ where $[\kappa]^{\le\omega} = \{A \subset \kappa : |A| \le \aleph_0\}$. It rests to verify that $\mathcal{M}$ is a cs-network at x.

Fix a neighborhood $U \subset X$ of x and a sequence $S \subset X$ convergent to x. For every $\alpha < \lambda$ choose a countable subset $\mathcal{C}_\alpha \subset \mathcal{N}_\alpha$ such that $\cup\mathcal{C}_\alpha \subset U$ and $S \cap (\cup \mathcal{C}_\alpha) = S \cap (\cup\{N \in \mathcal{N}_\alpha : N \subset U\})$. It follows that $\cup\mathcal{C}_\alpha \in \mathcal{M}$. Let $S_\alpha = S \cap (\cup\mathcal{C}_\alpha)$ and observe that $S_\alpha \subset S_\beta$ for $\alpha \le \beta < \lambda$. To finish the proof it suffices to show that $S \setminus S_\alpha$ is finite for some $\alpha < \lambda$. Then the element $\cup\mathcal{C}_\alpha \subset U$ of $\mathcal{M}$ will contain almost all members of the sequence S.

Separately, we shall consider the cases of countable and uncountable λ. If λ is uncountable, then it has uncountable cofinality and consequently, the transfinite sequence $(S_\alpha)_{\alpha<\lambda}$ eventually stabilizes, i.e., there is an ordinal $\alpha < \lambda$ such that $S_\beta = S_\alpha$ for all $\beta \ge \alpha$. We claim that the set $S \setminus S_\alpha$ is finite. Otherwise, $S \setminus S_\alpha$ would be a sequence convergent to x and there would exist an element $N \in \mathcal{N}$ with $N \subset U$ and infinite intersection $N \cap (S \setminus S_\alpha)$. Find now an ordinal $\beta \ge \alpha$ with $N \in \mathcal{N}_\beta$ and observe that $S \cap N \subset S_\beta = S_\alpha$ which contradicts to the choice of N.

If λ is countable and $S \setminus S_\alpha$ is infinite for any $\alpha < \lambda$, then we can find an infinite pseudo-intersection $T \subset S$ of the decreasing sequence $\{S \setminus S_\alpha\}_{\alpha<\lambda}$. Note that $T \cap S_\alpha$ is finite for every $\alpha < \lambda$. Since sequence T converges to x, there is an element $N \in \mathcal{N}$ such that $N \subset U$ and $N \cap T$ is infinite. Find $\alpha < \lambda$ with $N \in \mathcal{N}_\alpha$ and observe that $N \cap S \subset S_\alpha$. Then $N \cap T \subset N \cap T \cap S_\alpha \subset T \cap S_\alpha$ is finite, which contradicts to the choice of N.

Proof of Proposition 2. Let X be a topological space and fix a point $x \in X$.

(1) Suppose that $\mathrm{cs}^*_\chi(X) < \mathfrak{p}$ and fix a cs*-network $\mathcal{N}$ at the point x such that $|\mathcal{N}| < \mathfrak{p}$. Without loss of generality, we can assume that the family $\mathcal{N}$ is closed under finite unions. We claim that $\mathcal{N}$ is a cs-network at x. Assuming the converse we would find a neighborhood $U \subset X$ of x and a sequence $S \subset X$ convergent to x such that $S \setminus N$ is infinite for any element $N \in \mathcal{N}$ with $N \subset U$. Since $\mathcal{N}$ is closed under finite unions, the family $\mathcal{F} = \{S \setminus N : N \in \mathcal{N},\ N \subset U\}$ is closed under finite intersections. Since $|\mathcal{F}| \le |\mathcal{N}| < \mathfrak{p}$, the family $\mathcal{F}$ has an infinite pseudo-intersection $T \subset S$. Consequently, $T \cap N$ is finite for any $N \in \mathcal{N}$ with $N \subset U$. But this contradicts to the facts that T converges to x and $\mathcal{N}$ is a cs*-network at x.

The items (2) and (3) follow from Propositions 1(5) and 2(1). The item (4) follows from (1,2) and the inequality $\chi(X) \le \mathfrak{c}$ holding for any countable topological space X.

Finally, to derive (5) from (3) use the well-known fact that under GCH, $\lambda^{\aleph_0} \le \kappa$ for any infinite cardinals $\lambda < \kappa$, see [HJ, 9.3.8].

Proof of Theorem 1. Suppose that G is a non-metrizable sequential group with countable cs*-character. By Proposition 2(1), $\mathrm{cs}_\chi(G) = \mathrm{cs}^*_\chi(G) \le \aleph_0$.

First we show that each countably compact subspace K of G is first-countable. The space K, being countably compact in the sequential space G, is sequentially compact and so are the sets $K^{-1}K$ and $(K^{-1}K)^{-1}(K^{-1}K)$ in G. The sequential compactness of $K^{-1}K$ implies that it is an α_7-space. Since $\mathrm{cs}_\chi((K^{-1}K)^{-1}(K^{-1}K)) \le$

$\mathrm{cs}_\chi(G) \le \aleph_0$ we may apply Lemmas 3 and 2 to conclude that the space $K^{-1}K$ has countable sb-character and K has countable character.

Next, we show that G contains an open $\mathcal{MK}_\omega$-subgroup. By Lemma 5, G has a countable cs-network $\mathcal{K}$ consisting of countably compact subsets. Since the group product of two countably compact subspaces in G is countably compact, we may assume that $\mathcal{K}$ is closed under finite group products in G. We can also assume that $\mathcal{K}$ is closed under the inversion, i.e. $K^{-1} \in \mathcal{K}$ for any $K \in \mathcal{K}$. Then $H = \cup K$ is a subgroup of G. It follows that this subgroup is a sequential barrier at each of its points, and thus is open-and-closed in G. We claim that the topology on H is inductive with respect to the cover $\mathcal{K}$. Indeed, consider some $U \subset H$ such that $U \cap K$ is open in K for every $K \in \mathcal{K}$. Assuming that U is not open in H and using the sequentiality of H, we would find a point $x \in U$ and a sequence $(x_n)_{n\in\omega} \subset H \setminus U$ convergent to x. It follows that there are elements $K_1, K_2 \in \mathcal{K}$ such that $x \in K_1$ and K_2 contains almost all members of the sequence $(x^{-1}x_n)$. Then the product $K = K_1K_2$ contains almost all x_n and the set $U \cap K$, being an open neighborhood of x in K, contains almost all members of the sequence (x_n), which is a contradiction.

As it was proved before each $K \in \mathcal{K}$ is first-countable, and consequently H has countable pseudocharacter, being the countable union of first countable subspaces. Then H admits a continuous metric. Since any continuous metric on a countably compact space generates its original topology, every $K \in \mathcal{K}$ is a metrizable compactum, and consequently H is an $\mathcal{MK}_\omega$-subgroup of G.

Since H is an open subgroup of G, G is homeomorphic to $H \times D$ for some discrete space D.

Proof of Theorem 2. Suppose G is a non-metrizable sequential topological group with countable cs*-character. By Theorem 1, G contains an open $\mathcal{MK}_\omega$-subgroup H and is homeomorphic to the product $H \times D$ for some discrete space D . This implies that G has point-countable k-network. By a result of Shibakov [Shi], each sequential topological group with point-countable k-network and sequential order $< \omega_1$ is metrizable. Consequently, $\mathrm{so}(G) = \omega_1$. It is clear that $\psi(G) = \psi(H) \le \aleph_0$, $\chi(G) = \chi(H)$, $\mathrm{sb}_\chi(G) = \mathrm{sb}_\chi(H)$ and $ib(G) = c(G) = d(G) = l(G) = e(G) = nw(G) = knw(G) = |D| \cdot \aleph_0$.

To finish the proof it rests to show that $\mathrm{sb}_\chi(H) = \chi(H) = \mathfrak{d}$. It follows from Lemmas 2 and 3 that the group H, being non-metrizable, is not α_7 and thus contains a copy of the sequential fan S_ω. Then $\mathfrak{d} = \chi(S_\omega) = \mathrm{sb}_\chi(S_\omega) \le \mathrm{sb}_\chi(H) \le \chi(H)$. To prove that $\chi(H) \le \mathfrak{d}$ we shall apply a result of K. Sakai [Sa] asserting that the space $\mathbb{R}^\infty \times Q$ contains a closed topological copy of each $\mathcal{MK}_\omega$-space and the well-known equality $\chi(\mathbb{R}^\infty \times Q) = \chi(\mathbb{R}^\infty) = \mathfrak{d}$ (following from the fact that $\mathbb{R}^\infty$ carries the box-product topology, see [Sch, Ch.II, Ex.12]).

Proof of Theorem 5. First we describe two general constructions producing topologically homogeneous sequential spaces. For a locally compact space Z let $\alpha Z = Z \cup \{\infty\}$ be the one-point extension of Z endowed with the topology whose neighborhood base at ∞ consists of the sets $\alpha Z \setminus K$ where K is a compact subset of Z. Thus for a non-compact locally compacts space Z the space αZ is noting else but the one-point compactification of Z. Denote by $2^\omega = \{0,1\}^\omega$ the Cantor cube.

Consider the subsets

$$\Xi(Z) = \{(c,(z_i)_{i\in\omega}) \in 2^\omega \times (\alpha Z)^\omega : z_i = \infty \text{ for all but finitely many indices } i\} \text{ and}$$
$$\Theta(Z) = \{(c,(z_i)_{i\in\omega}) \in 2^\omega \times (\alpha Z)^\omega : \exists n \in \omega \text{ such that } z_i \neq \infty \text{ if and only if } i < n\}.$$

Observe that $\Theta(Z) \subset \Xi(Z)$.

Endow the set $\Xi(Z)$ (resp. $\Theta(Z)$) with the strongest topology generating the Tychonov product topology on each compact subset from the family $\mathcal{K}_\Xi$ (resp. $\mathcal{K}_\Theta$), where

$\mathcal{K}_\Xi = \{2^\omega \times \prod_{i\in\omega} C_i : C_i$ are compact subsets of αZ and almost all $C_i = \{\infty\}\}$;

$\mathcal{K}_\Theta = \{2^\omega \times \prod_{i\in\omega} C_i :\ \exists i_0 \in \omega$ such that $C_{i_0} = \alpha Z$, $C_i = \{\infty\}$ for all $i > i_0$ and C_i is a compact subsets of Z for every $i < i_0\}$.

Lemma 6. *Suppose Z is a zero-dimensional locally metrizable locally compact s-pace. Then*

1. *the spaces $\Xi(Z)$ and $\Theta(Z)$ are topologically homogeneous;*
2. *$\Xi(Z)$ is a regular zero-dimensional k_ω-space while $\Theta(Z)$ is a totally disconnected k-space;*
3. *if Z is Lindelöf, then $\Xi(Z)$ and $\Theta(Z)$ are zero-dimensional $\mathcal{MK}_\omega$-spaces with $\chi(\Xi(Z)) = \chi(\Theta(Z)) \le \mathfrak{d}$;*
4. *$\Xi(Z)$ and $\Theta(Z)$ contain copies of the space αZ while $\Theta(Z)$ contains a closed copy of Z;*
5. $\mathrm{cs}^*_\chi(\Xi(Z)) = \mathrm{cs}^*_\chi(\Theta(Z)) = \mathrm{cs}^*_\chi(\alpha Z)$, $\mathrm{cs}_\chi(\Xi(Z)) = \mathrm{cs}_\chi(\Theta(Z)) = \mathrm{cs}_\chi(\alpha Z)$, $\mathrm{sb}_\chi(\Theta(Z)) = \mathrm{sb}_\chi(\alpha Z)$, *and* $\psi(\Xi(Z)) = \psi(\Theta(Z)) = \psi(\alpha Z)$;
6. *the spaces $\Xi(Z)$ and $\Theta(Z)$ are sequential if and only if αZ is sequential;*
7. *if Z is not countably compact, then $\Xi(Z)$ contains a closed copies of S_2 and S_ω and $\Theta(Z)$ contains a closed copy of S_2.*

Proof. (1) First we show that the space $\Xi(Z)$ is topologically homogeneous.

Given two points $(c,(z_i)_{i\in\omega}), (c',(z'_i)_{i\in\omega})$ of $\Xi(Z)$ we have to find a homeomorphism h of $\Xi(Z)$ with $h(c,(z_i)_{i\in\omega}) = (c',(z'_i)_{i\in\omega})$. Since the Cantor cube 2^ω is topologically homogeneous, we can assume that $c \neq c'$. Fix any disjoint closed-and-open neighborhoods U, U' of the points c, c' in 2^ω, respectively.

Consider the finite sets $I = \{i \in \omega : z_i \neq \infty\}$ and $I' = \{i \in \omega : z'_i \neq \infty\}$. Using the zero-dimensionality and the local metrizability of Z, for each $i \in I$ (resp. $i \in I'$) fix an open compact metrizable neighborhood U_i (resp. U'_i) of the point z_i (resp. z'_i) in Z. By the classical Brouwer Theorem [Ke, 7.4], the products $U \times \prod_{i\in I} U_i$ and $U' \times \prod_{i\in I'} U'_i$, being zero-dimensional compact metrizable spaces without isolated points, are homeomorphic to the Cantor cube 2^ω. Now the topological homogeneity of the Cantor cube implies the existence of a homeomorpism $f : U \times \prod_{i\in I} U_i \to U' \times \prod_{i\in I'} U'_i$ such that $f(c,(z_i)_{i\in I}) = (c',(z'_i)_{i\in I'})$. Let

$W = \{(x,(x_i)_{i\in\omega}) \in \Xi(Z) : x \in U,\ x_i \in U_i \text{ for all } i \in I\}$ and

$W' = \{(x',(x'_i)_{i\in\omega}) \in \Xi(Z) : x' \in U',\ x'_i \in U'_i \text{ for all } i \in I'\}$.

It follows that W, W' are disjoint open-and-closed subsets of $\Xi(Z)$. Let $\chi : \omega \setminus I' \to \omega \setminus I$ be a unique monotone bijection.

Now consider the homeomorphism $\tilde{f} : W \to W'$ assigning to a sequence $(x,(x_i)_{i\in\omega}) \in W$ the sequence $(x',(x'_i)_{i\in\omega}) \in W'$ where $(x',(x'_i)_{i\in I'}) = f(x,(x_i)_{i\in I})$ and $x'_i =$

$x_{\chi(i)}$ for $i \notin I'$. Finally, define a homeomorphism h of $\Xi(Z)$ letting

$$h(x) = \begin{cases} x & \text{if } x \notin W \cup W'; \\ \tilde{f}(x) & \text{if } x \in W; \\ \tilde{f}^{-1}(x) & \text{if } x \in W' \end{cases}$$

and observe that $h(c, (z_i)_{i\in\omega}) = (c', (z'_i)_{i\in\omega})$ which proves the topological homogeneity of the space $\Xi(Z)$.

Replacing $\Xi(Z)$ by $\Theta(Z)$ in the above proof, we shall get a proof of the topological homogeneity of $\Theta(Z)$.

The items (2–4) follow easily from the definitions of the spaces $\Xi(Z)$ and $\Theta(Z)$, the zero-dimensionality of αZ, and known properties of k_ω-spaces, see [FST] (to find a closed copy of Z in $\Theta(Z)$ consider the closed embedding $e : Z \to \Theta(Z)$, $e : z \mapsto (z, z_0, z, \infty, \infty, \dots)$, where z_0 is any fixed point of Z).

To prove (5) apply Proposition 3(6,8,9,10). (To calculate the cs*-, cs-, and sb-characters of $\Theta(Z)$, observe that almost all members of any sequence $(a_n) \subset \Theta(Z)$ convergent to a point $a = (c, (z_i)) \in \Theta(Z)$ lie in the compactum $2^\omega \times \prod_{i\in\omega} C_i$, where C_i is a clopen neighborhood of z_i if $z_i \neq \infty$, $C_i = \alpha Z$ if $i = \min\{j \in \omega : z_j = \infty\}$ and $C_i = \{\infty\}$ otherwise. By Proposition 3(6), the cs*-, cs-, and sb-characters of this compactum are equal to the corresponding characters of αZ.)

(6) Since the spaces $\Xi(Z)$ and $\Theta(Z)$ contain a copy of αZ, the sequentiality of $\Xi(Z)$ or $\Theta(Z)$ implies the sequentiality of αZ. Now suppose conversely that the space αZ is sequential. Then each compactum $K \in \mathcal{K}_\Xi \cup \mathcal{K}_\Theta$ is sequential since a finite product of sequential compacta is sequential, see [En_1, 3.10.I(b)]. Now the spaces $\Xi(Z)$ and $\Theta(Z)$ are sequential because they carry the inductive topologies with respect to the covers $\mathcal{K}_\Xi$, $\mathcal{K}_\Theta$ by sequential compacta.

(7) If Z is not countably compact, then it contains a countable closed discrete subspace $S \subset Z$ which can be thought as a sequence convergent to ∞ in αZ. It is easy to see that $\Xi(S)$ (resp. $\Theta(S)$) is a closed subset of $\Xi(Z)$ (resp. $\Theta(Z)$). Now it is quite easy to find closed copies of S_2 and S_ω in $\Xi(S)$ and a closed copy of S_2 in $\Theta(S)$. □

With Lemma 6 at our disposal, we are able to finish the proof of Theorem 5. To construct the examples satisfying the conditions of Theorem 5(2,3), assume $\mathfrak{b} = \mathfrak{c}$ and use Proposition 4 to find a locally compact locally countable space Z whose one-point compactification αZ is sequential and satisfies $\aleph_0 = \mathrm{sb}_\chi(\alpha Z) < \psi(\alpha Z) = \mathfrak{c}$. Applying Lemma 6 to this space Z, we conclude that the topologically homogeneous k-spaces $X_2 = \Xi(Z)$ and $X_3 = \Theta(Z)$ give us required examples.

The example of a countable topologically homogeneous k_ω-space X_1 with $\mathrm{sb}_\chi(X_1) < \chi(X_1)$ can be constructed by analogy with the space $\Theta(\mathbb{N})$ (with that difference that there is no necessity to involve the Cantor cube) and is known in topology as the Ankhangelski-Franklin space, see [AF]. We briefly remind its construction. Let $S_0 = \{0, \frac{1}{n} : n \in \mathbb{N}\}$ be a convergent sequence and consider the countable space $X_1 = \{(x_i)_{i\in\omega} \in S_0^\omega : \exists n \in \omega \text{ such that } x_i \neq 0 \text{ iff } i < n\}$ endowed with the strongest topology inducing the product topology on each compactum $\prod_{i\in\omega} C_i$ for which there is $n \in \omega$ such that $C_n = S_0$, $C_i = \{0\}$ if $i > n$, and $C_i = \{x_i\}$ for some $x_i \in S_0 \setminus \{0\}$ if $i < n$. By analogy with the proof of Lemma 6 it can be shown that X_1 is a topologically homogeneous k_ω-space with $\aleph_0 = \mathrm{sb}_\chi(X_1) < \chi(X_1) = \mathfrak{d}$ and $\mathrm{so}(X_1) = \omega$.

Proof of Proposition 5. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) were proved by Lin [Lin, 3.13] in terms of (universally) csf-countable spaces. To prove the other equivalences apply

Lemma 7. *A Hausdorff topological space X is an α_4-space provided one of the following conditions is satisfied:*

1. *X is a Fréchet-Urysohn α_7-space;*
2. *X is a Fréchet-Urysohn countably compact space;*
3. *$\mathrm{sb}_\chi(X) < \mathfrak{p}$;*
4. *$\mathrm{sb}_\chi(X) < \mathfrak{d}$, each point of X is regular G_δ, and X is c-sequential.*

Proof. Fix any point $x \in X$ and a countable family $\{S_n\}_{n\in\omega}$ of sequences convergent to x in X. We have to find a sequence $S \subset X \setminus \{x\}$ convergent to x and meeting infinitely many sequences S_n. Using the countability of the set $\bigcup_{n\in\omega} S_n$ find a decreasing sequence $(U_n)_{n\in\omega}$ of closed neighborhoods of x in X such that $\big(\bigcap_{n\in\omega} U_n\big) \cap \big(\bigcup_{n\in\omega} S_n\big) = \{x\}$. Replacing each sequence S_n by its subsequence $S_n \cap U_n$, if necessary, we can assume that $S_n \subset U_n$.

(1) Assume that X is a Fréchet-Urysohn α_7-space. Let $A = \{a \in X : a$ is the limit of a convergent sequence $S \subset X$ meeting infinitely many sequences $S_n\}$. It follows from our assumption on (S_n) and (U_n) that $A \subset \bigcap_{n\in\omega} U_n$.

It suffices to consider the non-trivial case when $x \notin A$. In this case x is a cluster point of A (otherwise X would be not α_7). Since X is Fréchet-Urysohn, there is a sequence $(a_n) \subset A$ convergent to x. By the definition of A, for every $n \in \omega$ there is a sequence $T_n \subset X$ convergent to a and meeting infinitely many sequences S_n. Without loss of generality, we can assume that $T_n \subset \bigcup_{i>n} S_i$ (because $a \in A \setminus \{x\}$ and thus $a \notin \bigcup_{n\in\omega} S_n$). It is easy to see that x is a cluster point of the set $\bigcup_{n\in\omega} T_n$. Since X is Fréchet-Urysohn, there is a sequence $T \subset \bigcup_{n\in\omega} T_n$ convergent to x.

Now it rests to show that the set T meets infinitely many sequences S_n. Assuming the converse we would find $n \in \omega$ such that $T \subset \bigcup_{i\le n} S_n$. Then $T \subset \bigcup_{i\le n} T_n$ which is not possible since $\bigcup_{i\le n} T_i$ is a compact set failing to contain the point x.

(2) If X is Fréchet-Urysohn and countably compact, then it is sequentially compact and hence α_7, which allows us to apply the previous item.

(3) Assume that $\mathrm{sb}_\chi(X) < \mathfrak{p}$ and let $\mathcal{N}$ be a sb-network at x of size $|\mathcal{N}| < \mathfrak{p}$. Without loss of generality, we can assume that the family $\mathcal{N}$ is closed under finite intersections. Let $S = \bigcup_{n\in\omega} S_n$ and $F_{N,n} = N \cap (\bigcup_{i\ge n} S_i)$ for $N \in \mathcal{N}$ and $n \in \omega$. It is easy to see that the family $\mathcal{F} = \{F_{N,n} : N \in \mathcal{N},\ n \in \omega\}$ consists of infinite subsets of S, has size $|\mathcal{F}| < \mathfrak{p}$, and is closed under finite intersection. Now the definition of the small cardinal $\mathfrak{p}$ implies that this family $\mathcal{F}$ has an infinite pseudo-intersection $T \subset S$. Then T is a sequence convergent to x and intersecting infinitely many sequences S_n. This shows that X is an α_4-space.

(4) Assume that the space X is c-sequential, each point of X is regular G_δ, and $\mathrm{sb}_\chi(X) < \mathfrak{d}$. In this case we can choose the sequence (U_n) to satisfy $\bigcap_{n\in\omega} U_n = \{x\}$. Fix an sb-network $\mathcal{N}$ at x with $|\mathcal{N}| < \mathfrak{d}$. For every $n \in \omega$ write $S_n = \{x_{n,i} : i \in \mathbb{N}\}$. For each sequential barrier $N \in \mathcal{N}$ find a function $f_N : \omega \to \mathbb{N}$ such that $x_{n,i} \in N$ for every $n \in \omega$ and $i \ge f_N(n)$. The family of functions $\{f_N : N \in \mathcal{N}\}$ has size $< \mathfrak{d}$ and hence is not cofinal in $\mathbb{N}^\omega$. Consequently, there is a function $f : \omega \to \mathbb{N}$ such that $f \not\le f_N$ for each $N \in \mathcal{N}$. Now consider the sequence $S = \{x_{n,f(n)} : n \in \omega\}$. We claim that x is a cluster point of S. Indeed, given any neighborhood U of x,

find a sequential barrier $N \in \mathcal{N}$ with $N \subset U$. Since $f \not\leq f_N$, there is $n \in \omega$ with $f(n) > f_N(n)$. It follows from the choice of the function f_N that $x_{n,f(n)} \in N \subset U$.

Since $S \setminus U_n$ is finite for every n, $\{x\} = \bigcap_{n\in\omega} U_n$ is a unique cluster point of S and thus $\{x\} \cup S$ is a closed subset of X. Now the c-sequentiality of X implies the existence of a sequence $T \subset S$ convergent to x. Since T meets infinitely many sequences S_n, the space X is α_4. □

Proof of Proposition 6. Suppose a space X has countable cs*-character. The implications (1) ⇒ (2, 3, 4, 5) are trivial. The equivalence (1) ⇔ (2) follows from Proposition 1(2). To show that (3) ⇒ (2), apply Lemma 7 and Proposition 5(3 ⇒ 1).

To prove that (4) ⇒ (2) it suffices to apply Proposition 5(4 ⇒ 1) and observe that X is Fréchet-Urysohn provided $\chi(X) < \mathfrak{p}$ and X has countable tightness. This can be seen as follows.

Given a subset $A \subset X$ and a point $a \in \bar{A}$ from its closure, use the countable tightness of X to find a countable subset $N \subset A$ with $a \in \bar{N}$. Fix any neighborhood base $\mathcal{B}$ at x of size $|\mathcal{B}| < \mathfrak{p}$. We can assume that $\mathcal{B}$ is closed under finite intersections. By the definition of the small cardinal $\mathfrak{p}$, the family $\{B \cap N : B \in \mathcal{B}\}$ has infinite pseudo-intersection $S \subset N$. It is clear that $S \subset A$ is a sequence convergent to x, which proves that X is Fréchet-Urysohn.

(5) ⇒ (2). Assume that X is a sequential space containing no closed copies of S_ω and S_2 and such that each point of X is regular G_δ. Since X is sequential and contains no closed copy of S_2, we may apply Lemma 2.5 [Lin] to conclude that X is Fréchet-Urysohn. Next, Theorem 3.6 of [Lin] implies that X is an α_4-space. Finally apply Proposition 5 to conclude that X has countable sb-character and, being Fréchet-Urysohn, is first countable.

The final implication (6) ⇒ (2) follows from (5) ⇒ (2) and the well-known equality $\chi(S_\omega) = \chi(S_2) = \mathfrak{d}$.

Proof of Proposition 7. The first item of this proposition follows from Proposition 6(3 ⇒ 1) and the observation that each Fréchet-Urysohn countable compact space, being sequentially compact, is α_7.

Now suppose that X is a dyadic compact with $\mathrm{cs}^*_\chi(X) \leq \aleph_0$. If X is not metrizable, then it contains a copy of the one-point compactification αD of an uncountable discrete space D, see [En$_1$, 3.12.12(i)]. Then $\mathrm{cs}^*_\chi(\alpha D) \leq \mathrm{cs}^*_\chi(X) \leq \aleph_0$ and by the previous item, the space αD, being Fréchet-Urysohn and compact, is first-countable, which is a contradiction.

Proof of Proposition 8. Let D be a discrete space.

(1) Let $\kappa = \mathrm{cs}^*_\chi(\alpha D)$ and λ_1 (λ_2) is the smallest weight of a (regular zero-dimensional) space X of size $|X| = |D|$, containing no non-trivial convergent sequence. To prove the first item of proposition 8 it suffices to verify that $\lambda_2 \leq \kappa \leq \lambda_1$. To show that $\lambda_2 \leq \kappa$, fix any cs*-network $\mathcal{N}$ at the unique non-isolated point ∞ of αD of size $|\mathcal{N}| \leq \kappa$. The algebra $\mathcal{A}$ of subsets of D generated by the family $\{D \setminus N : N \in \mathcal{N}\}$ is a base of some zero-dimensional topology τ on D with $w(D,\tau) \leq \kappa$. We claim that the space D endowed with this topology contains no infinite convergent sequences. To get a contradiction, suppose that $S \subset D$ is an infinite sequence convergent to a point $a \in D \setminus S$. Then S converges to ∞ in αD and hence, there is an element $N \in \mathcal{N}$ such that $N \subset \alpha D \setminus \{a\}$ and $N \cap S$ is infinite. Consequently, $U = D \setminus N$ is a neighborhood of a in the topology τ such that $S \setminus U$

is infinite which contradicts to the fact that S converges to a. Now consider the equivalence relation $\sim$ on D: $x \sim y$ provided for every $U \in \tau$ $(x \in U) \Leftrightarrow (y \in U)$. Since the space (D, τ) has no infinite convergent sequences, each equivalence class $[x]_\sim \subset D$ is finite (because it carries the anti-discrete topology). Consequently, we can find a subset $X \subset D$ of size $|X| = |D|$ such that $x \not\sim y$ for any distinct points $x, y \in X$. Clearly that τ induces a zero-dimensional topology on X. It rests to verify that this topology is T_1. Given any two distinct point $x, y \in X$ use $x \not\sim y$ to find an open set $U \in \mathcal{A}$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Since $D \setminus U \in \mathcal{A}$, in both cases we find an open set $W \in \mathcal{A}$ such that $x \in W$ but $y \notin W$. It follows that X is a T_1-space containing no non-trivial convergent sequence and thus $\lambda_2 \le w(X) \le |\mathcal{A}| \le |N| \le \kappa$.

To show that $\kappa \le \lambda_1$, fix any topology τ on D such that $w(D, \tau) \le \lambda_1$ and the space (D, τ) contains no non-trivial convergent sequences. Let $\mathcal{B}$ be a base of the topology τ with $|\mathcal{B}| \le \lambda_1$, closed under finite unions. We claim that the collection $\mathcal{N} = \{\alpha D \setminus B : B \in \mathcal{B}\}$ is a cs*-network for αD at ∞. Fix any neighborhood $U \subset \alpha D$ of ∞ and any sequence $S \subset D$ convergent to ∞. Write $\{x_1, \dots, x_n\} = \alpha D \setminus U$ and by finite induction, for every $i \le n$ find a neighborhood $B_i \in \mathcal{B}$ of x_i such that $S \setminus \bigcup_{j \le i} B_j$ is infinite. Since $\mathcal{B}$ is closed under finite unions, the set $N = \alpha D \setminus (B_1 \cup \dots \cup B_n)$ belongs to the family $\mathcal{N}$ and has the properties: $N \subset U$ and $N \cap S$ is infinite, i.e., $\mathcal{N}$ is a cs*-network at ∞ in αD. Thus $\kappa \le |\mathcal{N}| \le |\mathcal{B}| \le \lambda_1$. This finishes the proof of (1).

An obvious modification of the above argument gives also a proof of the item (2).

Proof of Proposition 9. Let D be an uncountable discrete space.

(1) The inequalities $\aleph_1 \cdot \log|D| \le \mathrm{cs}^*_\chi(\alpha D) \le \mathrm{cs}_\chi(\alpha D)$ follows from Propositions 7(1) and 1(2,4) yielding $|D| = \chi(\alpha D) = \mathrm{sb}_\chi(\alpha D) \le 2^{\mathrm{cs}^*_\chi(\alpha D)}$. The inequality $\mathrm{cs}_\chi(\alpha D) \le \mathfrak{c} \cdot \mathrm{cof}([\log|D|]^{\le\omega})$ follows from proposition 8(2) and the observation that the product $\{0,1\}^{\log|D|}$ endowed with the $\aleph_0$-box product topology has weight $\le \mathfrak{c} \cdot \mathrm{cof}([\log|D|]^{\le\omega})$. Under the $\aleph_0$-*box product topology* on $\{0,1\}^\kappa$ we understand the topology generated by the base consisting of the sets $\{f \in \{0,1\}^\kappa : f|C = g|C\}$ where $g \in \{0,1\}^\omega$ and C is a countable subset of κ.

The item (2) follows from (1) and the equality $\aleph_1 \cdot \log\kappa = 2^{\aleph_0} \cdot \min\{\kappa, (\log\kappa)^\omega\}$ holding under GCH for any infinite cardinal κ, see [HJ, 9.3.8]

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