THE NIKODYM PROPERTY IN THE SACKS MODEL

DAMIAN SOBOTA AND LYUBOMYR ZDOMSKYY

Abstract. We prove that if $\mathcal{A}$ is a σ-complete Boolean algebra in a ground model V of set theory, then $\mathcal{A}$ has the Nikodym property in every side-by-side Sacks forcing extension $V[G]$, i.e. every pointwise bounded sequence of measures on $\mathcal{A}$ in $V[G]$ is uniformly bounded. This gives a consistent example of a class of infinite Boolean algebras with the Nikodym property and of cardinality strictly less than the continuum.

1. Introduction

Let $\mathcal{A}$ be a Boolean algebra. A sequence of measures $\langle\mu_n\colon n\in\omega\rangle$ on $\mathcal{A}$ is *pointwise bounded* if $\sup_{n\in\omega}|\mu_n(A)|<\infty$ for every $A\in\mathcal{A}$ and it is *uniformly bounded* if $\sup_{n\in\omega}\|\mu_n\|<\infty$. The Nikodym Boundedness Theorem states that if $\mathcal{A}$ is σ-complete, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded. This principle, due to its numerous applications, is one of the most important results in the theory of vector measures, see Diestel and Uhl [7, Section I.3].

Since σ-completeness is rather a strong property of Boolean algebras, Schachermayer [11] made a detailed study of the Nikodym theorem and introduced the Nikodym property for general Boolean algebras.

Definition 1.1. A Boolean algebra $\mathcal{A}$ has the *Nikodym property* if every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

The property has been studied by many authors, e.g. Darst [5], Seever [12], Haydon [9], Moltó [10], Freniche [8], Aizpuru [1, 2] or Valdivia [14].

Let us pose the following question. Let V be a model of ZFC+CH and $\mathcal{A}\in V$ be a σ-complete Boolean algebra of cardinality equal to the continuum $\mathfrak{c}$. Let $\mathbb{P}$ be a notion of forcing preserving ω_1 and G its generic filter over V. Assume that in the extension $V[G]$ the CH does not hold. Then, $\mathcal{A}$ will have cardinality ω_1 in $V[G]$, and hence it will no longer be σ-complete. However, will $\mathcal{A}$ still have the Nikodym property?

Brech [4, Theorem 3.1] proved that if $\mathbb{P}$ is the side-by-side Sacks forcing $\mathbb{S}^\kappa$ for some regular cardinal number κ, then $\mathcal{A}$ will have the *Grothendieck property* in $V[G]$, i.e. every sequence of measures in $V[G]$ which is weak* convergent on $\mathcal{A}$ is also weakly convergent. The Nikodym and Grothendieck properties are closely related to each other, see e.g. Schachermayer [11]. Thus, motivated by Brech's result, we studied the preservation of the Nikodym property by the Sacks forcing $\mathbb{S}^\kappa$ and proved that if $\mathcal{A}$ is a σ-complete Boolean algebra in V, then $\mathcal{A}$ has the Nikodym property in the $\mathbb{S}^\kappa$-generic extension $V[G]$ (Theorem 3.3).

Our result has one important consequence. In Sobota [13], the first author studied the relation between the Nikodym property and cardinal characteristics of the continuum. In particular, a construction of a Boolean algebra with the

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Nikodym property and of cardinality equal to $\operatorname{cof}(\mathcal{N})$, the cofinality of the σ-ideal $\mathcal{N}$ of subsets of the real line with zero Lebesgue measure, was presented. Since the construction was rather intricate, the natural question about the consistent existence of a *simple* example of Boolean algebra with the Nikodym property and cardinality strictly smaller than $\mathfrak{c}$ was posed. This paper answers this question.

1.1. **Terminology and notation.** Throughout the paper $\mathcal{A}$ will always denote a Boolean algebra. The Stone space of $\mathcal{A}$ is denoted by $K_{\mathcal{A}}$. Recall that by the Stone duality theorem $\mathcal{A}$ is isomorphic with the algebra of clopen subsets of $K_{\mathcal{A}}$; if $A \in \mathcal{A}$, then $[A]$ denotes the corresponding clopen subset of $K_{\mathcal{A}}$.

A subset X of a Boolean algebra $\mathcal{A}$ is an *antichain* if $x \wedge y = \mathbf{0}_{\mathcal{A}}$ for every distinct $x, y \in X$, i.e. every two distinct elements of X are *disjoint*. On the other hand, a subset X of a poset $\mathbb{P}$ is an *antichain* if no distinct $x, y \in X$ are compatible.

A *measure* $\mu\colon \mathcal{A} \to \mathbb{C}$ on $\mathcal{A}$ is always a finitely additive complex-valued function with finite variation. The measure μ has a unique Borel extension (denoted also by μ) onto the space $K_{\mathcal{A}}$, preserving the variation of μ. By the Riesz representation theorem the dual space $C\big(K_{\mathcal{A}}\big)^*$ of the Banach space of continuous complex-valued functions on $K_{\mathcal{A}}$ is isometrically isomorphic with the space of all measures on $\mathcal{A}$. For more information concerning measure theory and Banach spaces, see the book of Diestel [6].

V always denotes the set-theoretic universum. By $\mathbb{S}^\kappa$ we denote the side-by-side product of κ many Sacks forcings $\mathbb{S}$ for some uncountable regular cardinal number κ. Regarding all other notions related to the Sacks forcing, we follow the paper of Baumgartner [3]. If $s \in \mathbb{S}$ and $p \in s$, then $s|p = \big\{q \in s\colon\ q \subseteq p \text{ or } p \subseteq q\big\} \in \mathbb{S}$. If $n \in \omega$, then $l(n, s)$ denotes the *n-th forking level* of s.

Let $s, s' \in \mathbb{S}^\kappa, F \in [\operatorname{dom}(s)]^{<\omega}$ and $n \in \omega$. We put $l(F, n, s) = \big\{\sigma\colon\ \operatorname{dom}(\sigma) = F \ \&\ \forall \alpha \in F\colon\ \sigma(\alpha) \in l(n, s(\alpha))\big\}$. Note that $|l(F, n, s)| = 2^{n|F|}$. We write $s' \le_{F,n} s$ if $s' \le s$ and $l(F, n, s') = l(F, n, s)$. If $\sigma\colon F \to 2^{<\omega}$ is such that $\sigma(\alpha) \in s(\alpha)$ for every $\alpha \in F$, then we write $s|\sigma$ for a condition defined as $(s|\sigma)(\alpha) = s(\alpha)$ for $\alpha \in \operatorname{dom}(s) \setminus F$ and $(s|\sigma)(\alpha) = s(\alpha)|\sigma(\alpha)$.

2. Anti-Nikodym sequences in the Sacks model

In this section, assuming in a forcing extension the existence of sequences of measures on a ground model Boolean algebra $\mathcal{A}$ which are pointwise bounded but not uniformly bounded, we build (Proposition 2.9) in the ground model a special antichain in $\mathcal{A}$ which will be crucial in proving the main theorem of the paper — Theorem 3.3.

Definition 2.1. A sequence $\big\langle\mu_n\colon\ n \in \omega\big\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is called *anti-Nikodym* if it is pointwise bounded but not uniformly bounded.

Lemma 2.2. *If a sequence $\big\langle\mu_n\colon\ n \in \omega\big\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is anti-Nikodym, then there exists a point $t \in K_{\mathcal{A}}$ such that for every clopen neighborhood $U \in \mathcal{A}$ of t we have $\sup_{n\in\omega}\big\|\mu_n \restriction U\big\| = \infty$.*

The point t will be called *a Nikodym concentration point of the sequence* $\big\langle\mu_n\colon\ n \in \omega\big\rangle$.

Proof. Assume that for every point $t \in K_{\mathcal{A}}$ there exists $A_t \in \mathcal{A}$ such that $t \in [A_t]$ and $\big\langle\mu_n \restriction A_t\colon\ n \in \omega\big\rangle$ is uniformly bounded. Then, by compactness of $K_{\mathcal{A}}$ there exist $t_1, \ldots, t_n \in K_{\mathcal{A}}$ such that $A_{t_1} \vee \ldots \vee A_{t_m} = \mathbf{1}_{\mathcal{A}}$. This in turn implies that

$$\sup_{n\in\omega}\big\|\mu_n\big\| = \sup_{n\in\omega}\big|\mu_n\big|\big(\mathbf{1}_{\mathcal{A}}\big) \le \sup_{n\in\omega}\big|\mu_n\big|\big(A_{t_1}\big) + \ldots + \sup_{n\in\omega}\big|\mu_n\big|\big(A_{t_m}\big) =$$
$$\sup_{n\in\omega}\big\|\mu_n \restriction A_{t_1}\big\| + \ldots + \sup_{n\in\omega}\big\|\mu_n \restriction A_{t_m}\big\| < \infty,$$

which is a contradiction, since $\langle\mu_n\colon\ n\in\omega\rangle$ is not uniformly bounded. □

(Note that in the above proof we did not use the pointwise boundedness of $\langle\mu_n\colon\ n\in\omega\rangle$.)

Lemma 2.3. *Let $\langle\mu_n\colon\ n\in\omega\rangle$ be an anti-Nikodym sequence on $\mathcal{A}$ and let $t\in K_{\mathcal{A}}$ be its Nikodym concentration point. Assume that $t\in[A]$ for some $A\in\mathcal{A}$. Then, for every positive real number ρ and natural number M there exist an element $B\in\mathcal{A}$ and a natural number $n>M$ such that:*

- *$B\le A$ and $t\in[A\setminus B]$,*
- *$\big|\mu_n(B)\big|>\rho$.*

Proof. Since $\langle\mu_n\colon\ n\in\omega\rangle$ is anti-Nikodym and $t\in[A]$, there exist $C\le A$ and $n>M$ such that

$$\big|\mu_n(C)\big|>\sup_{m\in\omega}\big|\mu_m(A)\big|+\rho$$

and hence

$$\big|\mu_n(A\setminus C)\big|=\big|\mu_n(C)-\mu_n(A)\big|\ge\big|\mu_n(C)\big|-\big|\mu_n(A)\big|>\rho.$$

If $t\in[C]$, then put $B=A\setminus C$, otherwise put $B=C$. □

To the end of this section <u>let $\mathcal{A}$ be</u> a ground model infinite Boolean algebra.

Lemma 2.4. *Let $A_0,\dots,A_k\in\mathcal{A}$, K, M, $N\in\omega$. Let $\langle\dot\mu_n\colon\ n\in\omega\rangle$ be a sequence of names for measures on $\mathcal{A}$ and $\dot t$ a name for a point in $K_{\mathcal{A}}$. Let $s\in\mathbb{S}^\kappa$ force that $\langle\dot\mu_n\colon\ n\in\omega\rangle$ is anti-Nikodym, $\dot t$ is its Nikodym concentration point and $\dot t\notin\bigcup_{j=0}^k\big[\check A_j\big]$.*

Then, there exist a sequence $B_1,\dots,B_K$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\mathbf{1}_{\mathcal{A}}\setminus\bigvee_{j=0}^k A_j$, a sequence $n_K>\dots>n_1>M$ of natural numbers and a condition $s^*\le s$ forcing for every $1\le i\le K$ that $\dot t\notin\big[\check B_i\big]$ and

$$\big|\dot\mu_{n_i}\big(\check B_i\big)\big|>\sum_{j=0}^k\big|\dot\mu_{n_i}\big(\check A_j\big)\big|+\check N+2.$$

Proof. Use Lemma 2.3 inductively K times to obtain sequences $B_1,\dots,B_K\in\mathcal{A}$, $n_K>\dots>n_1>M$ and $s_K\le\dots\le s_1\le s$ such that for every $1\le i\le K$ the element B_i is disjoint with $\bigvee_{j=0}^k A_j\vee\bigvee_{l=1}^{i-1}B_l$ and the condition s_i forces that $\dot t\notin\big[\check B_i\big]$ and

$$\big|\dot\mu_{n_i}\big(\check B_i\big)\big|>\sum_{j=0}^k\big|\dot\mu_{n_i}\big(\check A_j\big)\big|+\check N+2.$$

Let $s^*=s_K$. □

Lemma 2.5. *Let $K,P\in\omega$. Let $\mu_1,\dots,\mu_K$ be a sequence of K measures on $\mathcal{A}$. Assume that $K\cdot\|\mu_j\|<P$ for every $1\le j\le K$. Then, for every $Q>K\cdot P$ and every pairwise disjoint elements $C_1,\dots,C_Q$ of $\mathcal{A}$ there exist natural numbers $k_1<\dots<k_{Q-K\cdot P}$ such that*

$$\big|\mu_j\big|\big(C_{k_l}\big)<1/K$$

for every $1\le j\le K$ and $1\le l\le Q-K\cdot P$.

Proof. Let $Q>K\cdot P$ and $C_1,\dots,C_Q$ be an antichain in $\mathcal{A}$. Assume that there exist $k_1<\dots<k_P$ such that

$$\big|\mu_j\big|\big(C_{k_l}\big)\ge1/K$$

for some $1 \leq j \leq K$ and every $1 \leq l \leq P$. Then, we have:

$$\|\mu_j\| \geq \sum_{l=1}^{P} \big|\mu_j\big|\big(C_{k_l}\big) \geq P \cdot 1/K > K \cdot \|\mu_j\| \cdot 1/K = \|\mu_j\|,$$

a contradiction, so for every $1 \leq j \leq K$ there must exist at most $P-1$ elements B_l's such that

$$\big|\mu_j\big|\big(C_{k_l}\big) \geq 1/K.$$

Hence, the thesis of the lemma holds for some $Q - K\cdot(P-1) \geq Q - K\cdot P$ elements B_l's. □

The following lemma is standard, cf. Baumgartner [3, Lemmas 1.5–1.8].

Lemma 2.6. *Let $s \in \mathbb{S}^\kappa$, $N \in \omega$ and $F_N \in [\mathrm{dom}(s)]^{<\omega}$.*

a) $\big\{s|\sigma\colon\ \sigma \in l(F_N, N, s)\big\}$ is an antichain in $\mathbb{S}^\kappa$ and $s = \bigcup_{\sigma\in l(F_N,N,s)} s|\sigma$.
b) If $\sigma \in l(F_N, N, s)$ and $p \leq s|\sigma$, then there exists $q \leq_{F,N} s$ such that $q|\sigma = p$.
c) If $D \subseteq \mathbb{S}^\kappa$ is open dense below s, then there exists $q \leq_{F_N,N} s$ such that $q|\sigma \in D$ for every $\sigma \in l(F_N, N, s)$.

□

Lemma 2.7. *Let $A_0, \ldots, A_k$, M, N, $\langle\dot\mu_n\colon\ n \in \omega\rangle$, $\dot t$ and s be as in the assumptions of Lemma 2.4. Let $F_N \in [\mathrm{dom}(s)]^{<\omega}$. Put $K = |l(F_N, N, s)|$ and enumerate $l(F_N, N, s) = \langle\sigma_i\colon\ 1 \leq i \leq K\rangle$.*

Then, there exist a condition $s^* \leq_{F_N,N} s$, a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\mathbf{1}_\mathcal{A} \setminus \bigvee_{j=0}^k A_j$ and a sequence $n_K > \ldots > n_1 > M$ such that for every $1 \leq i \leq K$ the condition $s^*|\sigma_i$ forces that:

- $\big|\dot\mu_{n_i}\big(\check B_i\big)\big| > \sum_{j=0}^k \big|\dot\mu_{n_i}\big(\check A_j\big)\big| + \sum_{j=1}^{i-1}\big|\dot\mu_{n_i}\big(\check B_j\big)\big| + \check N + 2,$
- $\big|\dot\mu_{n_i}\big|\Big(\bigvee_{j=i+1}^K \check B_j\Big) < 1,$
- $\dot t \notin \bigcup_{i=1}^K \big[\check B_i\big]$.

Proof. The proof basically goes by induction in K steps — each step for one σ_i $(1 \leq i \leq K)$. We start simply as follows — by Lemmas 2.4 and 2.6.b) there exist a condition $s_1 \leq_{F_N,N} s$, a family $\mathscr{B}_1^1 = \big\{B_1^1, \ldots, B_K^1\big\}$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\mathbf{1}_\mathcal{A} \setminus \bigvee_{j=0}^k A_j$, a sequence $n_K^1 > \ldots > n_1^1 > M$ of natural numbers and a natural number $P_1 > 0$ such that for every $1 \leq j \leq K$ we have:

$$s_1|\sigma_1 \Vdash \big|\dot\mu_{n_j^1}\big(\check B_j^1\big)\big| > \sum_{l=0}^{k} \big|\dot\mu_{n_j^1}\big(\check A_l\big)\big| + \check N + 2,$$

$$s_1|\sigma_1 \Vdash \check K \cdot \big\|\dot\mu_{n_j^1}\big\| < \check P_1, \text{ and}$$

$$s_1|\sigma_1 \Vdash \dot t \notin \bigcup_{B\in\check{\mathscr{B}}_1^1} [B]\,.$$

Assume now that for some $1 \leq L < K$ we have found:

- a sequence of conditions $s_L \leq_{F_N,N} \ldots \leq_{F_N,N} s_1 \leq_{F_N,N} s$,
- for every $1 \leq i \leq L$ a sequence of families $\mathscr{B}_L^i \subseteq \ldots \subseteq \mathscr{B}_i^i \subseteq \mathscr{B}^i \subseteq \mathcal{A}$ of pairwise disjoint non-zero elements of $\mathcal{A}$ with $\mathscr{B}_L^i \neq \emptyset$ and $\mathscr{B}^i = \big\{B_1^i, \ldots, B_K^i\big\}$,
- a sequence of natural numbers $n_K^L > \ldots > n_1^L > n_K^{L-1} > \ldots > n_1^{L-1} > \ldots > n_K^1 > \ldots > n_1^1 > M$, and
- a sequence of natural numbers $P_L > \ldots > P_1 > 0$,

such that:

(i) for every $1 \leq i \leq L$ and $1 \leq j \leq K$ we have:

(1) $$s_i|\sigma_i \Vdash \left|\dot{\mu}_{n^i_j}\big(\check{B}^i_j\big)\right| > \sum_{l=0}^{k} \left|\dot{\mu}_{n^i_j}\big(\check{A}_l\big)\right| + \sum_{l=1}^{i-1} \sum_{B\in\check{\mathscr{B}}^l_i} \left|\dot{\mu}_{n^i_j}(B)\right| + \check{N} + 2, \text{ and}$$

(2) $$s_i|\sigma_i \Vdash \check{K} \cdot \left\|\dot{\mu}_{n^i_j}\right\| < \check{P}_i;$$

(ii) for every $1 \leq j \leq i \leq L$ we have:

(3) $$s_i|\sigma_j \Vdash \dot{t} \notin \bigcup_{l=1}^{i} \bigcup_{B\in\check{\mathscr{B}}^l_i} [B]\,;$$

(iii) for every $1 \leq l < i \leq L$, $1 \leq j \leq K$ and $B \in \mathscr{B}^i$ we have:

(4) $$s_i|\sigma_l \Vdash \left|\dot{\mu}_{n^l_j}\right|(\check{B}) < 1/\check{K}.$$

Let us now construct $s_{L+1} \leq_{F_N,N} s_L$, $\mathscr{B}^1_{L+1} \subseteq \mathscr{B}^1_L, \ldots, \mathscr{B}^L_{L+1} \subseteq \mathscr{B}^L_L, \mathscr{B}^{L+1}_{L+1} \subseteq \mathscr{B}^{L+1} \subseteq \mathcal{A}$, $n^{L+1}_K > \ldots > n^{L+1}_1 > n^L_K$ and $P_{L+1} > P_L$ satisfying also the properties (i)–(iii).

First, we modify a bit the condition s_L. By density, there exists $p \leq s_L|\sigma_{L+1}$ such that for every $1 \leq i \leq L$ either there exists unique $1 \leq j_i \leq K$ such that $p \Vdash \dot{t} \in \big[\check{B}^i_{j_i}\big]$, or for every $B \in \mathscr{B}^i_L$ we have $p \Vdash \dot{t} \notin \big[\check{B}\big]$. In the former case put $\mathscr{B}^i_{L+1} = \mathscr{B}^i_L \setminus \big\{B^i_{j_i}\big\}$, in the latter — $\mathscr{B}^i_{L+1} = \mathscr{B}^i_L$. By Lemma 2.6.b), there exists $q \leq_{F_N,N} s_L$ such that $q|\sigma_{L+1} = p$. Note that

(5) $$q|\sigma_{L+1} \Vdash \dot{t} \notin \bigcup_{j=0}^{k} [\check{A}_j] \cup \bigcup_{l=1}^{L} \bigcup_{B\in\check{\mathscr{B}}^l_{L+1}} [B]\,.$$

By Lemmas 2.4 and 2.6.b), there exist a condition $r \leq_{F_N,N} q$, a family $\mathscr{C} = \big\{C_1, \ldots, C_Q\big\}$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\mathbf{1}_{\mathcal{A}} \setminus \Big(\bigvee_{j=1}^k A_j \vee \bigvee_{l=1}^L \bigvee \mathscr{B}^l_{L+1}\Big)$, where $Q = K \cdot L \cdot P_L + K$, a sequence $m_Q > \ldots > m_1 > n^L_K$ of natural numbers and a natural number $P_{L+1} > P_L$ such that for every $1 \leq j \leq Q$ we have:

(6) $$r|\sigma_{L+1} \Vdash \left|\dot{\mu}_{m_j}\big(\check{C}_j\big)\right| > \sum_{l=0}^{k} \left|\dot{\mu}_{m_j}\big(\check{A}_l\big)\right| + \sum_{l=1}^{L} \sum_{B\in\check{\mathscr{B}}^l_{L+1}} \left|\dot{\mu}_{m_j}(B)\right| + \check{N} + 2,$$

$$r|\sigma_{L+1} \Vdash \check{K} \cdot \left\|\dot{\mu}_{m_j}\right\| < \check{P}_{L+1}, \text{ and}$$

(7) $$r|\sigma_{L+1} \Vdash \dot{t} \notin \bigcup_{j=1}^{Q} [\check{C}_j]\,.$$

We now define s_{L+1} out of r in two steps. In the first step, by induction, the inequality (2) and Lemmas 2.5 and 2.6.b), we get a sequence $\mathscr{C}_L \subseteq \ldots \subseteq \mathscr{C}_1 \subseteq \mathscr{C}$ with $\left|\mathscr{C}_L\right| = K$, a sequence $k_K > \ldots > k_1$ of natural numbers and a sequence of conditions $p_L \leq_{F_N,N} \ldots \leq_{F_N,N} p_1 \leq_{F_N,N} r$ such that $\mathscr{C}_L = \big\{C_{k_1}, \ldots, C_{k_K}\big\}$ and for every $1 \leq i \leq L$, $1 \leq j \leq K$ and $C \in \mathscr{C}_i$ we have:

(8) $$p_i|\sigma_i \Vdash \left|\dot{\mu}_{n^i_j}\right|(\check{C}) < 1/\check{K}.$$

For every $1 \leq j \leq K$ write $B^{L+1}_j = C_{k_j}$ and $n^{L+1}_j = m_{k_j}$, and put $\mathscr{B}^{L+1} = \big\{B^{L+1}_1, \ldots, B^{L+1}_K\big\}$.

In the second step, by induction and again Lemma 2.6.b), we get a sequence $t_L \leq_{F_N,N} \ldots \leq_{F_N,N} t_1 \leq_{F_N,N} p_L$ such that for every $1 \leq i \leq L$ either there exists $1 \le j_i \le K$ such that $t_i|\sigma_i \Vdash \dot{t} \in \left[\check{B}^{L+1}_{j_i}\right]$, or for every $1 \le j \le K$ we have $t_i|\sigma_i \Vdash \dot{t} \notin \left[\check{B}^{L+1}_{j}\right]$. Put:

$$\mathscr{B}^{L+1}_{L+1} = \mathscr{B} \setminus \left\{B^{L+1}_{j_i}\colon\ t_i|\sigma_i \Vdash \dot{t} \in \left[\check{B}^{L+1}_{j_i}\right], 1 \le i \le L\right\} \tag{9}$$

and

$$s_{L+1} = t_L.$$

Note that by (7) and (9), for every $1 \le i \le L+1$ we have:

$$s_{L+1}|\sigma_i \Vdash \dot{t} \notin \bigcup_{B \in \check{\mathscr{B}}^{L+1}_{L+1}} [B]\,. \tag{10}$$

After the K-th step of the induction has been finished, we are left with the non-empty collections $\mathscr{B}^1_K, \ldots, \mathscr{B}^K_K$ (some of them may be singletons), the sequence $n^K_K > n^K_{K-1} > \ldots > n^1_2 > n^1_1 > M$ and the conditions $s_K \le_{F_N,N} \ldots \le_{F_N,N} s_1 \le_{F_N,N} s$. From each $\mathscr{B}^i_K$ pick one element $B^i_{l_i}$. Then, for every $1 \le i \le K$ by (1) and (6) we have:

$$s_K|\sigma_i \Vdash \left|\dot{\mu}_{n^i_{l_i}}\left(\check{B}^i_{l_i}\right)\right| > \sum_{j=0}^{k} \left|\dot{\mu}_{n^i_{l_i}}\left(\check{A}_j\right)\right| + \sum_{j=1}^{i-1} \left|\dot{\mu}_{n^i_{l_i}}\left(\check{B}^j_{l_j}\right)\right| + \check{N} + 2,$$

and by (4) and (8):

$$s_K|\sigma_i \Vdash \left|\dot{\mu}_{n^i_{l_i}}\right|\Big(\bigvee_{j=i+1}^{K} \check{B}^j_{l_j}\Big) = \sum_{j=i+1}^{K} \left|\dot{\mu}_{n^i_{l_i}}\right|\left(\check{B}^i_{l_i}\right) < \check{K} \cdot 1/\check{K} = 1,$$

and finally by (3), (5) and (10):

$$s_K|\sigma_i \Vdash \dot{t} \notin \bigcup_{j=1}^{K} \left[\check{B}^j_{l_j}\right].$$

Put:

$$s^* = s_K$$

and for every $1 \le i \le K$:

$$B_i = B^i_{l_i} \quad \text{and} \quad n_i = n^i_{l_i}.$$

□

By Lemma 2.6.a) we immediately obtain the following corollary.

Corollary 2.8. *Let $A_0, \ldots, A_k$, K, M, N, $\langle\dot{\mu}_n\colon\ n \in \omega\rangle$, $\dot{t}$, s and F_N be as in the assumptions of Lemma 2.7.*

Then, there exist a condition $s^* \le_{F_N,N} s$, a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\mathbf{1}_{\mathcal{A}} \setminus \bigvee_{j=0}^{k} A_j$ and a sequence $n_K > \ldots > n_1 > M$ such that s^* forces that $\dot{t} \notin \bigcup_{i=1}^{K} \left[\check{B}_i\right]$ and that there exists $1 \le i \le K$ for which it holds:

$$\left|\dot{\mu}_{n_i}\left(\check{B}_i\right)\right| > \sum_{j=0}^{k} \left|\dot{\mu}_{n_i}\left(\check{A}_j\right)\right| + \sum_{j=1}^{i-1} \left|\dot{\mu}_{n_i}\left(\check{B}_i\right)\right| + \check{N} + 2$$

and

$$\left|\dot{\mu}_{n_i}\right|\Big(\bigvee_{j=i+1}^{K} \check{B}_j\Big) < 1.$$

□

Proposition 2.9. *Let $\langle \dot\mu_n\colon\ n\in\omega\rangle$ be a sequence of names for measures on $\mathcal{A}$. Let $s\in\mathbb{S}^\kappa$ force that $\langle \dot\mu_n\colon\ n\in\omega\rangle$ is anti-Nikodym.*

Then, there exists:

- *an increasing sequence $\langle K_N\colon\ N\in\omega\rangle$ of natural numbers,*
- *a sequence $\langle B_i^N\colon\ 1\le i\le K_N,\ N\in\omega\rangle$ of pairwise disjoint elements of $\mathcal{A}$,*
- *a sequence $\langle n_i^N\colon\ 1\le 1\le K_N,\ N\in\omega\rangle$ in ω such that $n_1^N > n_{K_M}^M > \ldots > n_1^M$ for every $N>M$, and*
- *a condition $s^*\le s$ forcing for every $N\in\omega$ that there exist $1\le i\le K_N$ such that:*

$$\left|\dot\mu_{n_i^N}\left(\check B_i^N\right)\right| > \sum_{M=0}^{N-1}\sum_{j=1}^{K_M}\left|\dot\mu_{n_i^N}\left(\check B_j^M\right)\right| + \sum_{j=1}^{i-1}\left|\dot\mu_{n_i^N}\left(\check B_j^N\right)\right| + \check N + 2$$

and

$$\left|\dot\mu_{n_i^N}\right|\Big(\bigvee_{j=i+1}^{K_N}\check B_j^N\Big) < 1.$$

Proof. The conclusion follows by the inductive use of Corollary 2.8 (to obtain an appropriate fusion sequence $\langle s_N\colon\ N\in\omega\rangle$ of conditions in $\mathbb{S}^\kappa$) and the ultimate use of the fusion lemma (to obtain a fusion condition $s^*\in\mathbb{S}^\kappa$ such that $s^*\le_{F_N,N} s_N$ for every $N\in\omega$; see Baumgartner [3, Lemma 1.8]). □

3. Main result

Throughout this section $\mathcal{A}$ is a ground model σ-complete Boolean algebra, i.e. $\mathcal{A}\in V$ and $\mathcal{A}$ is σ-complete in V.

Lemma 3.1. *Let $X\in[\omega]^\omega$ and $X=\bigcup_{k\in\omega}X_k$ be an infinite partition of X into infinite subsets. For every measure μ on $\mathcal{A}$ and an antichain $\langle B_N\colon\ N\in\omega\rangle$ in $\mathcal{A}$ there exists $L\in\omega$ such that*

$$|\mu|\Big(\bigvee_{N\in X_k}B_N\Big)<1$$

for every $k>L$.

Proof. Since μ is finitely additive and bounded, we have:

$$\sum_{k\in\omega}|\mu|\Big(\bigvee_{N\in X_k}B_N\Big)\le|\mu|\Big(\bigvee_{N\in\omega}B_N\Big)\le|\mu|(\mathbf{1}_{\mathcal{A}})<\infty.$$

□

Lemma 3.2. *Let $\langle B_N\colon\ N\in\omega\rangle\in V$ be an antichain in $\mathcal{A}$ and $X\in[\omega]^\omega\cap V$. Let $s\in\mathbb{S}^\kappa$ be a condition, $N\in\omega$, $F_N\subseteq[\mathrm{dom}(s)]^{<\omega}$ and $\dot\mu_1,\ldots,\dot\mu_K$ names for measures on $\mathcal{A}$. Assume that s forces that $\dot\mu_1,\ldots,\dot\mu_K$ are measures. Then, there exists a condition $s^*\le_{F_N,N}s$ and a set $X'\in[X]^\omega\cap V$ such that for every $1\le i\le K$ we have:*

$$s^*\Vdash|\dot\mu_i|\Big(\bigvee_{M\in\check X'}\check B_M\Big)<1.$$

Proof. Let $X=\bigcup_{k\in\omega}X_k$ be an infinite partition of X into infinite sets. By Lemma 3.1 the following set is open dense below s:

$$D=\Big\{p\le s\colon\ \forall\ 1\le i\le K\ \exists\ L\in\omega\ \forall\ k>L\colon\ p\Vdash|\dot\mu_i|\Big(\bigvee_{M\in\check X_k}\check B_M\Big)<1\Big\}.$$

By Lemma 2.6.c) there exists $s^* \leq_{F_N,N} s$ such that $s^*|\sigma \in D$ for every $\sigma \in l(F_N,N,s)$. Hence, for every $\sigma \in l(F_N,N,s)$ there exists $L_\sigma \in \omega$ such that for every $k > L_\sigma$ the condition $s^*|\sigma$ forces that:

$$\left|\dot{\mu}_i\right|\Big(\bigvee_{M\in\check{X}_k}\check{B}_M\Big) < 1.$$

Let $L = \max\big(L_\sigma\colon\ \sigma \in l(F_N,N,s)\big)+1$. Put $X' = X_L$ and appeal to Lemma 2.6.a). □

We are now in the position to prove the main theorem of this paper.

Theorem 3.3. *Let G be an $\mathbb{S}^\kappa$-generic filter over V. Then, in $V[G]$ the Boolean algebra $\mathcal{A}$ has the Nikodym property.*

Proof. Working in $V[G]$ assume that $\mathcal{A}$ does not have the Nikodym property. Then, there exists an anti-Nikodym sequence $\big\langle\mu_n\colon\ n\in\omega\big\rangle$ of measures on $\mathcal{A}$. Let $t\in K_{\mathcal{A}}$ be its Nikodym concentration point.

Now and to the end of the proof, let us work in the ground model V. Let $\big\langle\dot{\mu}_n\colon\ n\in\omega\big\rangle$ be a sequence of names for measures in the sequence $\big\langle\mu_n\colon\ n\in\omega\big\rangle$ and $\dot{t}$ a name for t. There exists a condition $s\in G$ forcing that $\big\langle\dot{\mu}_n\colon\ n\in\omega\big\rangle$ is anti-Nikodym on $\check{\mathcal{A}}$ and $\dot{t}$ is its Nikodym concentration point.

Let $\big\langle K_N\colon\ N\in\omega\big\rangle$, $\big\langle B_i^N\colon\ 1\leq i\leq K_N,\ N\in\omega\big\rangle$, $\big\langle n_i^N\colon\ 1\leq i\leq K_N,\ N\in\omega\big\rangle$ and $s^*\leq s$ be given by Proposition 2.9. We will find a condition $s^{**}\leq s^*$ and a set $Y\in[\omega]^\omega\cap V$ such that s^{**} forces that

$$\dot{B} = \bigvee_{N\in Y}\bigvee_{i=1}^{K_N}\check{B}_i^N \in \check{\mathcal{A}}$$

and

$$\sup_{n\in\omega}\big|\dot{\mu}_n(\dot{B})\big| = \infty,$$

which will contradict the fact that s forces that $\big\langle\dot{\mu}_n\colon\ n\in\omega\big\rangle$ is pointwise bounded.

To obtain s^{**} and Y we follow by induction and use Lemma 3.2 to construct a fusion sequence $\big\langle s_N\colon\ N\in\omega\big\rangle$ of conditions such that $s_0 = s^*$ and for every $N\in\omega$ we have $s_{N+1}\leq_{F_N,N} s_N$, where $F_N = \big\{\alpha_i^k\colon\ i,k<N\big\}$ and $\operatorname{dom}(s_N) = \big\{\alpha_k^N\colon\ k\in\omega\big\}$, and a decreasing sequence $\big\langle X_N\colon\ N\in\omega\big\rangle$ of infinite subsets of ω such that:

- $X_0=\omega$ and for every $N\in\omega$ we have $\min X_N < \min X_{N+1}$, and
- for every $N\in\omega$ and $L=\min X_N$ the condition s_N forces that:

  $$\big|\dot{\mu}_{n_i^L}\big|\Big(\bigvee_{M\in\check{X}_{N+1}}\bigvee_{j=1}^{K_M}\check{B}_j^M\Big) < 1$$

  for every $1\leq i\leq K_L$.

Let $s^{**}\in\mathbb{S}^\kappa$ be such a condition that $s^{**}\leq_{F_N,N} s_N$ for every $N\in\omega$ (see Baumgartner [3, Lemma 1.8]). Put:

$$Y = \big\{\min X_N\colon\ N\in\omega\big\}$$

and

$$B = \bigvee_{N\in Y}\bigvee_{i=1}^{K_N} B_i^N.$$

Then, $B \in \mathcal{A}$ and, since $\langle X_N\colon\ N \in \omega\rangle$ is decreasing, s^{**} forces that for every $N \in Y$ and $1 \le i \le K_N$ the following inequality holds:

$$\left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{\substack{M\in Y\\ M>N}}\bigvee_{j=1}^{K_M}\check{B}_j^M\Big) < 1.$$

Finally, since $s^{**} \le s^*$, s^{**} forces for every $N \in Y$ that there exists $1 \le i \le K_N$ such that

$$\left|\dot{\mu}_{n_i^N}\big(\check{B}_i^N\big)\right| > \sum_{\substack{M\in Y\\ M<N}}\sum_{j=1}^{K_M}\left|\dot{\mu}_{n_i^N}\big(\check{B}_j^M\big)\right| + \sum_{j=1}^{i-1}\left|\dot{\mu}_{n_i^N}\big(\check{B}_j^N\big)\right| + \check{N} + 2$$

and

$$\left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{j=i+1}^{K_N}\check{B}_j^N\Big) < 1,$$

and hence:

$$\begin{aligned}
\left|\dot{\mu}_{n_i^N}(\check{B})\right| &= \Big|\dot{\mu}_{n_i^N}\Big(\bigvee_{\substack{M\in Y\\ M<N}}\bigvee_{j=1}^{K_M}\check{B}_j^M\Big) + \dot{\mu}_{n_i^N}\Big(\bigvee_{j=1}^{i-1}\check{B}_j^N\Big) + \dot{\mu}_{n_i^N}\big(\check{B}_i^N\big) + \\
&\quad + \dot{\mu}_{n_i^N}\Big(\bigvee_{j=i+1}^{K_N}\check{B}_j^N\Big) + \dot{\mu}_{n_i^N}\Big(\bigvee_{\substack{M\in Y\\ M>N}}\bigvee_{j=1}^{K_M}\check{B}_j^M\Big)\Big| \ge \\
&\ge \left|\dot{\mu}_{n_i^N}\big(\check{B}_i^N\big)\right| - \sum_{\substack{M\in Y\\ M<N}}\sum_{j=1}^{K_M}\left|\dot{\mu}_{n_i^N}\big(\check{B}_j^M\big)\right| - \sum_{j=1}^{i-1}\left|\dot{\mu}_{n_i^N}\big(\check{B}_j^N\big)\right| - \\
&\quad - \left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{j=i+1}^{K_N}\check{B}_j^N\Big) - \left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{\substack{M\in Y\\ M>N}}\bigvee_{j=1}^{K_M}\check{B}_j^M\Big) \ge \\
&\ge \check{N} + 2 - 1 - 1 = \check{N}.
\end{aligned}$$

Thus, s^{**} forces that for every $N \in \omega$ there exists n such that $\left|\dot{\mu}_n(\check{B})\right| > N$ and hence s^{**} forces that $\sup_{n\in\omega}\left|\dot{\mu}_n(\check{B})\right| = \infty$. □

Since the forcing $\mathbb{S}^\kappa$ preserves ω_1 and $\kappa = \mathfrak{c}$ in any $\mathbb{S}^\kappa$-generic extension (see Baumgartner [3, Theorems 1.11 and 1.14]), we immediately obtain the following corollary.

Corollary 3.4. *Assume that V is a model of ZFC+CH. If G is an $\mathbb{S}^\kappa$-generic filter, then in $V[G]$ the relations $\omega_1 < \kappa = \mathfrak{c}$ hold and $\mathcal{A}$ is an example of a Boolean algebra with the Nikodym property and of cardinality ω_1.*

Schachermayer [11, Theorem 2.5] proved that if a Boolean algebra $\mathcal{A}$ has simultaneously the Nikodym property and the Grothendieck property, then $\mathcal{A}$ has the *Vitali–Hahn-Saks property*, i.e. every pointwise convergent sequence of measures on $\mathcal{A}$ is uniformly exhaustive. Thus, Theorem 3.3 and Brech's result [4, Theorem 3.1] imply together that if $\mathcal{A}$ is a σ-complete Boolean algebra in the ground model V, then it has the Vitali–Hahn–Saks property in the $\mathbb{S}^\kappa$-generic extension $V[G]$. In particular, as in Corollary 3.4, this yields a simple consistent example of a Boolean algebra with the Vitali–Hahn–Saks property and of cardinality strictly less than $\mathfrak{c}$.

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Kurt Gödel Research Center for Mathematical Logic, Universität Wien, Währinger Strasse 25, 1090 Wien, Austria

E-mail address: damian.sobota@univie.ac.at

URL: www.logic.univie.ac.at/~dsobota

E-mail address: lzdomsky@gmail.com

URL: www.logic.univie.ac.at/~lzdomsky