ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2020. Випуск 89. С. 5–10
Visnyk of the Lviv Univ. Series Mech. Math. 2020. Issue 89. P. 5–10
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2020.89.005-010

УДК 510.3

BRENDLE'S PROOF OF THE CONSISTENCY OF $\mathfrak{b} < \mathfrak{a}$, WITHOUT RANKS, GAMES, AND COHEN REALS

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We present a simplified version of the proof of one of the main results of [3].

Key words: Mad family, filter, Mathias forcing.

1. Introduction

Our goal is to give a proof of the following result. We remind the reader of the definitions of notions involved in it at the beginning of the next section.

Theorem 1 (Brendle 98)**.** (GCH) *Let κ be an uncountable regular cardinal. Then there exists a ccc poset $\mathbb{P}$ which forces $\mathfrak{b} = \kappa < \mathfrak{a} = \kappa^+ = \mathfrak{c}$.*

We will follow the same strategy as in [3], the main technical ingredient thereof being simplified. More precisely, $\mathbb{P} = \mathbb{P}_{\kappa^+}$ comes from a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+ \rangle$ of ccc posets. The poset $\mathbb{Q}_0$ forces $\mathfrak{b} = \kappa = 2^\omega$ (e.g., one can take as $\mathbb{Q}_0$ the poset adding κ-many Hechler reals over V). Fix a dominating family $B = \{b_\xi : \xi < \kappa\} \subset \omega^{\uparrow\omega} \cap V^{\mathbb{Q}_0}$ such that $b_\xi \leqslant^* b_\eta$ for all $\xi < \eta$. If α has cofinality $< \kappa$, then $\dot{\mathbb{Q}}_\alpha$ is a name for a *partial* Hechler forcing producing a $\leqslant^*$-bound for certain $X_\alpha \subset \omega^{\uparrow\omega} \cap V^{\mathbb{P}_\alpha}$ of size $|X_\alpha| < \kappa$, supplied by a bookkeeping function fixed in advance. The purpose of these $\dot{\mathbb{Q}}_\alpha$'s is to make sure that $\mathfrak{b} = \mathfrak{c} = \kappa$ holds in $V^{\mathbb{P}_\gamma}$ for any γ of cofinality κ. Moreover, since the partial Hechler posets $\dot{\mathbb{Q}}_\alpha$ have size $< \kappa$, they preserve the unboundedness of B (it is well-known and easy to check that no poset of size $< \kappa$ can force B to be bounded), provided that $\mathbb{P}_\alpha$ did so, and the latter will be arranged with the help of Propositions 2 and 1 below. At stage γ of cofinality κ our bookkeeping function gives us a ($\mathbb{P}_\gamma$-name for an) almost disjoint family $\mathcal{A}_\gamma$. The poset $\dot{\mathbb{Q}}_\gamma$ forces $\mathcal{A}_\gamma$ to be non-maximal and preserves the unboundedness of B.

2020 Mathematics Subject Classification: 03E17, 03E40, 03E05

In order to prove Theorem 1 it is enough to accomplish the natural scenario discussed above. Propositions 2 and 1 along with a standard bookkeeping allow us to do this.

Proposition 2 is analogous to [3, 3.1. Theorem]. However, unlike in the proof of the latter result, in our proof of Proposition 2 we use neither auxiliary Cohen reals, nor tricky arguments involving ranks, which hopefully makes our proof somewhat more straightforward.

The proof given in [3] has inspired yet another construction of a model of $\mathfrak{b} < \mathfrak{a}$, see [6]. Their proof is rather different from the one we present in this note: They use countably closed non-ccc iterands which "forces" them to use countable supports and hence gives $\mathfrak{c} = \omega_2$, as well as they use some variants of games on filters considered in [8, 9].

There have been more attempts to simplify or to modify Brendle's proof from [3], see, e.g., [4]. Also, O. Guzmán has informed us in private communication that he knows how to eliminate Cohen reals. Moreover, Guzmán and Kalajdzievski have recently proved in [7] the consistency of $\omega_1 = \mathfrak{u} < \mathfrak{a} = \omega_2$. This yields $\mathfrak{b} < \mathfrak{a}$ since $\mathfrak{b} \leqslant \mathfrak{u}$ in ZFC and their posets do not add Cohen reals as these destroy ground model basis of ultrafilters. Nonetheless we believe that our approach might be still of some interest.

We thank the anonymous referee for careful reading and making very helpful comments.

2. Proofs

As usually, $\omega = \{0, 1, 2, \ldots\}$ denotes the set of natural numbers and $\omega^{\uparrow\omega}$ stands for non-decreasing elements of ω^ω. A family $\mathcal{A} \subset [\omega]^\omega$ is called *almost disjoint* if $A_0 \cap A_1$ is finite for any distinct $A_0, A_1 \in \mathcal{A}$. An infinite almost disjoint family $\mathcal{A}$ is called a *mad family* if $\mathcal{A} \cup \{X\}$ fails to be almost disjoint for any $X \in [\omega]^\omega \setminus \mathcal{A}$. The minimal cardinality of a mad family is denoted by $\mathfrak{a}$.

For $x, y \in \omega^\omega$ notation $x \leqslant^* y$ means that the set $\{n \in \omega : x(n) > y(n)\}$ is finite. $\mathfrak{b}$ denotes the minimal cardinality of $B \subset \omega^\omega$ which is unbounded with respect to $\leqslant^*$. It is known that $\omega_1 \leqslant \mathfrak{b} \leqslant \mathfrak{a}$, see [2, 10] for the information about $\mathfrak{a}, \mathfrak{b}$, and other combinatorial cardinal characteristics of the reals.

In what follows D denotes an unbounded subset of $\omega^{\uparrow\omega}$ which is σ-directed, i.e., for every $D_0 \in [D]^\omega$ there exists $g \in D$ such that $d \leqslant^* g$ for all $d \in D_0$. For instance, the dominating set B of Hechler generic reals mentioned above is like this.

The following fact follows from [1, Lemma 6.5.7].

Proposition 1. *Let δ be a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$ be a finite support iteration of ccc posets such that $\Vdash_{\mathbb{P}_\alpha}$"$D$ is unbounded" for all $\alpha < \delta$. Then $\Vdash_{\mathbb{P}_\delta}$"$D$ is unbounded".*

A subset $\mathcal{F}$ of $[\omega]^\omega$ is called a *filter* if $\mathcal{F}$ contains all co-finite sets, is closed under finite intersections of its elements, and under taking supersets. Every filter $\mathcal{F}$ gives rise to a natural forcing notion $\mathbb{M}_\mathcal{F}$ introducing a generic subset $X \in [\omega]^\omega$ such that $X \subset^* F$ for all $F \in \mathcal{F}$ as follows: $\mathbb{M}_\mathcal{F}$ consists of pairs $\langle s, F\rangle$ such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max s < \min F$. A condition $\langle s, F\rangle$ is stronger than $\langle t, G\rangle$ if $F \subset G$, s is an end-extension of t, and $s \setminus t \subset G$. $\mathbb{M}_\mathcal{F}$ is usually called *Mathias forcing associated with $\mathcal{F}$.*

Every almost disjoint family $\mathcal{A}$ generates a filter

$$\mathcal{F}(\mathcal{A}) = \left\{F \subset \omega \colon \exists \mathcal{B} \in [\mathcal{A}]^{<\omega} \ \left(\omega \setminus \bigcup \mathcal{B} \subset^* F\right)\right\}.$$

It is clear that any forcing producing an infinite pseudointersection of $\mathcal{F}(\mathcal{A})$ (or any other bigger filter) ruins the maximality of $\mathcal{A}$.

The next proposition yields the poset used at stages of iteration with cofinality κ.

Proposition 2. *($\mathfrak{b} = \mathfrak{c} = \kappa$.) Let $\mathcal{A}$ be an almost disjoint family. Then there exists a filter $\mathcal{U} \supset \mathcal{F}(\mathcal{A})$ such that $\mathbb{M}_{\mathcal{U}}$ preserves D unbounded.*

We shall need several auxiliary results. First of all, we shall assume in the sequel that $\mathcal{F}(\mathcal{A})$ is not contained in any filter $\mathcal{U}$ which is a union of $< \kappa$ many compacts, as otherwise $\mathcal{U}$ is as required: Any union of $< \mathfrak{b}$ many compacts has all of its continuous images under maps into $\omega^{\uparrow\omega}$ bounded, and $\mathbb{M}_{\mathcal{U}}$ preserves all ground model unbounded sets for any filters like that, see [5, Theorem 1.4].

For $\mathcal{X} \subset [\omega]^\omega$ and $Z \subset \omega$ we denote by $\mathcal{X} \upharpoonright Z$ the family $\{X \cap Z : X \in \mathcal{X}\}$. Also, $\mathcal{X}^+$ standardly stands for $\{Y \subset \omega : \forall X \in \mathcal{X}\ (|X \cap Y| = \omega)\}$.

Lemma 1. *$\mathcal{A} \cap \mathcal{U}^+$ is infinite for every filter $\mathcal{U} \subset \mathcal{F}(\mathcal{A})^+$ which is a union of $< \kappa$ many compacts.*

Proof. Suppose on the contrary that $\mathcal{A}' = \mathcal{A} \cap \mathcal{U}^+$ is finite and set $F = \omega \setminus \cup\mathcal{A}' \in \mathcal{F}(\mathcal{A})$. Then $\mathcal{F}(\mathcal{A}) \upharpoonright F \subset \mathcal{U} \upharpoonright F$. Indeed, $\mathcal{F}(\mathcal{A}) \upharpoonright F$ is the filter on F generated by $\{(\omega \setminus A) \cap F : A \in \mathcal{A} \setminus \mathcal{A}'\}$ and $\omega \setminus A \in \mathcal{U}$ for every $A \in \mathcal{A} \setminus \mathcal{A}'$. Thus $\mathcal{F}(\mathcal{A})$ is contained in a filter on ω which is a union of $< \kappa$ many compacts (namely $\{X : \exists U \in \mathcal{U}(F \cap U \subset^* X)\}$), which contradicts our assumption on $\mathcal{A}$. □

In what follows the family of filters $\mathcal{U}$ on ω which are unions of $< \kappa$ many compacts will be denoted by $\mathcal{C}_\kappa$. Let us denote by $\mathcal{E}$ the family of all subsets E of $\textsc{fin} := [\omega]^{<\omega} \setminus \{\varnothing\}$ such that for every $n \in \omega$ there exists $e \in E$ with $\min e > n$. For any $A \subset \textsc{fin}$ we denote by $\mathcal{K}(A)$ the family $\{X \subset \omega : X \cap a \neq \varnothing$ for all $a \in A\}$. It is clear that $\mathcal{K}(A)$ is compact for all A as above, and $\mathcal{K}(E) \subset [\omega]^\omega$ if $E \in \mathcal{E}$. We shall call $E \in \mathcal{E}$ *centered* if so is $\mathcal{K}(E)$, where a family $\mathcal{X} \subset [\omega]^\omega$ is called centered if $\cap\mathcal{X}' \in [\omega]^\omega$ for all $\mathcal{X}' \in [\mathcal{X}]^{<\omega}$.

For a filter $\mathcal{F}$ on ω we denote by $\mathcal{F}^{<\omega}$ the filter on $\textsc{fin}$ generated by $\{\mathcal{P}(F) \cap \textsc{fin} : F \in \mathcal{F}\}$ as a base. Note that this notation is unusual since $\mathcal{F}^{<\omega}$ “should” denote the family of all finite sequences of elements of $\mathcal{F}$, which is not the object we have defined in the previous sentence. However, we shall use this notation since it is standard in the current literature.

Observation 1. *Let $E \in \mathcal{E}$. Then $X \in \mathcal{K}(E)^+$ iff for every $n \in \omega$ there exists $e \in E$, $\min e \geqslant n$, such that $e \subset X$.*

In particular, for a filter $\mathcal{F}$ on ω, $\{\uparrow e : e \in E\}$ covers $\mathcal{F}$ iff $\mathcal{F} \subset \mathcal{K}(E)^+$ iff $\mathcal{K}(E) \subset \mathcal{F}^+$ iff $E \in (\mathcal{F}^{<\omega})^+$. (Here $\uparrow X = \{Y \subset \omega : X \subset Y\}$ for any $X \subset \omega$.)

Proof. The “if” part is obvious. For the “only if” one, assume to the contrary that $X \not\supset e$ for any $e \in E$ with $\min e \geqslant n$. For every $e \in E$ select $n_e \in e$ as follows: if $e \cap n \neq \varnothing$, pick $n_e \in e \cap n$, and otherwise pick $n_e \in e \setminus X$. Then $Y = \{n_e : e \in E\} \in \mathcal{K}(E)$ and $Y \cap X \subset n$ thus contradicting our assumption that $X \in \mathcal{K}(E)^+$. □

Lemma 2. *Let $\mathcal{R} \in \mathcal{C}_\kappa$ be such that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is centered. Suppose that $\langle E_n : n \in \omega\rangle \in \mathcal{E}^\omega$ is a decreasing sequence such that $E_n \subset \mathcal{P}(\omega \setminus n)$ and $\mathcal{K}(E_n) \subset \langle \mathcal{F}(\mathcal{A}) \cup \mathcal{R}\rangle^+$ for all n. Then one of the following two options holds:*

(i) *There exists $n \in \omega$ and $X \in \mathcal{K}(E_n)$ such that $\{\omega \setminus X\} \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is centered. In particular, for any filter $\mathcal{U}$ containing the latter union, $E_n \notin (\mathcal{U}^{<\omega})^+$.*

(ii) *There exists $g \in D$ such that letting $H' = \bigcup_{n\in\omega} E_n \cap \mathcal{P}(g(n))$, we have that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H')$ is centered. In particular, for any filter $\mathcal{U}$ containing the latter union, $H' \in (\mathcal{U}^{<\omega})^+$, i.e., for every $U \in \mathcal{U}$ there exists $e \in H'$ such that $e \subset U$.*

Proof. Suppose that (i) fails. It follows that

$$\bigcup_{n\in\omega} \mathcal{K}(E_n) \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R} \text{ is centered.} \tag{1}$$

Indeed, otherwise there exists $n \in \omega$ such that $\mathcal{K}(E_n) \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is not centered because the sequence $\langle E_n : n \in \omega\rangle \in \mathcal{E}^\omega$ is decreasing. Since $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}$ is centered by our assumption, there are $F \in \langle \mathcal{F}(\mathcal{A}) \cup \mathcal{R}\rangle$ and $\{X_0, \dots, X_m\} \in \mathcal{K}(E_n)$ such that $\bigcap_{i\leqslant m} X_i \cap F = \varnothing$. Thus $F \subset \bigcup_{i\leqslant m}(\omega \setminus X_i)$, which implies $\omega \setminus X_i \in \langle \mathcal{F}(\mathcal{A}) \cup \mathcal{R}\rangle^+$ for some $i \leqslant m$, i.e., (i) takes place, which contradicts our assumption. Thus Equation (1) is true.

Applying now Lemma 1 and Observation 1 to $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{K}(E_n) \cup \mathcal{R}$, we can find mutually distinct $\{A_i : i \in \omega\} \subset \mathcal{A}$ such that for every $n, i \in \omega$, $X \in \mathcal{R}$, and $\vec{Y} = \langle Y_j : j < n\rangle \in \mathcal{K}(E_n)^n$ there exists $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Y_j \cap A_i$. Since $\mathcal{K}(E_n)$ is compact, there exists $k \in \omega$ such that for every $\vec{Y} \in \mathcal{K}(E_n)^n$ and $i \leqslant n$ there exists $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Y_j \cap A_i \cap k$. Let $k_{X,n}$ be the minimal k with this property.

Claim 1. *Let $X \in \mathcal{R}$ and $n \in \omega$. Then for every $\vec{Z} \in \mathcal{K}(E_n \cap \mathcal{P}(k_{X,n}))^n$ and $i \leqslant n$ there exists $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Z_j \cap A_i \cap k_{X,n}$.*

Proof. Suppose that the claim is wrong and pick $i < n$ and $\vec{Z} \in \mathcal{K}(E_n \cap \mathcal{P}(k_{X,n}))^n$ witnessing its failure. For every $e \in E_n \setminus \mathcal{P}(k_{X,n})$ select $n_e \in e \setminus k_{X,n}$ and set $Y_j = Z_j \cup \{n_e : e \in E_n \setminus \mathcal{P}(k_{X,n})\}$. It follows that $Y_j \in \mathcal{K}(E_n)$ for all $j < n$ and there is no $e \in E_n$ such that $e \subset X \cap \bigcap_{j<n} Y_j \cap A_i \cap k_{X,n}$, a contradiction to our choice of $k_{X,n}$. □

Observe that the map $\mathcal{R} \ni X \mapsto \langle k_{X,n} : n \in \omega\rangle$ is continuous, and consequently there exists $f \in \omega^\omega$ such that for all X and all but finitely many $n \in \omega$ we have $k_{X,n} < f(n)$, because $\mathcal{R} \in \mathcal{C}_\kappa$ and $\kappa = \mathfrak{b}$.

Claim 2. *$\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H_I)$ is centered for any $I \in [\omega]^\omega$, where $H_I = \bigcup_{n\in I} E_n \cap \mathcal{P}(f(n))$.*

Proof. Let us fix $\mathcal{A}' \in [\mathcal{A}]^{<\omega}$, $n \in \omega$, $I \in [\omega]^\omega$, $X \in \mathcal{R}$, and $\langle Y_j : j < n\rangle \in \mathcal{K}(H_I)^n$. It suffices to prove that $(\omega \setminus \cup\mathcal{A}') \cap X \cap \bigcap_{j<n} Y_j \setminus n \neq \varnothing$. Let us fix $i \in \omega$ such that $A_i \notin \mathcal{A}'$ and let $m \in I \setminus \max\{i, n\}$ be such that $A_i \cap (\cup\mathcal{A}') \subset m$, and $k_{X,m} < f(m)$. Note that all but finitely many $m \in I$ are as above. By Claim 1 there exists $e \in E_m$ such that

$$e \subset X \cap \bigcap_{j<n} Y_j \cap A_i \cap f(m),$$

and hence also $e \subset \omega \setminus \cup\mathcal{A}'$ because $\omega \setminus \cup\mathcal{A}' \supset A_i \setminus m$ by our choice of m and $\min e \geqslant m$ for all $e \in E_m$. □

Now let $g \in D$ be such that $[f < g] := \{n \in \omega : f(n) < g(n)\}$ is infinite. It suffices to note that H' defined in item (ii) of the formulation contains $H_{[f<g]}$ and hence $\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H')$ is centered because so is $\mathcal{F}(\mathcal{A}) \cup \mathcal{R} \cup \mathcal{K}(H_{[f<g]})$ by Claim 2. □

We shall also need the following result proved in [6] (it is Proposition 1 there, stated in a slightly different terminology) which allows us to work in the proof of Proposition 2 directly with a filter instead of working with the Mathias forcing associated to it.

Теорема 1 (Guzmán–Hrušák–Martínez, 2014). *Let $\mathcal{F}$ be a filter and $D \subset \omega^{\uparrow\omega}$ be unbounded and σ-directed. Then $\mathbb{M}_{\mathcal{F}}$ preserves the unboundedness of D iff for every decreasing sequence $\langle E_n : n \in \omega\rangle$ of elements of $(\mathcal{F}^{<\omega})^+$ there exists $g \in D$ such that $\bigcup_{n\in\omega}(E_n \cap \mathcal{P}(g(n))) \in (\mathcal{F}^{<\omega})^+$. Moreover, in this characterization we may assume that $E_n \subset \mathcal{P}(\omega \setminus n)$ for all $n \in \omega$.*

We are in a position now to present the

Proof of Proposition 2. Let $\{\langle E^\alpha_n : n \in \omega\rangle \ : \ \alpha \in \kappa\}$ be an enumeration of all decreasing sequences $\langle E_n : n \in \omega\rangle \in \mathcal{E}^\omega$ such that $E_n \subset \mathcal{P}(\omega \setminus n)$ for all n. Set $\mathcal{R}^0 = \{\omega\}$ and assume that for some $\alpha \in \kappa$ we have already constructed $\mathcal{R}^\alpha \in \mathcal{C}_\kappa$ such that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$ is centered. Now consider the sequence $\langle E^\alpha_n : n \in \omega\rangle$. Three cases are possible.

1. There exists $n_\alpha \in \omega$ such that $\mathcal{K}(E^\alpha_{n_\alpha}) \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$ is not centered. Given any ultrafilter $\mathcal{G}$ containing $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$, we can find $X_\alpha \in \mathcal{K}(E^\alpha_{n_\alpha})$ such that $X_\alpha \notin \mathcal{G}$, and therefore $\{\omega \setminus X_\alpha\} \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$ is centered being a subset of $\mathcal{G}$. Now we set $\mathcal{R}^{\alpha+1} = \langle \mathcal{R}^\alpha \cup \{\omega \setminus X_\alpha\}\rangle$.

2. For $\mathcal{R} := \mathcal{R}^\alpha$ and $\langle E_n : n \in \omega\rangle := \langle E^\alpha_n : n \in \omega\rangle$, item (i) from Lemma 2 takes place. This means that there exist $n_\alpha \in \omega$ and $X_\alpha \in \mathcal{K}(E^\alpha_{n_\alpha})$ such that $\{\omega \setminus X_\alpha\} \cup \mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha$ is centered. As in item 1 we set $\mathcal{R}^{\alpha+1} = \langle \mathcal{R}^\alpha \cup \{\omega \setminus X_\alpha\}\rangle$.

3. For $\mathcal{R} := \mathcal{R}^\alpha$ and $\langle E_n : n \in \omega\rangle := \langle E^\alpha_n : n \in \omega\rangle$, item (ii) from Lemma 2 takes place. Then there exists $g_\alpha \in D$ such that letting $H_\alpha = \bigcup_{n\in\omega} E^\alpha_n \cap \mathcal{P}(g_\alpha(n))$, we have that $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\alpha \cup \mathcal{K}(H_\alpha)$ is centered. In this case we set $\mathcal{R}^{\alpha+1} = \langle \mathcal{R}^\alpha \cup \mathcal{K}(H_\alpha)\rangle$.

This completes our inductive construction of the sequence $\langle \mathcal{R}^\alpha : \alpha < \kappa\rangle$. Set $\mathcal{R}^\kappa = \bigcup_{\alpha<\kappa} \mathcal{R}^\alpha$ and let $\mathcal{U}$ be the filter generated by $\mathcal{F}(\mathcal{A}) \cup \mathcal{R}^\kappa$. We claim that $\mathcal{U}$ is as required. Indeed, consider any $\langle E^\alpha_n : n \in \omega\rangle$. If in the construction of $\mathcal{R}^{\alpha+1}$ one of the first two alternatives took place, we know that $E^\alpha_{n_\alpha} \notin (\mathcal{U}^{<\omega})^+$ as witnessed by $\omega \setminus X_\alpha \in \mathcal{U}$. So let us assume that the third alternative took place. Then $H_\alpha = \bigcup_{n\in\omega} E^\alpha_n \cap \mathcal{P}(g_\alpha(n)) \in (\mathcal{U}^{<\omega})^+$ by the definition of $\mathcal{U}$. It remains to use Theorem 1. □

Acknowledgements

The author would like to thank the Austrian Science Fund FWF (Grants I 2374-N35 and I 3709-N35) for generous support for this research.

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Стаття: надійшла до редколегії 24.12.2020
доопрацьована 09.01.2021
прийнята до друку 23.03.2021

ДОВЕДЕННЯ БРЕНДЛА НЕСУПЕРЕЧНОСТІ $\mathfrak{b} < \mathfrak{a}$, ЯКЕ НЕ ВИКОРИСТОВУЄ РАНГІВ, ІГОР І ЧИСЕЛ КОЕНА

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Наведено спрощене доведення головного результату статті [3]. На відміну від оригінального доведення, ми не використовуємо рангів і допоміжних чисел Коена. Також не використовуються ігри на ідеалах, які фігурують у інших відомих автору спрощеннях доведення вищезгаданого результату Брендла.

Ключові слова: максимальна майже диз'юнктна сім'я, фільтр, форсінг Матіаса.