MEASURABLE CARDINALS AND THE COFINALITY OF THE SYMMETRIC GROUP

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ABSTRACT. Assuming the existence of a $P_2\kappa$-hypermeasurable cardinal, we construct a model of Set Theory with a measurable cardinal κ such that $2^\kappa = \kappa^{++}$ and the group $Sym(\kappa)$ of all permutations of κ cannot be written as a union of a chain of proper subgroups of length $< \kappa^{++}$. The proof involves the iteration of a suitably defined uncountable version of the Miller forcing poset as well as the "tuning fork" argument introduced by the first author and K. Thompson in [11].

1. INTRODUCTION

A deep theorem of Macpherson and Neumann [16] states that if the symmetric group $Sym(\kappa)$ consisting of all permutations of a cardinal κ can be written as a union of an increasing chain $\langle G_i : i < \lambda\rangle$ of proper subgroups G_i, then $\lambda > \kappa$. Throughout this paper the minimal λ with this property will be denoted by $\mathrm{cf}(Sym(\kappa))$. It was proven in [22] that for $\kappa = \kappa^{<\kappa}$ the pair $(\mathrm{cf}(Sym(\kappa)), 2^\kappa)$ can be anything not obviously wrong. More precisely, for every regular $\lambda > \kappa$ and θ such that $\mathrm{cf}(\theta) \geq \lambda$, there exist a cardinal preserving forcing extension V^P such that $\mathrm{cf}(Sym(\kappa)) = \lambda$ and $2^\kappa = \theta$ in V^P. Moreover, for inaccessible κ we can assume [17, § 1] that P is κ-directed closed. Therefore if κ is supercompact, then it remains so in V^{Q*P}, where Q is a Laver preparation forcing making the supercompactness of κ indestructible by κ-directed closed forcing notions. The main result of this paper states that consistency of $\mathrm{cf}(Sym(\kappa)) > \kappa^+$ at a measurable κ can be obtained assuming much less than supercompactness.

Theorem 1.1. *Suppose GCH holds and there exists an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $(H(\kappa^{++}))^V = (H(\kappa^{++}))^M$. Then there exists a forcing extension V' of V such that κ is still measurable in V' and $V' \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$.*

Date: June 16, 2009.

2000 *Mathematics Subject Classification.* Primary: 03E35. Secondary: 03E55, 03E99.

Key words and phrases. Cofinality of the symmetric group, hypermeasurable cardinal, elementary embedding, lifting, Miller forcing, Sacks forcing.

The authors would like to thank FWF grant P19898-N18 for support for this research.

By work of Gitik [12] a cardinal κ of Mitchell order κ^{++} is required for GCH to fail at a measurable cardinal; thus the hypothesis of our result is close to optimal (it is in fact equiconsistent with the existence of a cardinal κ whose Mitchell order for extenders is $\kappa^{++}+1$).

A cardinal κ for which there exists an embedding such as in Theorem 1.1 will be called *$P_2\kappa$-hypermeasurable*. To the best knowledge of the authors, $\mathrm{cf}(Sym(\kappa)) = \kappa^+$ for measurable κ in all other known models of Set Theory constructed under assumptions weaker than (a certain degree of) supercompactness; see Remark 5.6 for a more detailed discussion.

The idea of the proof of Theorem 1.1 resembles that of the consistency of $\mathfrak{u} < \mathrm{cf}(Sym(\omega))$ established in [23]. In particular, in section 2 we introduce a variant of Miller forcing and a (slightly more general than in [14]) variant of Sacks forcing at an inaccessible cardinal κ. According to Theorem 2.9, iterated forcing constructions where at each stage we take any of these forcing notions do not collapse κ^+. In section 3 we introduce a new cardinal characteristic $\mathfrak{g}_{cl}(\kappa)$, which is a version for κ of the classical groupwise density number $\mathfrak{g}$. Section 4 is devoted to the proof of the fact that suitably arranged iterated forcing constructions considered in section 2 of length κ^{++} make $\mathrm{cf}(Sym(\kappa))$ equal to κ^{++}. More precisely, the Miller forcing is responsible for $\mathrm{cf}^*(Sym(\kappa)) = \kappa^{++}$, while the Sacks forcing makes $\mathrm{cf}(Sym(\kappa))$ and $\mathrm{cf}^*(Sym(\kappa))$ equal. (Here $\mathrm{cf}^*(Sym(\kappa))$ is the minimal length of a special chain of proper subgroups of $Sym(\kappa)$ introduced in Definition 4.1.) And finally, in section 5 we show how to extend elementary embeddings to forcing extensions considered in section 2, and thus prove Theorem 1.1. The idea of the proof in section 5 can be traced back to the work [11], where the "tuning fork" argument was introduced.

2. A variant of Miller forcing for uncountable cardinals. Basic properties. Alternation with Sacks.

In this section we suggest one of the possible ways to generalize the Miller forcing introduced in [18] to uncountable cardinals and study some basic properties of iterated forcing constructions, where at each stage we take either the Miller or Sacks forcing poset. The discussion is patterned after Kanamori [14]. It is worth mentioning here that there are other generalizations of Miller forcing, see e.g. [5].

Throughout this section κ denotes a strongly inaccessible cardinal.

Definition 2.1. Let $p \subset \kappa^{<\kappa}$. For $s \in p$ we denote by $C(p,s)$ (or simply by $C(s)$ if p is clear from the context) the set $\{\alpha \in \kappa : s^\frown\alpha \in p\}$.

Miller(κ) denotes the following forcing. A condition is a subset p of $\kappa^{<\kappa}$ such that

(i) $s \in p, t \subset s \longrightarrow t \in p$.

(ii) Each $s \in p$ is increasing and has a proper extension in p.

(iii) For every $\alpha < \kappa$ limit, $s \in \kappa^\alpha$, if $s \restriction \beta \in p$ for arbitrary large $\beta < \alpha$, then $s \in p$.

(iv) For every $s \in p$ there is $t \in p$ with $s \subset t$ which splits in p (i.e., $C(p,t)$ has more than one element). Moreover, if t_0, t_1 split in p and $t_0 \subset t_1$, then $C(p,t_1) \subset C(p,t_0)$.

(v) If $s \in p$ splits in p, then the set $C(p,s)$ is club.

(vi) If α is a limit ordinal, $s \in \kappa^\alpha$, and $s \restriction \beta$ splits in p for arbitrary large $\beta < \alpha$, then s splits in p and $C(s)$ is the intersection of $C(s \restriction \beta)$ for all β such that $s \restriction \beta$ splits in p.

We order Miller(κ) by declaring p to be stronger than q (and write $p \leq q$) iff $p \subset q$. □

It is clear that Miller(κ) is κ-closed. For every subtree p of $\kappa^{<\kappa}$ we denote by Split(p) the family of all $s \in p$ which split in p. Given $s \in \kappa^{<\kappa}$, $\ell(s)$ denotes the length of s, i.e. the (unique) α such that $s \in \kappa^\alpha$. If $p \in$ Miller(κ) and $\alpha \in \kappa$, then we denote by $\mathrm{Split}_\alpha(p)$ the set

$$\{s \in p \,:\, o.t.(\{t \subsetneqq s : t \in \mathrm{Split}(p)\}) \leq \alpha,\ \forall t \subsetneqq s (o.t.(s(\ell(t)) \cap C(p,t)) \leq \alpha)\}.$$

In what follows we shall heavily apply a *fusion argument* to Miller(κ) as well as to the Sacks forcing.

Definition 2.2. For $q \leq p \in$ Miller(κ) and $\alpha \in \kappa$ the notation $q \leq_\alpha p$ means that $\mathrm{Split}_\alpha(p) = \mathrm{Split}_\alpha(q)$. A sequence $\langle p_\alpha : \alpha \in \kappa \rangle$, where $p_\alpha \in$ Miller(κ), is called a *fusion sequence*, if

(i) If $\beta \leq \alpha$, then $p_\alpha \leq p_\beta$.

(ii) $p_{\alpha+1} \leq_\alpha p_\alpha$.

(iii) $p_\delta = \cap_{\alpha<\delta} p_\alpha$ for limit $\delta \in \kappa$. □

The following lemma is straightforward.

Lemma 2.3. *Let $\langle p_\alpha : \alpha \in \kappa \rangle$ be a fusion sequence. Then $q = \bigcap_{\alpha \in \kappa} p_\alpha \in$ Miller(κ) and $q \leq_\alpha p_\alpha$ for all $\alpha \in \kappa$.*

Next, we recall the definition of the Sacks forcing for uncountable cardinals.

Definition 2.4. Let us fix a sequence $\vec{A} = \langle A_\alpha : \alpha < \kappa \rangle$ such that $|A_\alpha| < \kappa$ for all α. Let $\mathcal{T}$ be the set of all functions t which satisfy the following conditions.

(i) There exists α such that the domain of t equals α.

(ii) For all $\beta \in \mathrm{dom}(t)$, $t(\beta) \in A_\beta$.

Sacks($\vec{A}$) stands for the forcing whose conditions are subsets T of $\mathcal{T}$ such that:

(1) $s \in T$, $t \subset s \to t \in T$.

(2) Each t has a proper extension in T.

(3) If $t \in \mathcal{T}$ and the set of such β that $t \restriction \beta \in T$ is unbounded in $\ell(t)$, then $t \in T$.

(4) There exists a club $C(T)$ such that the set $\mathrm{succ}_T(t)$ of immediate successors of an element $t \in T$ with domain α coincides with $\{t^\frown a : a \in A_\alpha\}$ provided $\alpha \in C(T)$, and $|\mathrm{succ}_T(t)| = 1$ otherwise.

Extension is defined by $S \leq T$ iff S is a subset of T. □

When each A_α is $\{0, 1\}$ we get the usual Sacks forcing considered in [7, 11, 14]. Some other sequences $\vec{A}$ are employed in [9]. Yet another sequence will be used in Section 4. But the basic properties (e.g. chain condition, fusion) of Sacks($\vec{A}$) does not really depend on $\vec{A}$.

Given any $T \in$ Sacks($\vec{A}$) and $i \in \kappa$, we denote by $\mathrm{Split}_i(T)$ the set $\{t \in T : (\exists j \leq i)\ell(t) = \alpha_j\}$, where $\langle \alpha_i : i \in \kappa \rangle$ is the increasing enumeration of $C(T)$. Now the notions of $\leq_\alpha$ and of a fusion sequence can be introduced for Sacks($\vec{A}$) in the same way as for Miller(κ).

If γ is an ordinal and S^0, S^1 are disjoint subsets of γ such that $S^0 \bigcup S^1 = \gamma$, then we denote by $ST_{S^0,S^1,\vec{A}}$ the forcing poset $\mathbb{P}_\gamma$ from the iterated forcing construction $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \gamma, \eta < \gamma \rangle$ with supports of size $\leq \kappa$ defined as follows:

$$\{\eta < \gamma : \mathbb{P}_\eta \Vdash \dot{\mathbb{Q}}_\eta = \mathrm{Miller}(\kappa)\} = S^0 \text{ and } \{\eta < \gamma : \mathbb{P}_\eta \Vdash \dot{\mathbb{Q}}_\eta = \mathrm{Sacks}(\vec{A})\} = S^1.$$

Definition 2.5. Suppose that $\alpha \leq \kappa$ and $\langle p_\beta : \beta \in \alpha \rangle$ is a decreasing sequence of elements of $ST_{S^0,S^1,\vec{A}}$. The "meet" $q = \bigwedge_{\beta\in\alpha} p_\beta \in ST_{S^0,S^1,\vec{A}}$ is defined as follows: $\mathrm{supp}(q) = \bigcup_{\beta\in\alpha} \mathrm{supp}(p_\beta)$ and for every $\xi \in \mathrm{supp}(q)$, $q \restriction \xi \Vdash q(\xi) = \bigcap_{\beta\in\alpha} p_\beta(\xi)$. (Note that in case $\alpha = \kappa$ there could be no such q.) □

In order to prove that κ^+ is preserved by $ST_{S^0,S^1,\vec{A}}$ and κ^{++} is preserved for $\gamma = \kappa^{++}$ we need to employ a suitable variant of fusion.

Definition 2.6. Suppose that $\alpha \in \kappa$, $F \in [\gamma]^{<\kappa}$, and $q, p \in ST_{S^0,S^1,\vec{A}}$. $q \leq_{F,\alpha} p$ means that $q \leq p$ and $q \restriction \xi \Vdash q(\xi) \leq_\alpha p(\xi)$ for all $\xi \in F$.[1]

A sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$ is a *generalized fusion sequence (for* $ST_{S^0,S^1,\vec{A}}$*)*, iff

(*i*) $|F_\alpha| < \kappa$ for all $\alpha \in \kappa$.
(*ii*) $F_\alpha \supset F_\beta$ for all $\beta \leq \alpha < \kappa$.
(*iii*) $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$ for all α.
(*iv*) If δ is limit, then $F_\delta = \bigcup_{\beta<\alpha} F_\beta$ and $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$.
(*v*) $\bigcup\{F_\alpha : \alpha \in \kappa\} = \bigcup\{\mathrm{supp}(p_\alpha) : \alpha < \kappa\}$. □

The easy but technical proof of the following lemma is left to the reader.

Lemma 2.7. *Let* $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$ *be a generalized fusion sequence for* $ST_{S^0,S^1,\vec{A}}$*. Then* $q = \bigwedge_{\alpha<\kappa} p_\alpha \in ST_{S^0,S^1,\vec{A}}$ *and* $q \leq_{F_\alpha,\alpha} p_\alpha$ *for all* $\alpha \in \kappa$.

There is no loss of generality to assume that each A_α is an element of κ.

Definition 2.8. Suppose that $p \in ST_{S^0,S^1,\vec{A}}$, $F \subset \mathrm{supp}(p)$ with $|F| < \kappa$, and $\sigma : F \to \kappa^{<\kappa}$. Then $p|\sigma$ is a function with the same domain as p such that $(p|\sigma)(\xi)$ equals $p(\xi)$ if $\xi \notin F$ and $p(\xi)_{\sigma(\xi)}$ otherwise, where for $q \in \mathrm{Miller}(\kappa) \cup \mathrm{Sacks}(\vec{A})$ and $t \in \kappa^{<\kappa}$, q_t denotes the set of all elements of q compatible with t. □

It is clear that $p|\sigma \in ST_{S^0,S^1,\vec{A}}$ if and only if for every $\xi \in F$ we have $(p|\sigma) \restriction \xi \Vdash_\xi \sigma(\xi) \in p(\xi)$. If $p|\sigma \in ST_{S^0,S^1,\vec{A}}$, then we say that σ *lies on* p.

Theorem 2.9. *For every ordinal* γ *and decomposition* $\gamma = S^0 \cup S^1$ *the forcing* $ST_{S^0,S^1,\vec{A}}$ *preserves cardinals* $\leq \kappa^+$.

Suppose that $2^\kappa = \kappa^+$ *in* V*. If* $\gamma < \kappa^{++}$*, then there exists a dense subset* $W_\gamma \subset ST_{S^0,S^1,\vec{A}}$ *of size* $|W_\gamma| \leq \kappa^+$*. If* $\gamma = \kappa^{++}$*, then* $ST_{S^0,S^1,\vec{A}}$ *has the* κ^{++}*-chain condition.*

Similar results were discussed in [14] and [7] for the Sacks forcing. Nevertheless, we give complete proofs here. Our exposition follows [14]. The first part of Theorem 2.9 follows from the lemma below.

Lemma 2.10. (1) *Assume that* $p \in ST_{S^0,S^1,\vec{A}}$ *and* $p \Vdash \dot{z} \in V$*. Then for every* $F \in [\gamma]^{<\kappa}$ *and* $\alpha_0 \in \kappa$ *there exists* $q \leq_{F,\alpha_0} p$ *and* $x \in V$ *with* $|x| \leq \kappa$ *such that* $q \Vdash \dot{z} \in x$.

(2) *Assume that* $p \in ST_{S^0,S^1,\vec{A}}$ *and* $p \Vdash$ *"*$\dot{z} \in V$ *and* $|\dot{z}| \leq \kappa$*". Then for every* $F \in [\gamma]^{<\kappa}$ *and* $\alpha_0 \in \kappa$ *there exists* $q \leq_{F,\alpha_0} p$ *and* $x \in V$ *with* $|x| \leq \kappa$ *such that* $q \Vdash \dot{z} \subset x$.

[1]The preorder $\leq_\alpha$ here depends on whether $\xi \in S^0$ or $\xi \in S^1$.

Proof. It is well-known how to obtain the second item from the first one, see [14, Theorem 2.3].

In order to prove the first item we shall inductively construct a generalized fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa\rangle$ with $(p_\beta, F_\beta) = (p, F)$ for all $\beta \leq \alpha_0$, and $x \in V$ of size $|x| \leq \kappa$ such that $q = \bigwedge_{\alpha\in\kappa} p_\alpha$ and x are as required. The trivial description of how to construct F_α's is omitted. The limit step of the construction is obvious, so we concentrate on the successor case.

Let us enumerate as $\{\sigma_{\alpha,i} : i \in \eta\}$ all ground model functions $\sigma : F_\alpha \to \kappa^{\alpha+1}$ which lie on some $r \leq p_\alpha$ so that $r = r|\sigma$, $r \restriction \xi \Vdash \sigma(\xi) \restriction \alpha \in \mathrm{Split}_\alpha(p_\alpha(\xi))$ for all $\xi \in F_\alpha$, and $\sigma(\xi)(\alpha) = \alpha$ for all $\xi \in F_\alpha \cap S^0$. (Here $\eta < \kappa$ is a cardinal.) We shall construct a sequence $\langle p_{\alpha,i} : i \in \eta\rangle$ as follows. Set $p_{\alpha,-1} = p_\alpha$ and suppose that we have already constructed a decreasing sequence $\langle p_{\alpha,j} : j < i\rangle$ such that $p_{\alpha,j} \leq_{F_\alpha,\alpha} p_{\alpha,k}$ for all $k \leq j < i$. If i is limit, we set $p_{\alpha,i} = \bigwedge_{j<i} p_{\alpha,j}$. Suppose that $i = j + 1$. If there is no $r \leq p_{\alpha,j}$ such that $r = r|\sigma_{\alpha,j}$ and $r \restriction \xi \Vdash \sigma(\xi) \restriction \alpha \in \mathrm{Split}_\alpha(p_\alpha(\xi))$ for all $\xi \in F_\alpha$, we set $p_{\alpha,i} = p_{\alpha,j}$. And if there is such r, let $r_{\alpha,j} \leq r$ and $x_{\alpha,j} \in V$ be such that $r_{\alpha,j} \Vdash \dot z = x_{\alpha,j}$. Now, using the Maximal Principle we define $p_{\alpha,j+1}$ to be the amalgamation of $p_{\alpha,j}$ and $r_{\alpha,j}$ as in the proof of [14, Theorem 2.2]. More precisely,

(a) $\mathrm{supp}(p_{\alpha,j+1}) = \mathrm{supp}(r_{\alpha,j})$.

(b) If $\xi \in F_\alpha$, then $p_{\alpha,j+1}(\xi)$ is such that

$$r_{\alpha,j} \restriction \xi \Vdash p_{\alpha,j+1}(\xi) = (p_{\alpha,j}(\xi) \setminus p_{\alpha,j}(\xi)_{\sigma_{\alpha,j}(\xi)}) \bigcup r_{\alpha,j}(\xi),$$

and for any condition $c \leq p_{\alpha,j+1} \restriction \xi$ incompatible with $r_{\alpha,j} \restriction \xi$,

$$c \Vdash_\xi p_{\alpha,j+1}(\xi) = p_{\alpha,j}(\xi).$$

(c) if $\xi \notin F_\alpha$, then $p_{\alpha,j+1}(\xi)$ is such that

$$r_{\alpha,j} \restriction \xi \Vdash p_{\alpha,j+1}(\xi) = r_{\alpha,j}(\xi),$$

and for any condition $c \leq p_{\alpha,j+1} \restriction \xi$ incompatible with $r_{\alpha,j} \restriction \xi$,

$$c \Vdash_\xi p_{\alpha,j+1}(\xi) = p_{\alpha,j}(\xi).$$

Now we let $p_{\alpha+1} = \bigwedge_{i\in\eta} p_{\alpha,i}$. It follows that $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$. This completes our construction of $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa\rangle$. Set $x = \{x_{\alpha,i}\}$.

Claim 2.11. *Suppose that $r \leq q$. Then there exists a sequence $\langle r_\alpha : \alpha \in \kappa\rangle$ of elements of $ST_{S^0,S^1,\vec A}$ with $r_0 = r$, a sequence $\langle \sigma_\alpha : F_\alpha \to \kappa^{<\kappa} | \alpha < \kappa\rangle$, and sequences $\langle \mu_{\alpha,\xi}, \nu_{\alpha,\xi} : \alpha \in \kappa, \xi \in F_\alpha\rangle$ of ordinals less than κ such that*

(*i*) *If $\beta < \alpha$, then $r_\alpha \leq r_\beta$.*

(*ii*) *If $\xi \in F_\alpha$, then $\ell(\sigma_\alpha(\xi)) = \mu_{\alpha,\xi} + 1$.*

(*iii*) *If $\beta < \alpha$, then $\sigma_\beta(\xi) \subsetneq \sigma_\alpha(\xi)$ for all $\xi \in F_\beta$.*

(iv) *For every* $\xi \in F_{\alpha+1}$ *we have* $r_{\alpha+1} \restriction \xi \Vdash$ "$r_{\alpha+1}(\xi) = r_{\alpha+1}(\xi)_{\sigma_{\alpha+1}(\xi)}$, $\sigma_{\alpha+1}(\xi) \restriction \mu_{\alpha+1,\xi}$ *splits in* $r_\alpha(\xi)$, *and* $\sigma_{\alpha+1}(\xi) \restriction \mu_{\alpha+1,\xi} \in \mathrm{Split}_{\nu_{\alpha+1,\xi}}(q(\xi))$".

(v) *If* δ *is limit, then*
- $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} = \bigcup_{\alpha<\delta} \sigma_\alpha(\xi)$;
- $\sigma_\delta(\xi)(\mu_{\delta,\xi}) = \sup\{\sigma_\alpha(\xi)(\mu_{\alpha,\xi}) : \alpha < \delta\}$ *for all* $\xi \in F_\delta \cap S^0$ *(We assume that* $\sigma_\alpha(\xi) = \emptyset$ *for all* $\xi \notin F_\alpha$*);*
- $\nu_{\delta,\xi} = \sup_{\alpha<\delta} \nu_{\alpha,\xi}$ *for all* $\xi \in F_\delta$;
- $r_\delta \restriction \xi \Vdash$ "$\sigma_\delta(\xi) \in r_\delta(\xi)$, $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi}$ *splits in* $r_\delta(\xi)$, *and* $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} \in \mathrm{Split}_{\nu_{\delta,\xi}}(q(\xi))$" *for all* $\xi \in F_\delta$.

Proof. The construction proceeds by induction. For limit δ we simply set $\sigma_\delta(\xi)$ and $\nu_{\delta,\xi}$ to be so as it is required in (v) and $r_\delta = \bigwedge_{\alpha<\delta} r_\alpha$. Thus $\mu_{\delta,\xi} = \sup_{\alpha<\delta} \mu_{\alpha,\xi}$. Let us fix any $\alpha < \delta$ and $\xi \in F_\alpha \cap S^0$. From the above it follows that $r_\delta \restriction \xi \Vdash$"$\sigma_\beta(\xi) \restriction \mu_{\beta,\xi}$ splits in $r_\alpha(\xi)$ for all $\alpha < \beta < \delta$", and hence $r_\delta \restriction \xi \Vdash$"$\sigma_\delta(\xi) \restriction \mu_{\delta,\xi}$ splits in $r_\alpha(\xi)$", and consequently $r_\delta \restriction \xi \Vdash$"$\sigma_\delta(\xi) \restriction \mu_{\delta,\xi}$ splits in $r_\delta(\xi) = \bigcap_{\beta<\delta} r_\beta(\xi)$". By the definition of Miller(κ), $r_\delta \restriction \xi \Vdash C(r_\alpha(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}) = \bigcap_{\alpha<\beta<\delta} C(r_\alpha(\xi), \sigma_\beta(\xi) \restriction \mu_{\beta,\xi})$, and hence $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi)(\mu_{\delta,\xi}) = \sup_{\alpha<\beta<\delta} \sigma_\beta(\xi)(\mu_{\beta,\xi}) \in C(r_\alpha(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}))$, which implies that $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi)(\mu_{\delta,\xi}) \in C(r_\delta(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}) = \bigcap_{\alpha<\delta} C(r_\alpha(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}))$, which gives us that $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi) \in r_\delta(\xi)$. Finally, equalities $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} = \bigcup_{\alpha<\delta} \sigma_\alpha(\xi) \restriction \mu_{\alpha,\xi}$ and $\nu_{\delta,\xi} = \sup_{\alpha<\xi} \nu_{\alpha,\xi}$ combined with $r_\delta \restriction \xi \Vdash \sigma_\alpha(\xi) \restriction \mu_{\alpha,\xi} \in \mathrm{Split}_{\nu_{\alpha,\xi}}(q(\xi))$ imply $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi) \restriction \mu_{\delta,\xi} \in \mathrm{Split}_{\nu_{\delta,\xi}}(q(\xi))$, which completes the limit step.

At successor step $\alpha+1$ consider the increasing enumeration $\langle \xi_i : i < \eta \rangle$ of $F_{\alpha+1}$ and find a decreasing sequence $\langle u_i : i < \eta \rangle$ of elements of $ST_{S^0,S^1,\vec{A}}$ as follows: Set $u_i = \bigwedge_{j<i} u_j$ for limit i. Now given u_i, find $v \leq u_i \restriction \xi_i$, $\pi \in \kappa^{\mu+1}$ for some $\mu \in \kappa$, and $\nu \in \kappa$ such that $\pi \supset \sigma_\alpha(\xi_i)$ if $\xi_i \in F_\alpha$ and

$$v \Vdash_{\xi_i} \pi \in r_\alpha(\xi_i),\ \pi \restriction \mu \in \mathrm{Split}_\nu(q(\xi_i)) \cap \mathrm{Split}(r_\alpha(\xi_i)).$$

Then we set

$$u_{i+1} = v^\frown r_\alpha(\xi_i)_\pi{}^\frown r_\alpha \restriction (\gamma \setminus (\xi_i + 1)),$$

$\sigma_{\alpha+1}(\xi_i) = \pi$. ($\mu_{\alpha+1,\xi_i}$ and $\nu_{\alpha+1,\xi_i}$ automatically become equal to μ and ν respectively.) With u_i's thus defined, we set $r_{\alpha+1} = \bigwedge_{i<\eta} u_i$. This completes the inductive construction, hence the proof of the claim. □

The following claim is obvious.

Claim 2.12. *There exists a club* $C \subset \kappa$ *such that* $\mu_{\alpha,\xi} = \nu_{\alpha,\xi} = \alpha$ *and* $\sigma_\alpha(\xi)(\mu_{\alpha,\xi}) = \alpha$ *for every* $\alpha \in C$ *and* $\xi \in F_\alpha$. *Consequently,* $r_\alpha \restriction \xi \Vdash \sigma_\alpha(\xi) \restriction \alpha \in \mathrm{Split}_\alpha(q(\xi))$ *for every such* $\alpha \in C$ *and* $\xi \in F_\alpha$.

We are in a position now to finish the proof of Lemma 2.10. Let C be such as in Claim 2.12 and $\alpha \in C$. Then $\sigma_\alpha = \sigma_{\alpha,i}$ for some $i < \eta$ (see the construction of $p_{\alpha+1}$ at the beginning of the proof of Lemma 2.10). Since $r_{\alpha+1} \leq q \leq p_{\alpha,i}$, Claim 2.11$(iv)$ implies that for every $\xi \in F_{\alpha+1} \supset F_\alpha$ we have $r_{\alpha+1} \restriction \xi \Vdash r_{\alpha+1}(\xi) = r_{\alpha+1}(\xi)_{\sigma_\alpha(\xi)}$. Therefore the construction of $p_{\alpha,i+1}$ is nontrivial. Since $r_{\alpha+1} \leq q \leq p_{\alpha,i+1}$, $r_{\alpha+1} = r_{\alpha+1}|\sigma_\alpha \leq p_{\alpha,i+1}|\sigma_{\alpha,i} = r_{\alpha,i}$, and hence $r_{\alpha+1} \Vdash \dot{z} = x_{\alpha,i}$. Therefore for every $r \leq q$ there exists $r' \leq r$ such that $r' \Vdash \dot{z} \in x$, which finishes our proof. □

Proof of Theorem 2.9. The proof is analogous to that of [14, Lemma 3.1]. Let W_γ be the set of those $q \in ST_{S^0,S^1,\vec{A}}$ such that:

(i) There is an increasing sequence $\langle F_\alpha : \alpha \in \kappa \rangle$ of subsets of γ such that $|F_\alpha| < \kappa$ for all α, $F_\delta = \bigcup_{\alpha\in\delta} F_\alpha$ for limit δ, and $\bigcup_{\alpha\in\kappa} F_\alpha = \mathrm{supp}(q)$.

(ii) For every α there exists a (possibly empty) collection Σ_α of ground model functions $\sigma : F_\alpha \to \kappa^{\alpha+1}$ of size $|\Sigma_\alpha| < \kappa$ such that $q|\sigma \in ST_{S^0,S^1,\vec{A}}$ for all $\sigma \in \bigcup_{\alpha\in\kappa} \Sigma_\alpha$, and whenever $\beta \in \kappa$ and $r \leq q$, there exists $\alpha > \beta$ and $\sigma \in \Sigma_\alpha$ so that r and $q|\sigma$ are compatible.

The proof of Lemma 2.10 gives that W_γ is dense in $ST_{S^0,S^1,\vec{A}}$. In addition, almost literal repetition of the proof of [14, Lemma 3.1] gives that if a pair of sequences

$$\langle \langle F_\alpha : \alpha \in \kappa \rangle, \langle \Sigma_\alpha : \alpha \in \kappa \rangle \rangle$$

is a witness for $q_i \in W_\gamma$, $i \in 2$, then $q_0 \leq q_1 \leq q_0$ in $ST_{S^0,S^1,\vec{A}}$. It suffices to note that there are at most κ^+-many such pairs.

Finally, the fact that $ST_{S^0,S^1,\vec{A}}$ has κ^{++}-chain condition provided $\gamma = \kappa^{++}$ is a direct consequence of [1, Theorem 2.2]. □

At this point we would like to note that there has been extensive work by Eisworth, Roslanowski, Shelah and perhaps others on possible generalizations of proper forcing to uncountable cardinals, see, e.g., [8, 19]. It is plausible that Theorem 2.9 follows from one of the general results about uncountable versions of proper forcing. However Claim 2.11 will play a central role in the proof of Claim 5.4, and for this reason we gave a complete proof of Theorem 2.9 instead of trying to put it into the framework of the results from [8] or [19].

3. Miller(κ) and a variant of the groupwise density number

Throughout this section κ is strongly inaccessible, $2^\kappa = \kappa^+$ in V, $\kappa^{++} = S^0 \sqcup S^1$, $\vec{A} = \langle A_\alpha : \alpha \in \kappa \rangle$ is a sequence of ordinals below κ, and S^0 is κ^+-stationary (we use $\sqcup$ as notation for disjoint union). Here we define a

new cardinal characteristic of κ and show that iteration of Miller(κ) pushes it to κ^{++}.

Definition 3.1. We say that $\mathcal{G} \subset [\kappa]^\kappa$ is a *cgd-family* (abbreviated from club groupwise dense), if for every continuous increasing function $\phi : \kappa \to \kappa$ there exists a club $C \subset \kappa$ such that $\bigcup_{\alpha\in C} \phi(\alpha+1) \setminus \phi(\alpha) \in \mathcal{G}$, and for every $A \in \mathcal{G}$ and $B \in [\kappa]^\kappa$ such that $|B \setminus A| < \kappa$ we have $B \in \mathcal{G}$. In what follows the minimal size of a collection $\mathfrak{G}$ of cgd-families with empty intersection is denoted by $\mathfrak{g}_{cl}(\kappa)$. □

Theorem 3.2. *Suppose that G is a $ST_{S^0,S^1,\vec{A}}$-generic filter. Then $V[G] \models \mathfrak{g}_{cl}(\kappa) = \kappa^{++}$.*

The proof of Theorem 3.2 is divided into a sequence of lemmas.

Lemma 3.3. *Suppose that G is a $ST_{S^0,S^1,\vec{A}}$-generic filter. Then for every subset x of κ such that $x \in V[G]$ there exists $\gamma < \kappa^{++}$ such that $x \in V[G_\gamma]$, and the smallest such γ has cofinality $\leq \kappa$.*

Proof. Let $\dot{x}$ be a $ST_{S^0,S^1,\vec{A}}$-name of x. Note that the set $D \subset W_{\kappa^{++}}$ of all $q \in ST_{S^0,S^1,\vec{A}}$ such as in the proof of Theorem 2.9 with additional property that for every $\sigma \in \Sigma_\alpha$ the condition $q|\sigma$ decides $\dot{x}(\beta)$ for all $\beta < \alpha$, is dense in $ST_{S^0,S^1,\vec{A}}$. (Any q obtained along the lines of the proof of Lemma 2.10 with an extra requirement that $r_{\alpha,j}$ decides $\dot{x}(\beta)$ for all $\beta < \alpha$ belongs to D.) Item (ii) from the proof of Theorem 2.9 implies that $\{q|\sigma : \sigma \in \bigcup_{\alpha\in\kappa} \Sigma_\alpha\}$ is predense below q. Therefore for every $q \in D$ and $\beta \in \kappa$ there exists a subset $E_{q,\beta}$ predense below q of size $\leq \kappa$ and such that each element of $E_{q,\beta}$ decides $\dot{x}(\beta)$. From the above it follows that for every $q \in D$ we have $q \Vdash \dot{x} = \pi$, where $\pi = \{\langle\langle\beta, \check{i}_{\beta,r}\rangle, r\rangle : \beta \in \kappa, r \in E_{q,\beta}\}$ and $r \Vdash \dot{x}(\beta) = i_{\beta,r}$. The rest of the proof is straightforward. □

The following lemma resembles [3, Lemma 5.10].

Lemma 3.4. *Let G be a $ST_{S^0,S^1,\vec{A}}$-generic filter and $\mathcal{F} \in V[G]$ be a cgd-family. There is a κ^+-closed unbounded set of ordinals $\eta < \kappa^{++}$ for which $\mathcal{F} \cap V[G_\eta] \in V[G_\eta]$ and $\mathcal{F} \cap V[G_\eta]$ is a cgd-family in $V[G_\eta]$.*

Proof. Let $\dot{\mathcal{F}}$ be a $ST_{S^0,S^1,\vec{A}}$-name for $\mathcal{F}$ and $p \in G$ be a condition which forces that $\dot{\mathcal{F}}$ is a cgd-family, and $\gamma < \kappa^{++}$ be such that $p \in \mathbb{P}_\gamma$. The proof of Lemma 3.3 yields a set Π_γ of $\mathbb{P}_\gamma$-names of size $|\Pi_\gamma| = \kappa^+$ such that for every $\mathbb{P}_\gamma$-generic filter H and $x \in \mathcal{P}(\kappa) \cap V[H]$ there exists $\pi \in \Pi_\gamma$ with the property $x = \pi^H$. For every $\pi \in \Pi_\gamma$ we denote by $B(\pi)$ a maximal antichain of conditions in $\mathbb{P}_{\kappa^{++}}$ that decide whether $\pi \in \dot{\mathcal{F}}$. Let $\eta_1 = \eta_1(\gamma)$ be the

supremum of the union of supports of all conditions appearing in some $B(\pi)$, $\pi \in \Pi_\gamma$. (Recall that $ST_{S^0,S^1,\vec{A}}$ has κ^{++}-c.c..) Then $\mathcal{F} \cap V[G_\gamma] \in V[G_{\eta_1}]$.

For every $\pi \in \Pi_\gamma$ we can find a maximal antichain $A(\pi)$ below p whose elements decide whether π is (the range of) a continuous increasing function, and if $q \in A(\pi)$ decides that π is such, then for some $\xi(\pi,q) > \gamma$ and $\theta_{\pi,q} \in \Pi_{\xi(\pi,q)}$, q forces $\theta_{\pi,q}$ to be a club and $\bigcup_{\alpha\in\theta_{\pi,q}}[\pi(\alpha),\pi(\alpha+1)) \in \dot{\mathcal{F}}$. Let η_2 be the upper bound of the set

$$\big\{\xi(\pi,q) : \pi \in \Pi_\gamma, q \in \bigcup_{\pi\in\Pi_\gamma} A(\pi)\big\} \ \cup \ \big\{\mathrm{supp}(q) : q \in \bigcup_{\pi\in\Pi_\gamma} A(\pi)\big\}.$$

Then $\eta(\gamma) := \max\{\eta_1,\eta_2\}$ has the properties $\dot{\mathcal{F}}^H \cap V[H_\gamma] \in V[H_{\eta(\gamma)}]$, and if $\psi \in V[H_\gamma]$ is any continuous increasing sequence, then there is a club $C \in V[H_{\eta(\gamma)}]$ so that $\bigcup_{\alpha\in C}[\psi(\alpha),\psi(\alpha+1)) \in \dot{\mathcal{F}}^H$, where H is any $ST_{S^0,S^1,\vec{A}}$-generic filter containing p.

Let $E \subset \kappa^{++}$ be the κ^+-closed unbounded set of those η such that $\eta(\gamma) \leq \eta$ for all $\gamma < \eta$. We claim that E is as required. Indeed, by Lemma 3.3 for every $\eta \in E$ we have $\mathcal{F} \cap V[G_\eta] = \{\pi^{G_\eta} : \exists\gamma < \eta(\pi \in \Pi_\gamma \wedge B(\pi) \cap G_\eta \neq \emptyset)\}$, and the last set is obviously in $V[G_\eta]$. Now suppose that $\psi \in V[G_\eta]$ is a continuous increasing function from κ to κ. Applying Lemma 3.3 we can find $\gamma < \eta$ such that $\psi \in V[G_\gamma]$. From the above it follows that there is a club $C \in V[G_{\eta(\gamma)}] \subset V[G_\eta]$ so that $\bigcup_{\alpha\in C}[\psi(\alpha),\psi(\alpha+1)) \in \dot{\mathcal{F}}^G \cap V[G_\eta]$, which finishes our proof. □

Lemma 3.5. *For every $p \in \mathrm{Miller}(\kappa)$ there exists a continuous increasing sequence $\langle \nu_\alpha : \alpha \in \kappa\rangle$ such that for every club C there exists $q \leq p$ such that the range of every branch through q is almost (=modulo a subset of size $< \kappa$) contained in $\bigcup_{\alpha\in C}[\nu_\alpha,\nu_{\alpha+1})$.*

Proof. We inductively define a desired sequence $\langle \nu_\alpha\rangle_{\alpha\in\kappa}$. Choose ν_0 arbitrary. For limit $\delta \in \kappa$ we set $\nu_\delta = \sup_{\alpha\in\delta}\nu_\alpha$. After ν_α is defined, let $\beta > \nu_\alpha$ be such that for every $s \in p$ whose range is a subset of ν_α and $\xi \in [\nu_\alpha,\beta)$, if $s^\frown\xi \in p$, then the range of the smallest extension $t \in \mathrm{Split}(p)$ of $s^\frown\xi$ is contained in β. We set $\nu_{\alpha+1} = \beta$.

We claim that the sequence $\langle \nu_\alpha : \alpha \in \kappa\rangle$ is as required. Indeed, it is continuous by the construction. Let C and $D \subset \bigcup_{\alpha\in C}[\nu_\alpha,\nu_{\alpha+1})$ be clubs. (The role of D here is to ensure that the splitting nodes of the condition q constructed below split into clubs rather than into sets containing clubs. We could take, e.g., $D = \{\alpha \in C : \nu_\alpha = \alpha\}$.) Let q be the tree generated by the set of those $s = s_1^\frown\xi \in p$ such that $s_1 \in \mathrm{Split}(p)$ and for every $t \leq s$, if $t \in \mathrm{Split}(p)$, then $s(\ell(t)) \in \bigcup_{\alpha\in C\setminus\mu(t)}[\nu_\alpha,\nu_{\alpha+1}) \cap D$, where $\mu(t)$ is the

minimal ordinal μ such that ν_μ contains the range of t. Then $q \in \mathrm{Miller}(\kappa)$. It suffices to note that the range of each branch through q is a subset of $\bigcup_{\alpha\in C}[\nu_\alpha, \nu_{\alpha+1}) \bigcup \beta$, where β is the range of the stem (=smallest splitting element) of p. □

Proof of Theorem 3.2. The simple density argument based on Lemma 3.5 gives us that if H is a Miller(κ)-generic filter and $\mathcal{F}$ is a cgd-family in V, then the range of $\bigcap H \in \kappa^\kappa$ is almost included in some $F \in \mathcal{F}$.

Now let $\mathfrak{F} \in V[G]$ be a collection of cgd-families of size κ^+. For every $\mathcal{F} \in \mathfrak{F}$ Lemma 3.4 yields a κ^+-closed unbounded set $C_\mathcal{F} \in \mathcal{P}(\kappa^{++}) \cap V$ such that $\mathcal{F} \cap V[G_\eta] \in V[G_\eta]$ and $\mathcal{F} \cap V[G_\eta]$ is a cgd-family in $V[G_\eta]$ for every $\eta \in C_\mathcal{F}$. Let us fix $\eta \in S^0 \cap \bigcap_{\mathcal{F}\in\mathfrak{F}} C_\mathcal{F}$. From the above it follows that $G_{\eta+1} = G_\eta * H$, where H is a Miller(κ)-generic filter over $V[G_\eta]$. As we already noted, H gives rise to a subset $X \in V[G_{\eta+1}]$ of κ such that for every $\mathcal{F} \in \mathfrak{F}$ there exists $F \in \mathcal{F} \cap V[G_\eta]$ such that $X \subset F$. Therefore $X \in \bigcap \mathfrak{F}$, which finishes our proof. □Theorem 3.2.

4. A new lower bound for the cofinality of symmetric group

In this section κ denotes a strongly inaccessible cardinal. The main result of this section says that for a certain sequence $\vec{A}$, if both S^0 and S^1 are κ^+-stationary and G is $ST_{S^0,S^1,\vec{A}}$-generic, then $V[G] \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$. The motivation for this is given in section 5. We follow the strategy of the proof of [23, Theorem 2.2]. In its turn that proof relies upon the methods developed in [21, § 2].

Following [23] we give the following definition.

Definition 4.1. For a subset A of κ we shall identify the group $Sym(A)$ with the subgroup of $Sym(\kappa)$ consisting of permutations σ such that $\sigma \restriction (\kappa \setminus A) = \mathrm{id}_{\kappa\setminus A}$.

For every increasing $\psi \in \kappa^\kappa$ we denote by P_ψ the group $\prod_{\alpha\in\kappa} Sym(\psi(\alpha+1) \setminus \psi(\alpha))$, which will be identified with a subgroup of $Sym(\kappa)$. $\mathrm{cf}^*(Sym(\kappa))$ is the least cardinal λ such that it is possible to express $Sym(\kappa) = \bigcup_{i<\lambda} \Gamma_i$ as the union of a chain of proper subgroups such that for every increasing *continuous* $\phi \in \kappa^\kappa$ there exists $i \in \lambda$ such that P_ϕ is a subgroup of Γ_i.

For an increasing function $\theta : \kappa \to \kappa$ we set $\tilde{\theta}(\alpha) = \sup_{\xi\in\alpha} \theta(\xi)$ and $Q_\theta = P_{\tilde{\theta}}$. (Note that $\tilde{\theta}$ is continuous and $P_\theta \subset Q_\theta$.) □

The following lemma resembles [23, Theorem 2.6]. But the proofs of Lemma 4.2 and Theorem 2.6 from [23] are completely different.

Lemma 4.2. $\mathrm{cf}^*(Sym(\kappa)) \geq \mathfrak{g}_{cl}(\kappa)$.

Proof. The proof is divided into two steps.

Claim 4.3. *For every $\pi \in Sym(\kappa)$ there exists continuous increasing $\psi \in \kappa^\kappa$ such that $\pi \in P_\psi$.*

Proof. For any $\alpha \in \kappa$ we set $\beta(\alpha) = \min\{\pi(\xi) : \xi \geq \alpha\}$ and $\gamma(\alpha) = \sup\{\pi(\xi) : \xi \in \alpha\}$. Since π is a bijection, the Fodor's lemma implies that $\beta(\alpha) \geq \alpha$ for club many α's. Therefore there exists a club $C \subset \kappa$ such that $\gamma \restriction C = \mathrm{id}_C$ and $\beta(\alpha) \geq \alpha$ for all $\alpha \in C$. Now, the increasing bijective enumeration $\psi : \kappa \to C \cup \{0\}$ is as required. □

Given any $B \in [\kappa]^\kappa$, we denote by $e_B : \kappa \to B$ the increasing bijective enumeration of B. Note that continuous strictly increasing functions from κ to κ are exactly those of the form e_C for a club C.

Claim 4.4. *Let Γ be a subgroup of $Sym(\kappa)$ containing $Sym_0(\kappa) = \{\pi : \pi(\alpha) = \alpha$ for all but $< \kappa$ many α's$\}$ and such that $\langle \Gamma, g\rangle \neq Sym(\kappa)$ for all $g \in Sym(\kappa)$, and $\mathcal{G}_\Gamma = \{A \in [\kappa]^\kappa : \forall B\, (|B \setminus A| < \kappa \to Q_{e_B} \not\subset \Gamma)\}$. Then $\mathcal{G}_\Gamma$ is a cgd-family.*

Proof. Let $\phi : \kappa \to \kappa$ be a continuous increasing function. Since $Sym_0(\kappa) \subset \Gamma$, the family $\mathcal{G}_\Gamma$ is closed under modifications of size $< \kappa$ of its elements. Thus it is enough to show that there exists a club C such that, letting $C_\phi = \bigcup_{\alpha\in C}[\phi(\alpha), \phi(\alpha+1))$, we have $C_\phi \in \mathcal{G}_\Gamma$, which means $Q_{e_{C_\phi}} \not\subset \Gamma$. Assume to the contrary that $Q_{e_{C_\phi}} \subset \Gamma$ for every club $C \subset \kappa$. Set $O = \bigcup_{\alpha \text{ odd}}[\phi(\alpha), \phi(\alpha+1))$. We claim that $Sym(O) \subset \Gamma$. Once this is established, we get a contradiction with [16, Lemma 2.4]. Let us fix $\sigma \in Sym(O)$. Claim 4.3 yields a continuous increasing $\psi : \kappa \to \kappa$ such that $\sigma \in \prod_{\xi\in\kappa} Sym([e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) \bigcap O)$. Set $C = \{\alpha : \alpha$ is limit and $\phi(\alpha) = \sup_{\xi\in\alpha} e_O \circ \psi(\xi)\}$. It is clear that C is club. Since elements of C are limit ordinals, the choice of O ensures that $C_\phi \cap O = \emptyset$. We claim that

$$[\phi(\alpha), \phi(\alpha+1)) \bigcap [e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) = \emptyset \tag{1}$$

for every $\alpha \in C$ and $\xi \in \kappa$. Indeed, if $\xi < \alpha$, then $e_O \circ \psi(\xi+1) < \sup_{\eta<\alpha} e_O \circ \psi(\eta) = \phi(\alpha)$. Now suppose $\xi \geq \alpha$. Then $O \ni e_O \circ \psi(\xi) \geq \phi(\alpha)$, and therefore $[\phi(\alpha), \phi(\alpha+1)) \cap O = \emptyset$ implies $e_O \circ \psi(\xi) \geq \phi(\alpha+1)$, which proves (1). For any $\xi \in \kappa$ consider $\alpha(\xi), \beta(\xi) \in \kappa$ such that $\alpha(\xi) = \min\{\alpha \in C : \phi(\alpha) \geq e_O \circ \psi(\xi+1)\}$ and $\phi(\alpha(\xi))$ is the $\beta(\xi)$'th element of C_ϕ. Equation

(1) gives

$$[e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) \subset [\sup_{\beta<\beta(\xi)} e_{C_\phi}(\beta), e_{C_\phi}(\beta(\xi))) = \\ = [\tilde{e}_{C_\phi}(\beta(\xi)), \tilde{e}_{C_\phi}(\beta(\xi)+1)),$$

and therefore

$$\prod_{\xi\in\kappa} Sym([e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) \bigcap O) \subset Q_{e_{C_\phi}} \subset \Gamma,$$

which implies $\sigma \in \Gamma$ and thus completes our proof. □

Let us express $Sym(\kappa) = \bigcup_{i<\lambda} \Gamma_i$ as a union of an increasing chain of proper subgroups such that each P_ψ is contained in some Γ_i. Since $|Sym_0(\kappa)| = \kappa$ and $\lambda > \kappa$, we can assume that $Sym_0(\kappa) \subset \Gamma_0$. For every $A \in [\kappa]^\kappa$ there exists $i \in \lambda$ such that $Q_{e_A} = P_{\tilde{e_A}} \subset \Gamma_i$, consequently $\bigcap_{i\in\lambda} \mathcal{G}_{\Gamma_i} = \emptyset$, and therefore $\mathfrak{g}_{cl}(\kappa) \leq \lambda$, which finishes our proof. □Lemma 4.2

Definition 4.5. Let $\phi_0 : \kappa \to \kappa$ be the continuous increasing function such that $\phi_0(0) = \omega$ and $\phi_0(\alpha+1) = \phi_0(\alpha) + \alpha$ for all $\alpha \in \kappa$. We set $N_\alpha = Sym(\phi_0(\alpha+1) \setminus \phi_0(\alpha))$, $\vec{N} = \langle N_\alpha : \alpha < \kappa\rangle$, and $ST_{S^0,S^1} \stackrel{\text{def}}{=} ST_{S^0,S^1,\vec{N}}$.

Each branch $\vec{t} = \langle t(\alpha)\rangle_{\alpha\in\kappa}$ of $T \in \mathrm{Sacks}(\vec{N})$ can be naturally identified with an element of $\sigma_{\vec{t}} \in P_{\phi_0}$ such that $\sigma_{\vec{t}} \restriction (\phi_0(\alpha+1) \setminus \phi_0(\alpha)) = t(\alpha)$. We also need the following

Definition 4.6. $[\kappa]^{\kappa,\kappa}$ denotes the set $\{A \subset \kappa : |A| = |\kappa \setminus A| = \kappa\}$. If $A \in [\kappa]^{\kappa,\kappa}$ and $\sigma \in Sym(\kappa)$, then σ^A is defined by $\sigma^A(e_A(\alpha)) = e_A(\sigma(\alpha))$. If Γ is a subgroup of $Sym(\kappa)$, then $\Gamma^A = \{\sigma^A : \sigma \in \Gamma\}$ and $\Gamma(A) = \{\sigma \restriction A : \sigma \in \Gamma, \sigma[A] = A\}$. □

The next lemma is of crucial importance for the proof of the equality $\mathrm{cf}^*(Sym(\kappa)) = \mathrm{cf}(Sym(\kappa))$ in $V^{ST_{S^0,S^1}}$ for κ^+-stationary subsets S^0, S^1 of κ^{++}.

Lemma 4.7. *Let $\psi : \kappa \to \kappa$ be a continuous increasing function. Then for every $T \in \mathrm{Sacks}(\vec{N})$ there exists $A \in [\kappa]^{\kappa,\kappa}$ such that for every $\pi \in P_\psi$ there exists $S \leq T$ such that $\sigma_{\vec{s}} \restriction A = \pi^A$ for all branches $\vec{s}$ of S.*

Proof. Let $B = \{\alpha \in \kappa : \psi(\alpha) = \phi_0(\alpha) = \alpha\}$. Then B is obviously a club. Consider $D \in [C(T) \cap B]^\kappa$ such that D is a discrete subspace of κ and find a club $C' \subset C(T) \setminus D$. Set $A = \bigcup_{\alpha\in D}(\phi_0(\alpha+1) \setminus \phi_0(\alpha)) = \bigcup_{\alpha\in D}[\alpha, \alpha+\alpha)$

and let $h : \kappa \to A$ be the monotone bijection. We claim that for every $\beta < \kappa$ there exists $\alpha(\beta) \in D$ such that

$$h[\psi(\beta), \psi(\beta+1)) \subset [\phi_0(\alpha(\beta)), \phi_0(\alpha(\beta)+1)) = [\alpha(\beta), \alpha(\beta)\cdot 2). \tag{2}$$

Indeed, let us fix β and find $\alpha(\beta)$ such that $h(\psi(\beta)) \in [\alpha(\beta), \alpha(\beta)\cdot 2)$. Since $\alpha(\beta)$ is a fixed point of ϕ_0, it is indecomposable, i.e. it is not equal to a sum of any two smaller ordinals. By our choice of D, $\alpha(\beta) > \sup(D \cap \alpha(\beta))$, and hence $o.t.(A \cap \phi_0(\alpha(\beta))) < \phi_0(\alpha(\beta)) = \alpha(\beta)$. Therefore $o.t.(A \cap \alpha(\beta)\cdot 2) = o.t.(A \cap \phi_0(\alpha(\beta))) + \alpha(\beta) = \alpha(\beta)$, which means that $h(\alpha(\beta)) = \alpha(\beta)\cdot 2$. From the above it follows that $\psi(\beta) = o.t.(A \cap h(\psi(\beta))) < o.t.(A \cap \alpha(\beta)\cdot 2) = \alpha(\beta)$. Since $\alpha(\beta)$ is a fixed point of ψ, we conclude that $\psi(\beta+1) < \alpha(\beta)$. In other words, $o.t.(A \cap h(\psi(\beta+1))) < o.t.(A \cap \phi_0(\alpha(\beta)+1))$, which implies the inequality $h(\psi(\beta+1)) < \phi_0(\alpha(\beta)+1)$ and thus proves (2).

Now, let us fix $\pi \in P_\psi$. A direct verification shows that $S \in \mathrm{Sacks}(\vec{N})$ such that $C(S) = C'$ and for every $\beta \in \kappa$ and $s \in S$ we have

$$h^{-1} \circ s(\alpha(\beta)) \circ h \restriction [\psi(\alpha), \psi(\alpha+1)) = \pi \restriction [\psi(\alpha), \psi(\alpha+1))$$

is as required. □

Lemma 4.8. *Suppose that $\lambda = \mathrm{cf}(Sym(\kappa)) < \mathrm{cf}^*(Sym(\kappa))$ and $\langle \Gamma_i : i \in \lambda \rangle$ is an increasing chain of proper subgroups of $Sym(\kappa)$ such that $Sym(\kappa) = \bigcup_{i<\lambda} \Gamma_i$. Then there exists a continuous increasing $\psi : \kappa \to \kappa$ such that $P^A_\psi \not\subset \Gamma_i(A)$ for all $i < \lambda$ and $A \in [\kappa]^{\kappa,\kappa}$.*

Proof. Let us fix $A_0 \in [\kappa]^{\kappa,\kappa}$. The same argument as in the proof of [4, Lemma 2.7] gives us a continuous increasing $\psi : \kappa \to \kappa$ such that $P^{A_0}_\psi \not\subset \Gamma_i(A_0)$ for all $i < \lambda$. We claim that this ψ is as required. Indeed, let $A \in [\kappa]^{\kappa,\kappa}$, $\pi \in Sym(\kappa)$ be such that $\pi \restriction A$ is the monotone bijection between A and A_0, and $j \in \lambda$ be such that $\pi \in \Gamma_i$. It is easy to check that if If $P^A_\psi \subset \Gamma_i(A)$ for some i, then $P^{A_0}_\psi \subset \Gamma_{\max\{i,j\}}(A_0)$, which contradicts our choice of ψ. □

The next lemma can be proven by the same methods as Lemma 3.4.

Lemma 4.9. *Suppose that $2^\kappa = \kappa^+$ in V, $\kappa^{++} = S^0 \sqcup S^1$ is a decomposition into two κ^+-stationary subsets, and G is ST_{S^0,S^1}-generic filter. For every $\Pi \subset Sym(\kappa)$ of size $|\Pi| \leq \kappa^+$ and every sequence $\langle \Gamma_i : i < \kappa^+ \rangle \in V[G]$ of subgroups of $Sym(\kappa)$ there is a κ^+-closed unbounded set of ordinals $\eta < \kappa^{++}$ for which $\Pi \in V[G_\eta]$, $\langle \Gamma_i \cap V[G_\eta] : i < \kappa^+ \rangle \in V[G_\eta]$, and for every $A \in [\kappa]^{\kappa,\kappa} \cap V[G_\eta]$ and $i < \kappa^+$ we have $\Gamma_i(A) \cap V[G_\eta] = (\Gamma_i \cap V[G_\eta])(A)$.*

Finally, we are in a position to prove the following theorem, which is the main result of this section.

Theorem 4.10. *Let S^0, S^1, and G be as in Lemma 4.9. Then $V[G] \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$.*

Proof. Suppose to the contrary that $V[G] \vDash Sym(\kappa) = \kappa^+$. Let $\langle \Gamma_i : i < \kappa^+ \rangle \in V[G]$ be an increasing chain of subgroups of $Sym(\kappa)$ such that $Sym(\kappa) = \bigcup_{i<\kappa^+} \Gamma_i$. By Theorem 3.2 and Lemma 4.2 we have $V[G] \vDash \mathrm{cf}^*(Sym(\kappa)) = \kappa^{++}$. Lemma 4.8 yields a continuous increasing $\psi : \kappa \to \kappa$ such that for every $A \in [\kappa]^{\kappa,\kappa}$ and $i < \kappa^+$ we have $P^A_\psi \not\subset \Gamma_i(A)$. Fix $A_* \in [\kappa]^{\kappa,\kappa}$ and for every $i < \kappa^+$ find $\pi_i \in P_\psi$ such that $\pi_i^{A_*} \in P^{A_*}_\psi \setminus \Gamma_i(A_*)$. Observe that $\Pi^A \not\subset \Gamma_i(A)$ for any $A \in [\kappa]^{\kappa,\kappa}$ and $i < \kappa^+$, where $\Pi = \{\pi_i : i < \kappa^+\}$. (The condition $\pi_i^A \notin \Gamma_i(A)$ holds at least starting from i such that Γ_i contains an extension of the order-preserving bijection between A_* and A.)

Let $\eta < \kappa^{++}$ be such an element of the κ^+-closed unbounded subset provided by Lemma 4.9 for $\langle \Gamma_i : i < \kappa^+ \rangle$ and Π for which $\dot{\mathbb{Q}}_\eta = \mathrm{Sacks}(\vec{N})$, i.e. $\eta \in S^1$. We can additionally require $A_* \in V[G_\eta]$. Suppose that H is the $\mathrm{Sacks}(\vec{N})$-generic filter over $V[G_\eta]$ such that $G_{\eta+1} = G_\eta * H$ and $\vec{h}$ is the common branch of all trees in H. Applying Lemma 4.7 we conclude that the set

$$\{S \in \mathrm{Sacks}(\vec{N}) : \exists A \in [\kappa]^{\kappa,\kappa} \cap V[G_\eta]\ \exists \pi \in \Pi$$
$$(\pi^A \notin (\Gamma_i \cap V[G_\eta])(A)\ \wedge\ S \Vdash \sigma_{\vec{h}} \restriction A = \pi^A)\}$$

is dense for all $i < \kappa^+$. Therefore for every i there exists $A_i \in [\kappa]^{\kappa,\kappa} \cap V[G_\eta]$ and $j(i) < \kappa^+$ such that $\sigma = \sigma_{\vec{h}} \restriction A_i = \pi^{A_i}_{j(i)} \notin (\Gamma_i \cap V[G_\eta])(A_i)$. Let $i < \kappa^+$ be such that $\sigma \in \Gamma_i$. Then

$$(\Gamma_i \cap V[G_\eta])(A_i) \not\ni \pi^{A_i}_{j(i)} = \sigma \restriction A_i \in \Gamma_i(A_i) \cap V[G_\eta],$$

which contradicts our choice of η. □

Now it is natural to ask whether we needed to employ $\mathrm{Sacks}(\vec{N})$ at all.

Question 4.11. *Is $\mathrm{cf}(Sym(\kappa)) \geq \mathfrak{g}_{cl}(\kappa)$?*

The cardinal characteristic $\mathfrak{g}_{cl}(\kappa)$ seems to be a natural generalization of the classical groupwise density number $\mathfrak{g}$ introduced in [2] and it was proved in [4] that $\mathrm{cf}(Sym(\omega)) \geq \mathfrak{g}$. But the methods of [4] do not seem to be applicable to Question 4.11.

5. Proof of Theorem 1.1

Without loss of generality, $j = j_E$ for some (κ, κ^{++})-extender E (such embeddings will be called (κ, κ^{++})-*extender ultrapowers* in what follows) so

that $M = \{j(f)(a) : f \in V, f : H(\kappa) \to V, \text{ and } a \in H(\kappa^{++})\}$, see, e.g, [13, pp.381–384][2].

Claim 5.1. *There exists a cardinal preserving forcing extension V' of V such that GCH holds in V' and j can be extended to an elementary embedding $j' : V' \to M'$ satisfying the following conditions:*

(i) $H(\kappa^{++})^{V'} = H(\kappa^{++})^{M'}$;

(ii) j' *is given by a* (κ, κ^{++})*-extender ultrapower so that* $M' = \{j'(f)(a) : f \in V', f : H(\kappa)^{V'} \to V', \text{ and } a \in H(\kappa^{++})^{V'}\}$;[3]

(iii) There exist disjoint κ^+*-stationary in* V' *(and hence in* M'*) subsets* $S^0, S^1 \in M'$ *of* κ^{++} *such that* $S^0 \cup S^1 = \kappa^{++}$*, and a sequence* $\langle (S^0_k, S^1_k) : k \in \kappa\rangle$*, where* S^0_k *and* S^1_k *are disjoint* ρ_k^+*-stationary subsets of* ρ_k^{++} *for which* $\rho_k^{++} = S^0_k \cup S^1_k$*, such that* $j'\langle (S^0_k, S^1_k) : k \in \kappa\rangle(\kappa) = (S^0, S^1)$*. (Here* ρ_k *denotes the* k*-th inaccessible cardinal below* κ*,* $k < \kappa$*.)*

Proof. We define a forcing poset $\mathbb{R}$ as follows. Let $\mathbb{R}_0 = \{\mathbf{1}_0\}$. For $k \leq \kappa$ we denote by $\dot{\mathbb{S}}_k$ a $\mathbb{R}_k$-name for the poset $Fn(\rho_k^{++}, 2, \rho_k^{++})$ adding one Cohen subset to ρ_k^{++}, see [15]. Proceeding this way along all inaccessible cardinals $\leq \kappa$ and using reverse Easton supports we define $\mathbb{R}$. Let G be a $\mathbb{R}_\kappa$-generic over V, g be a $\mathbb{S}_\kappa = \dot{\mathbb{S}}^G_\kappa$-generic over $V[G]$, $G_k = G \cap \mathbb{R}_k$ and g_k be such that $G_{k+1} = G_k * g_k$ for all $k < \kappa$. Note that g_k is the characteristic function of some subset of S^0_k of ρ^{++}. It is clear that S^0_k as well as its complement meet all subsets of ρ_k^{++} of size ρ_k^{++} which appear in $V[G_k]$. Since $\mathbb{R}_{k+1}$ has ρ_k^{++}-c.c., each ρ_k^+-closed unbounded in ρ_k^{++} subset $C' \in V[G_{k+1}]$ contains a ρ_k^+-closed unbounded in ρ_k^{++} subset $C \in V$ (the proof of [13, Lemma 22.25] works in this case as well), and hence S^0_k as well as $\rho_k^{++} \setminus S^0_k$ are ρ_k^+-stationary subsets of ρ_k^{++} in $V[G_{k+1}]$. The rest of our forcing is ρ_k^{+++}-closed, and hence S^0_k and $\rho_k^{++} \setminus S^0_k$ remain ρ_k^+-stationary in $V[G * g]$. Let S^0 be such that g is the characteristic function of S^0 and $S^1 = \kappa^{++} \setminus S^0$. Again, S^0 and S^1 are κ^+-stationary subsets of κ^{++} in $V[G * g]$.

$j(\mathbb{R})$ is the iteration with reverse Easton supports of length $j(\kappa) + 1$. A standard argument gives us that $j(\mathbb{R})_\kappa = \mathbb{R}_\kappa$, and hence G is $j(\mathbb{R})_\kappa$-generic over M and $(H(\kappa)^{++})^{V[G]} = (H(\kappa)^{++})^{M[G]}$, see [7, Lemma 4.4]. From the above it follows that $Fn(\kappa^{++}, 2, \kappa^{++})^{V[G]} = Fn(\kappa^{++}, 2, \kappa^{++})^{M[G]}$, and therefore $\mathbb{R} = j(\mathbb{R})_{\kappa+1}$ and g is $Fn(\kappa^{++}, 2, \kappa^{++})$-generic over M as well.

[2]What we actually use here is the following analogue of [13, Lemma 20.30]: *A cardinal κ is $P_2\kappa$-hypermeasurable iff there exists a (κ, κ^{++})-extender E such that $H(\kappa^{++}) \subset \mathrm{Ult}_E$ and $\kappa^{++} < j_E(\kappa)$.*

[3]We could assume here that the domain of f is still $H(\kappa)^V$ and $a \in H(\kappa^{++})^V$, but this is irrelevant.

Suppose that there exists a $j(\mathbb{R})$-generic filter $G' = G * g * H * h \in V[G * g]$ over M such that H is a $j(\mathbb{R})_{\kappa,j(\kappa)}$-generic over $M[G * g]$, h is $\mathbb{S}_{j(\kappa)} = j(\dot{\mathbb{S}}_\kappa)^{G*g*H}$-generic over $M[G*g*H]$, and $j[G*g] \subset G'$. Then j can be extended to an elementary embedding $j' : V[G * g] \to M[G']$ such that $j'(G * g) = G'$, see [6, Proposition 9.1]. Therefore $j'\langle S^0_k : k \in \kappa\rangle(\kappa) = S^0$. In addition, conditions (i) and (ii) hold by [6, Proposition 9.3]. Thus j', $V' = V[G * g]$, and $M' = M[H]$ are as required.

It suffices to note that such H and h exist: the construction of H is standard, see, e.g., fourth, fifth and sixth paragraphs of the proof of [7, Theorem 4.2]; the existence of h follows from the κ^+-distributivity of $\mathbb{S}_\kappa$ by virtue of [6, Proposition 15.1], which implies that the subfilter h of $\mathbb{S}_{j(\kappa)}$ generated by $j[g]$ is as required. □

There is no loss of generality in assuming $j = j'$, $V = V'$, and $M = M'$. We define a forcing poset $\mathbb{P}$ as follows. Let $\mathbb{P}_0 = \{\mathbf{1}_0\}$. For $k \leq \kappa$ we denote by $\dot{\mathbb{Q}}_k$ a $\mathbb{P}_k$-name for $ST_{S^0_k,S^1_k}$[4]. Proceeding this way along all inaccessible cardinals $\leq \kappa$ and using reverse Easton supports we define $\mathbb{P}$. Observe that $\mathbb{P}_k$ has ρ^+_k-c.c., and hence S^0_k, S^1_k are still ρ^+_k-stationary in $V^{\mathbb{P}_k}$. From the above and Theorem 4.10 we have that $V^{\mathbb{P}} \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$. Thus it suffices to prove that κ is measurable in $V^{\mathbb{P}}$. In order to do this we shall extend j to an elementary embedding from $V^{\mathbb{P}}$ into $M^{j(\mathbb{P})}$.

$j(\mathbb{P})$ is an iteration of length $j(\kappa)+1$ in M with reverse Easton support. It is clear that $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$. Let G be a $\mathbb{P}_\kappa$-generic filter over V. Since M and V have the same $H(\kappa^{++})$ and $j\langle(S^0_k, S^1_k) : k \in \kappa\rangle(\kappa) = (S^0, S^1)$, we have $(\kappa^{++})^{M[G]} = (\kappa^{++})^{V[G]}$ (see [7, Lemma 4.4]) and $j(\mathbb{P})_{\kappa+1} = \mathbb{P}$. Note that $j(\mathbb{P}) = j(\mathbb{P}_\kappa) * j(\dot{\mathbb{Q}}_\kappa)$. Let g be generic for $\dot{\mathbb{Q}}^G_\kappa$ over $V[G]$. We need to find a suitable $j(\mathbb{P})$-generic filter over M in order to lift j to $V[G * g]$. The following claim is analogous to [1, Lemma 6.4].

Claim 5.2. *If $x \subset M[G]$ (resp. $x \subset M[G * g]$), $x \in V[G]$ (resp. $x \in V[G * g]$), and $V[G] \vDash |x| \leq \kappa$ (resp. $V[G * g] \vDash |x| \leq \kappa$), then $x \in M[G]$ (resp. $x \in M[G * g]$).*

Proof. We present the proof of the $G * g$ part only. The other part is analogous. Without loss of generality, x is a set of ordinals. Let $\dot{x}$ be a $\mathbb{P}$-name such that $\dot{x}^{G*g} = x$. The κ^+-c.c. of $\mathbb{P}$ yields a set of ordinals $y \in V$ of size $|y| \leq \kappa^+$ in V and a condition $q \in \mathbb{P}$ such that $q \Vdash \dot{x} \subset y$. For every $\alpha \in y$ there exists a maximal in $\{p \in \mathbb{P} : p \leq q\}$ antichain A_α of conditions p such that $p \Vdash \alpha \in \dot{x}$ for every $p \in A_\alpha$. Applying

[4]Here $S^0_\kappa = S^0$ and $S^1_\kappa = S^1$.

Theorem 2.9, we conclude that $|A_\alpha| \leq \kappa^+$ for every $\alpha \in y$. It is clear that $\langle A_\alpha : \alpha \in y\rangle \in H(\kappa^{++})$, and hence $\langle A_\alpha : \alpha \in y\rangle \in (H(\kappa^{++}))^M$. It suffices to note that $x = \{\alpha \in y : G * g \cap A_\alpha \neq \emptyset\}$. □

In the same way as in the proof of [7, Theorem 4.2] (using Claim 5.2 instead of [11, Lemma 3]) we can find a $j(\mathbb{P}) \restriction (\kappa, j(\kappa))$-generic filter $H \in V[G * g]$ over $M[G * g]$. Thus $j[G] = G \subset G * g * H$, and hence j lifts to an embedding $j^* : V[G] \to M[G * g * H]$ definable in $V[G * g]$, see [6, Proposition 9.1]. Let M^* denote $M[G * g * H]$.

We give Definition 5.3 and Claim 5.4 in full generality for any iteration of Miller and Sacks forcings.

Definition 5.3. Let ρ be a strongly inaccessible cardinal and γ be an ordinal, S^0, S^1 be disjoint sets such that $S^0 \cup S^1 = \gamma$, and $\vec{A} = \langle A_\alpha : \alpha < \rho\rangle$ be a sequence of elements of ρ. Suppose that $\langle (p_\alpha, F_\alpha) : \alpha \in \rho\rangle$ is a generalized fusion sequence for $ST_{S^0,S^1,\vec{A}}$, $q = \bigwedge_{\alpha<\rho} p_\alpha$, and $i \in \rho$. We say that a function $\sigma : F \to \rho^{i+1}$ *is i-properly situated on q (with respect to the fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \rho\rangle$)*, if $F_i \subset F$, σ lies on some $r \leq q$ such that $r \restriction \xi \Vdash \sigma(\xi) \restriction i \in \max \mathrm{Split}_i(q(\xi))$ for all $\xi \in F$, and $\sigma(\xi)(i) = i$ for all $\xi \in F \cap S^0$.

Claim 5.4. *Let ρ, S^0, S^1, $\vec{A}$, $\langle (p_\alpha, F_\alpha) : \alpha \in \rho\rangle$, q, i be such as in Definition 5.3, $u \leq q$, $F, T \in [\gamma]^{<\rho}$ with $F \subset T$, and $C \subset \rho$ be a club. Then there exists $v \leq_{F,i} u$ satisfying the following conditions:*

*For every $\sigma : F \to \rho^{i+1}$ which lies on v and has the property $\sigma(\xi)(i) = i$ for all $\xi \in F \cap S^0$, **there exist** $j \in C$ and $\pi : T \cup F_j \to \rho^{(j+1)}$ such that $\pi(\xi) \restriction (i+1) = \sigma(\xi)$ for all $\xi \in F$, π lies on v, $v|\sigma = v|\pi$, and $v|\pi$ is a witness for π being j-properly situated on q with respect to $\langle (p_\alpha, F_\alpha) : \alpha \in \rho\rangle$.*

Proof. Let us enumerate as $\{\sigma^\zeta : \zeta \in \eta\}$ all $\sigma : F \to \rho^{i+1}$ with the property $\sigma(\xi)(i) = i$ for all $\xi \in F \cap S^0$ and which lie on some $r \leq u$. Set $u_0 = u$ and suppose that for some $\zeta < \eta$ and all $\zeta' < \zeta$ we have already defined $u_{\zeta'} \in ST_{S^0,S^1,\vec{A}}$ such that $u_{\zeta'} \leq_{F,i} u_{\zeta''}$ for all $\zeta'' \leq \zeta' < \zeta$. If ζ is limit, we set $u_\zeta = \bigwedge_{\zeta' \in \zeta} u_{\zeta'}$.

Let us consider the case $\zeta = \zeta' + 1$. If there is no $r \leq u_{\zeta'}$ such that $\sigma^{\zeta'}$ lies on $r = r|\sigma^{\zeta'}$, then we set $u_\zeta = u_{\zeta'}$. Otherwise set $r_0^{\zeta'} = r$, $\sigma_0^{\zeta'} = \sigma^{\zeta'}$, and $F_\alpha^{\zeta'} = F_\alpha \bigcup T$. Repeating the same argument as in Claim 2.11, we can construct a sequence $\langle r_\alpha^{\xi'} : \alpha \in \rho\rangle$ of elements of $ST_{S^0,S^1,\vec{A}}$, a sequence $\langle \sigma_\alpha^{\zeta'} : F_\alpha^{\zeta'} \to \rho^{<\rho} | \alpha < \rho\rangle$, and sequences $\langle \mu_{\alpha,\xi}^{\zeta'}, \nu_{\alpha,\xi}^{\zeta'} : \alpha \in \rho, \xi \in F_\alpha^{\zeta'}\rangle$ of ordinals less than ρ fulfilling the items $(i) - (v)$ of Claim 2.11. Claim 2.12

yields a club $C^{\zeta'} \subset \rho$ such that $\mu^{\zeta'}_{\alpha,\xi} = \nu^{\zeta'}_{\alpha,\xi} = \alpha$ and $\sigma^{\zeta'}_\alpha(\xi)(\mu^{\zeta'}_{\alpha,\xi}) = \alpha$ for every $\alpha \in C^{\zeta'}$ and $\xi \in F^{\zeta'}_\alpha \cap S^0$. Let us fix $j^{\zeta'} \in C^{\zeta'} \cap C$ and set $\pi^{\zeta'} = \sigma^{\zeta'}_{j^{\zeta'}}$ and $r^{\zeta'} = r^{\zeta'}_{j^{\zeta'}+1}$. By Claim 2.11$(iii), (iv)$ we have $r^{\zeta'}|\pi^{\zeta'} = r^{\zeta'}$ and $r^{\zeta'}$ is a witness for $\pi^{\zeta'}$ being $j^{\zeta'}$-properly situated on q. Now let u_ζ be the amalgamation of $u_{\zeta'}$ and $r^{\zeta'}$ defined as follows:

(a) $\mathrm{supp}(u_\zeta) = \mathrm{supp}(r^{\zeta'})$.

(b) If $\xi \in F$, then $u_\zeta(\xi)$ is such that

$$r^{\zeta'} \restriction \xi \Vdash u_\zeta(\xi) = (u_{\zeta'}(\xi) \setminus u_{\zeta'}(\xi)_{\sigma^{\zeta'}(\xi)}) \bigcup r^{\zeta'}(\xi),$$

and for any condition $c \leq u_\zeta \restriction \xi$ incompatible with $r^{\zeta'} \restriction \xi$, $c \Vdash_\xi u_\zeta(\xi) = u_{\zeta'}(\xi)$.

(c) if $\xi \notin F$, then $u_\zeta(\xi)$ is such that $r^{\zeta'} \restriction \xi \Vdash u_\zeta(\xi) = r^{\zeta'}(\xi)$, and for any condition $c \leq u_\zeta \restriction \xi$ incompatible with $r^{\zeta'} \restriction \xi$, $c \Vdash_\xi u_\zeta(\xi) = u_{\zeta'}(\xi)$.

By the definition of u_ζ we have

$$u_\zeta|\sigma^{\zeta'} = r^{\zeta'} = r^{\zeta'}|\pi^{\zeta'} = (u_\zeta|\sigma^{\zeta'})|\pi^{\zeta'} = u_\zeta|\pi^{\zeta'}$$

and $u_\zeta \leq_{F,i} u_{\zeta'}$.

We claim that $v = \bigwedge_{\zeta<\eta} u_\zeta$ is as required. Indeed, let $\sigma : F \to \rho^{i+1}$ be such as in the formulation. Since $v \leq u$, $\sigma = \sigma^\zeta$ for some $\zeta \in \eta$ and the construction of $u_{\zeta+1}$ is nontrivial. From the above it follows that $v|\sigma \leq u_\zeta|\sigma^\zeta = u_\zeta|\pi^\zeta$, consequently π^ζ lies on v and $v|\sigma = v|\pi^\zeta \leq u_\zeta|\pi^\zeta = r^\zeta$. Now it is easy to see that $j = j^\zeta$ and $\pi = \pi^\zeta$ are as required. □

Claim 5.5. *Let ρ, S^0, S^1, and $\vec{A}$ be such as in Definition 5.3, and $p \in ST_{S^0,S^1,\vec{A}}$. Then for every sequence $\langle D_\alpha : \alpha \in \rho\rangle$ of open dense subsets of $ST_{S^0,S^1,\vec{A}}$ there exists a generalized fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \rho\rangle$ with $p_0 = p$ and such that, letting $q = \bigwedge_{\alpha\in\rho} p_\alpha$, for every limit $i \in \rho$ and $\sigma : F_i \to \rho^{i+1}$ which is i-properly situated on q, σ lies on q and $q|\sigma \in D_i$.*

Proof. Take $r_{\alpha,j} \in D_\alpha$ in the construction of a fusion sequence from the proof of Lemma 2.10 (the part before Claim 2.11) instead of demanding that $r_{\alpha,j}$ decides $\dot{z}$ as a ground model object. The resulting fusion sequence is easily seen to be as required. □

Let us come back to our main task, namely to extend j^* to an elementary embedding $j^{**} : V[G*g] \to M^*[h]$ for some $\mathbb{Q}_{j(\kappa)} := j^*(\dot{\mathbb{Q}}^G_\kappa) = ST^{M^*}_{j^*(S^0),j^*(S^1),j^*(\vec{N})}$-generic filter h over M^* so that j^{**} is definable in $V[G*g]$. By [6, Proposition 9.1] it is enough to find such a $\mathbb{Q}_{j(\kappa)}$-generic $h \in V[G*g]$ over M^* for which $j^*[g] \subset h$.

For every $\xi < \kappa^{++}$ we denote by $x(\xi) \in \kappa^\kappa \cap V[G*g]$ the (unique!) branch through all trees in $g(\xi)$ and let $a_\xi = \kappa$ (resp. $a_\xi = 0$) for all $\xi \in S^0$

(resp. $\xi \in S^1$). We claim that

$$h = \{j^*(p)|\sigma_I \ : \ p \in g, I \in M^*, I \subset j[\kappa^{++}], |I| = \kappa\},$$

where $\sigma_I(j(\xi)) = x(\xi)^\frown a_\xi$ for all $j(\xi) \in I$, is $\mathbb{Q}_{j(\kappa)}$-generic over M^*. The proof below is a generalization of the "tuning fork" argument invented in [11]. Let $\bar{D} \in M^*$ be an open dense subset of $\mathbb{Q}_{j(\kappa)}$. Write $\bar{D}$ as $j^*(f)(\bar{a})$, where f has domain $H(\kappa)^V$, $f \in V[G]$, and $\bar{a} \in H(\kappa^{++})^V$. There is no loss of generality to assume that $f(a)$ is open dense in $\mathbb{Q}_\kappa := \dot{\mathbb{Q}}_\kappa^G$ for all $a \in H(\kappa)^V$. Let us enumerate $H(\kappa)^V$ as $\langle a_k : k \in \kappa\rangle$ and set $D_k = \bigcap_{k' \leq k} f(a_{k'})$.

Let $p \in \mathbb{Q}_\kappa$ be arbitrary. Claim 5.5 yields a generalized fusion sequence $\langle (p_k, F_k) : k \in \kappa\rangle$ such that $p_0 = p$ and, letting $q = \bigwedge_{k\in\kappa} p_k$, for every limit $k \in \kappa$ and σ which is k-properly situated on q, σ lies on q and $q|\sigma \in D_k$.

Let $\langle \bar{F}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ and $\langle \bar{p}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ be the results of applying j^* to $\langle F_k : k \in \kappa\rangle$ and $\langle p_k : k \in \kappa\rangle$ respectively. By elementarity of j^*, $\langle (\bar{p}_{\bar{k}}, \bar{F}_{\bar{k}}) : \bar{k} \in j(\kappa)\rangle$ is a generalized fusion sequence for $\mathbb{Q}_{j(\kappa)}$, $\bar{q} := j^*(q) = \bigwedge_{\bar{k}<j(\kappa)} \bar{p}_{\bar{k}}$, and there exists $\bar{\beta} \in j(\kappa)$ so that for each limit $\bar{\alpha} \geq \bar{\beta}$ and $\bar{\sigma}$ which is $\bar{\alpha}$-properly situated on $\bar{q}$, $\bar{\sigma}$ lies on $\bar{q}$ and $\bar{q}|\bar{\sigma} \in \bar{D}$. We can additionally assume that $\bar{\beta} > \kappa$.

Fix $u \leq q$ and a club $C \subset \kappa$ such that $j(C) \cap (\kappa, \bar{\beta}] = \emptyset$ (its existence is established, e.g., in the proof of [11, Lemma 4]). Using Claim 5.4, we can construct a fusion sequence $\langle (u_k, T_k) : k \in \kappa\rangle$ with $u_0 = u$ satisfying the following conditions:

(i) $F_k \subset T_k$;

(ii) For every $\sigma : T_k \to \kappa^{k+1}$ which lies on u_k and has the property $\sigma(\xi)(k) = k$ for all $\xi \in T_k \cap S^0$, there exist a limit ordinal $m \in C \setminus (k+1)$ and $\pi : T_{k+1} \cup F_m \to \kappa^{(m+1)}$ such that $\pi(\xi) \restriction (k+1) = \sigma(\xi)$ for all $\xi \in T_k$, π lies on u_{k+1}, $u_{k+1}|\sigma = u_{k+1}|\pi$, and $u_{k+1}|\pi$ is a witness for π being m-properly situated on q with respect to $\langle (p_k, F_k) : k \in \kappa\rangle$.

Let $\langle \bar{T}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ and $\langle \bar{u}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ be the results of applying j^* to $\langle T_k : k \in \kappa\rangle$ and $\langle u_k : k \in \kappa\rangle$ respectively, $v = \bigwedge_{k<\kappa} u_k$, and $\bar{v} = j^*(v) = \bigwedge_{\bar{k}<j(\kappa)} \bar{u}_{\bar{k}}$. By elementarity of j^*, for every $\bar{\sigma} : \bar{T}_\kappa \to j(\kappa)^{\kappa+1}$ which lies on $\bar{u}_\kappa$ and has the property $\bar{\sigma}(\bar{\xi})(\kappa) = \kappa$ for all $\bar{\xi} \in \bar{T}_\kappa \cap j(S^0)$, there exist a limit ordinal $\bar{m} \in j(C) \setminus (\kappa+1)$ and $\bar{\pi} : \bar{T}_{\kappa+1} \cup \bar{F}_{\bar{m}} \to j(\kappa)^{(\bar{m}+1)}$ such that $\bar{\pi}(\bar{\xi}) \restriction (\kappa+1) = \bar{\sigma}(\bar{\xi})$ for all $\bar{\xi} \in \bar{T}_\kappa$, $\bar{\pi}$ lies on $\bar{u}_{\kappa+1}$, $\bar{u}_{\kappa+1}|\bar{\sigma} = \bar{u}_{\kappa+1}|\bar{\pi}$, and $\bar{u}_{\kappa+1}|\bar{\sigma}$ is a witness for $\bar{\pi}$ being $\bar{m}$-properly situated on $\bar{q}$ with respect to $\langle (\bar{p}_{\bar{k}}, \bar{F}_{\bar{k}}) : \bar{k} \in j(\kappa)\rangle$.

Since p and $u \leq q$ were chosen arbitrarily, we can assume that $v \in g$. Observe that $\bar{T}_\kappa = \bigcup_{k\in\kappa} j[T_k] \subset j[\kappa^{++}]$, $|\bar{T}_\kappa| = \kappa$, and $\bar{T}_\kappa \in M^*$. The elementarity of j^* implies that $\bar{\sigma} := \sigma_{\bar{T}_\kappa}$ lies on $j^*(w)$ for any $w \in g$. In

particular, $\bar{\sigma}$ lies on $\bar{u}_k = j^*(u_k)$ for all $k \in \kappa$, and hence it lies on $\bar{u}_\kappa = \bigwedge_{k\in\kappa} \bar{u}_k$ as well. Therefore we can find $\bar{m} \in j(C) \setminus (\kappa+1)$ and $\bar{\pi} : \bar{T}_{\kappa+1} \cup \bar{F}_{\bar{m}} \to j(\kappa)^{(\bar{m}+1)}$ as above, i.e. $\bar{u}_{\kappa+1}|\bar{\sigma}$ is a witness for $\bar{\pi}$ being $\bar{m}$-properly situated on $\bar{q}$ with respect to $\langle (\bar{p}_{\bar{k}}, \bar{F}_{\bar{k}}) : \bar{k} \in j(\kappa)\rangle$. By the construction of $\langle (p_k, F_k) : k \in \kappa\rangle$, elementarity of j^*, the equalities $j(C) \cap (\kappa, \bar{\beta}) = \emptyset$ and $\bar{m} \in j(C) \setminus (\kappa+1)$, and our choice of $\bar{\beta}$, we conclude that $\bar{\pi} \restriction \bar{F}_{\bar{m}}$ lies on $\bar{q}$ and $\bar{q}|(\bar{\pi} \restriction \bar{F}_{\bar{m}}) \in \bar{D}$. On the other hand,

$$\bar{q}|(\bar{\pi} \restriction \bar{F}_{\bar{m}}) \geq \bar{u}_{\kappa+1}|\bar{\pi} = \bar{u}_{\kappa+1}|\bar{\sigma} \geq \bar{v}|\bar{\sigma} = j^*(v)|\sigma_{\bar{T}_\kappa} \in h,$$

which means that $h \cap \bar{D} \neq \emptyset$ and thus finishes the proof of Theorem 1.1. □

Remark 5.6. 1. To the best knowledge of the authors there are essentially three other different forcing extensions V^P of V which preserve the measurability of κ and kill the GCH at κ under the assumption that κ is $P_2\kappa$-hypermeasurable, see [6, § 24], [11], and [7, § 4]. In all three cases we have $\mathrm{cf}(Sym(\kappa)) = \kappa^+$ in V^P. The historically first of them is due to Woodin [6, § 24]. His P can be written as $P_0 * P_1 * P_2$, where P_0 is iteration of Cohen posets below κ with reverse Easton support, and thus $|P_0| = \kappa$ and P_0 has κ-c.c.; P_1 is the poset adding κ^{++}-many Cohen subsets of κ, and P_2 adds no new subsets of κ. It is clear that $V^{P_0 * P_1} \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^+$ (see the last paragraph in [22, p. 894]), and every forcing which does not add new subsets of κ cannot enlarge $\mathrm{cf}(Sym(\kappa))$.

In forcing extensions constructed in [11] and [7] the equality $\mathfrak{d}(\kappa) = \kappa^+$ holds, and it is well-known (the proof of [21, Proposition 1.4] works for every regular κ) that $\mathrm{cf}(Sym(\kappa)) \leq \mathfrak{d}(\kappa)$ for every regular κ.

2. It is known [22] that the equality $\mathrm{cf}(Sym(\kappa)) = \kappa^{++}$ (and much more) is consistent for every inaccessible κ. But the authors were not able to lift elementary embeddings to forcing extensions used in [22][5] assuming considerably less than supercompactness. However such a possibility is not formally excluded. On the other hand, applying the methods developed in [10] to forcing extensions from [22] we could obtain the following result:

Suppose $0^\sharp$ *exists. Then there is an inner model in which* $\mathrm{cf}(Sym(\kappa)) = \kappa^{++}$ *for every regular cardinal* κ *of the form* $\aleph_{2\alpha}$.

It is worth mentioning here that for every cardinal κ the inequality $\mathrm{cf}(Sym(\kappa)) > \kappa^+$ implies $\mathrm{cf}(Sym(\kappa^+)) \leq \mathrm{cf}(Sym(\kappa))$, and it is not known even how to obtain $\mathrm{cf}(Sym(\kappa)) > \kappa^+$ at two consecutive κ simultaneously, see [22].

[5]The forcing posets used in [22] were developed in [17, § 2,3]

3. In order to show that $j(\mathbb{P})_{\kappa+1} = \mathbb{P}$ in the proof of Theorem 1.1 we needed suitable stationary sets S^0, S^1 and $\langle S^0_k, S^1_k : k \in \kappa\rangle$. Instead of using the auxiliary forcing introducing such sets we could apply the same inner model argument as in the proof of [9, Theorem 11].

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