Random planar maps decomposed into blocks: a phase study

Journées Aléa 2022

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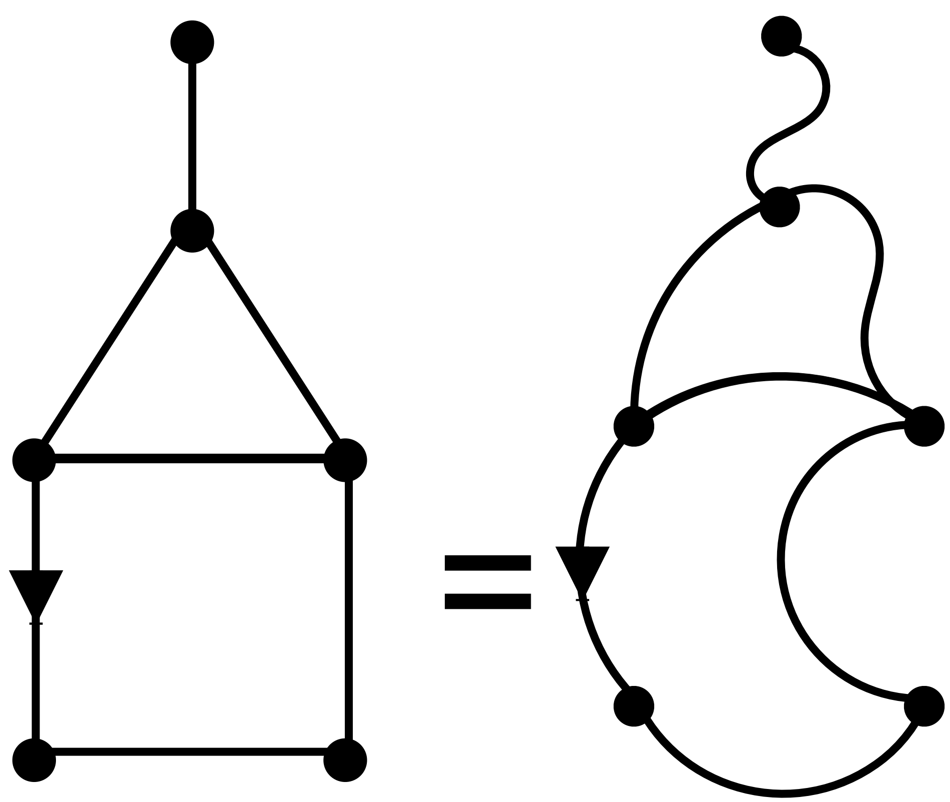
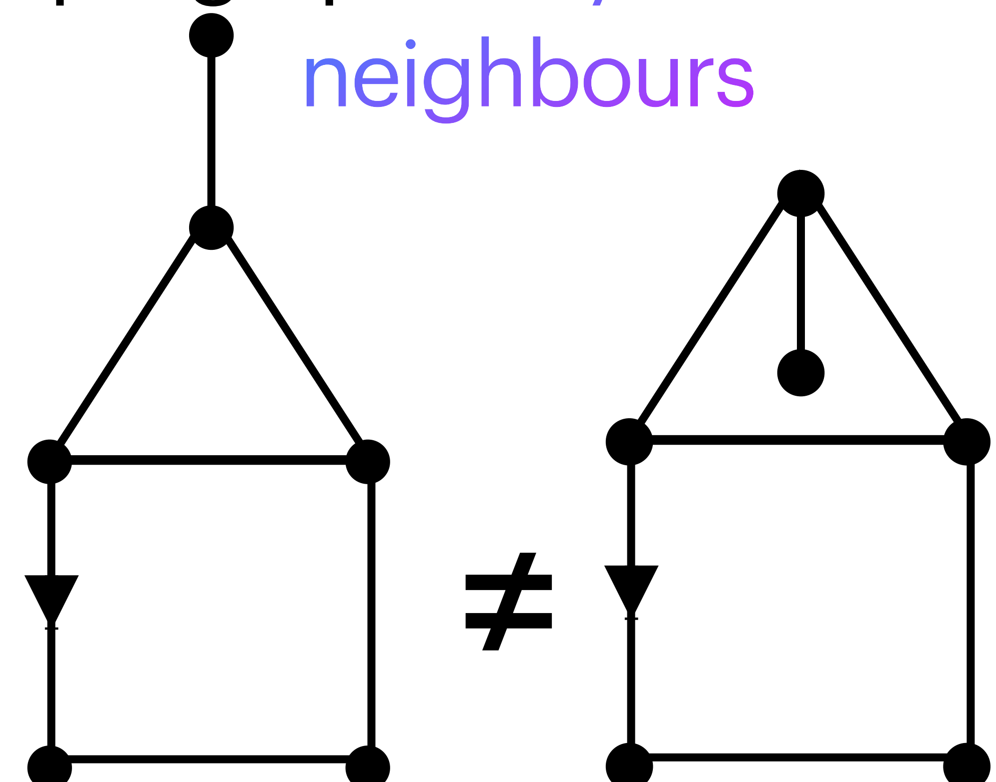
PhD under the supervision of Marie Albenque and Éric Fusy
Joint work with William Fleurat (ENS de Lyon)

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms

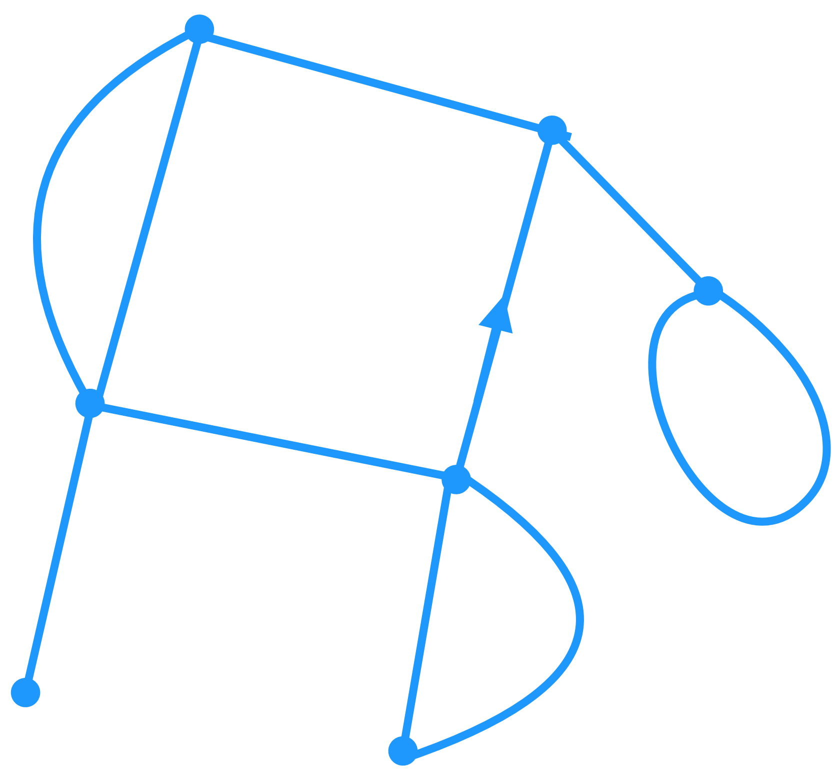


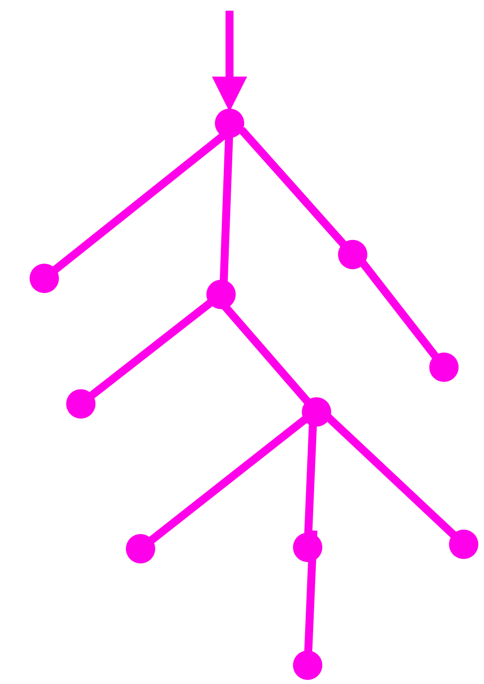
Map = graph + cyclic order on neighbours



- Rooted planar map = map endowed with a marked oriented edge;
- Size $|\mathfrak{m}|$ = number of edges;
- Corner (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Motivation



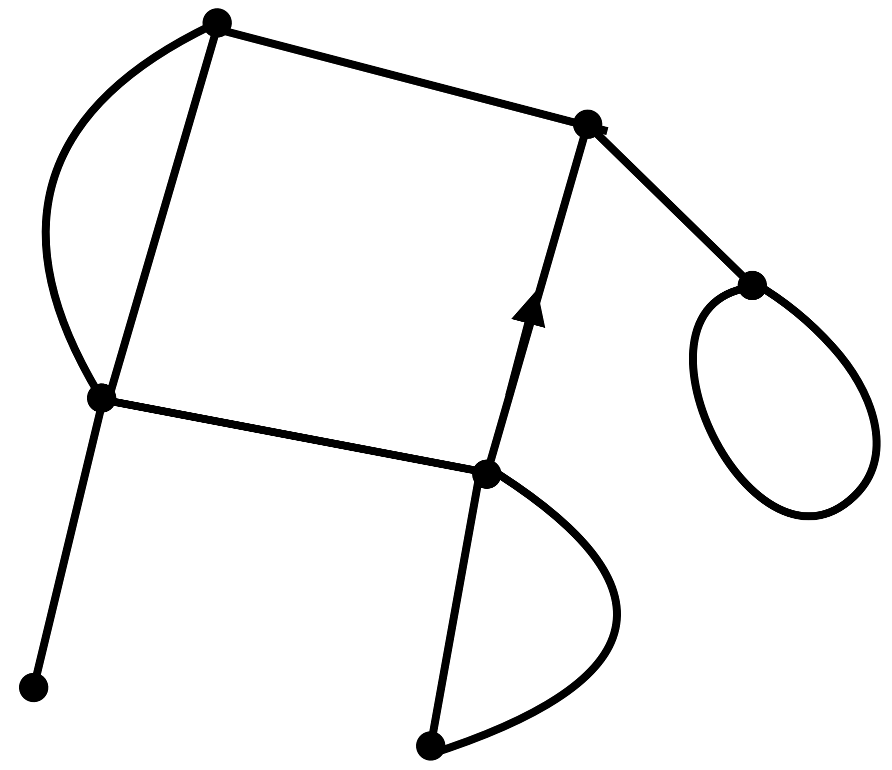


- Enumeration $\kappa\rho^{-n}n^{-5/2}$ (Tutte 1963);
- Distance between vertices $n^{1/4}$ (Chassaing, Schaeffer 2004);
- Enumeration $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices $\sqrt{n}$ (Flajolet, Odlyzko 1982);

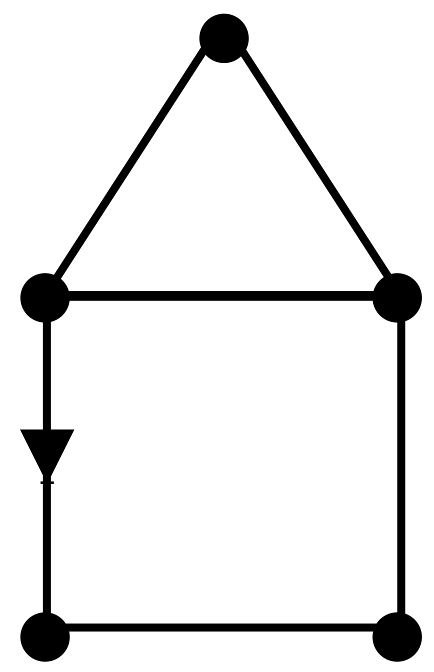
Interpolating model?

2-connectivity

Connected: to be able to disconnect, at least one vertex must be removed



2-connected: to be able to disconnect, at least two vertices must be removed



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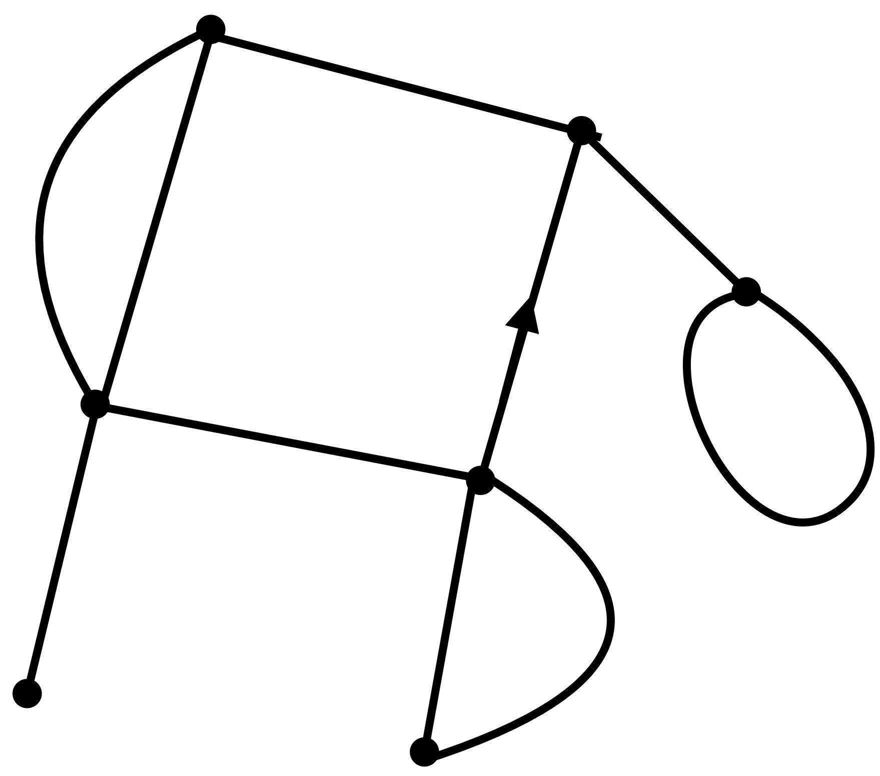
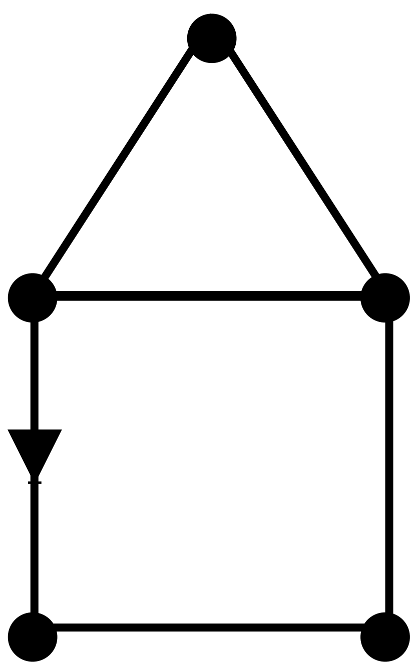
Block = maximal (for inclusion) 2-connected submap





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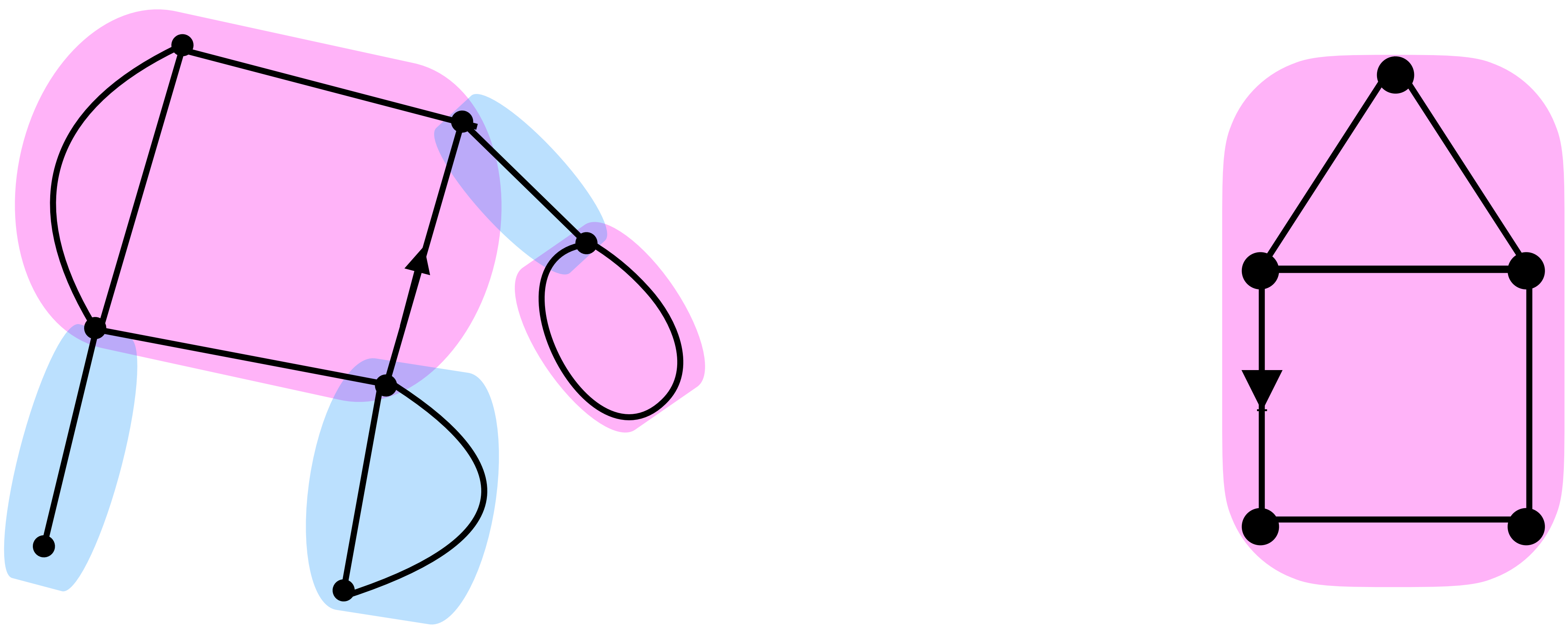

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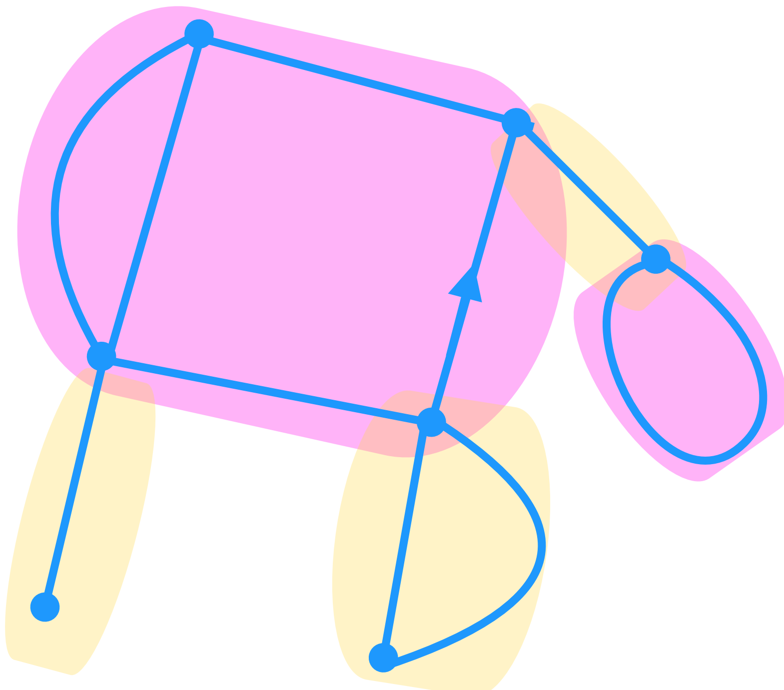
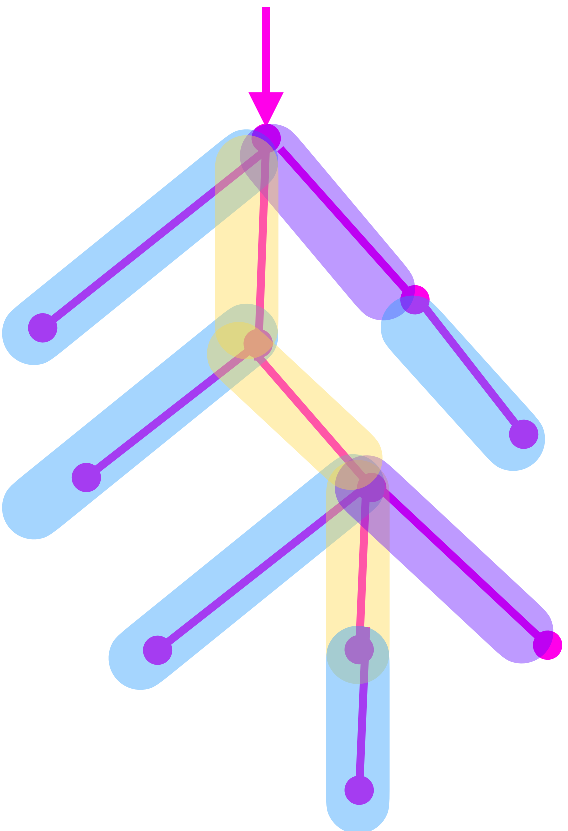




- Enumeration $\kappa\rho^{-n}n^{-5/2}$
  (Tutte 1963);
- Distance between vertices $n^{1/4}$
  (Chassaing, Schaeffer 2004);
- Condensation phenomenon: a large block concentrates a macroscopical part of the mass
  (Banderier, Flajolet, Schaeffer, Soria 2001).

- Enumeration $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices $n^{1/2}$
  (Flajolet, Odlyzko 1982);
- Only small blocks.

Interpolating model?

I. Approach

Model

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,

$\mathscr{M}_n$ = {maps of size n},

$\mathfrak{m} \in \mathscr{M}_n$,

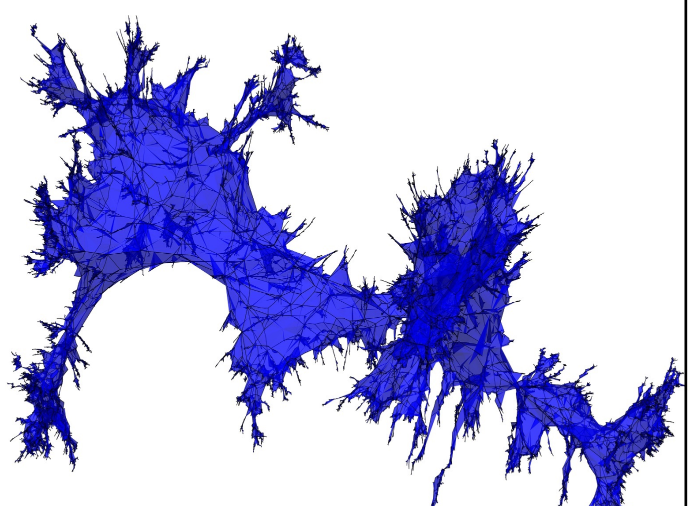
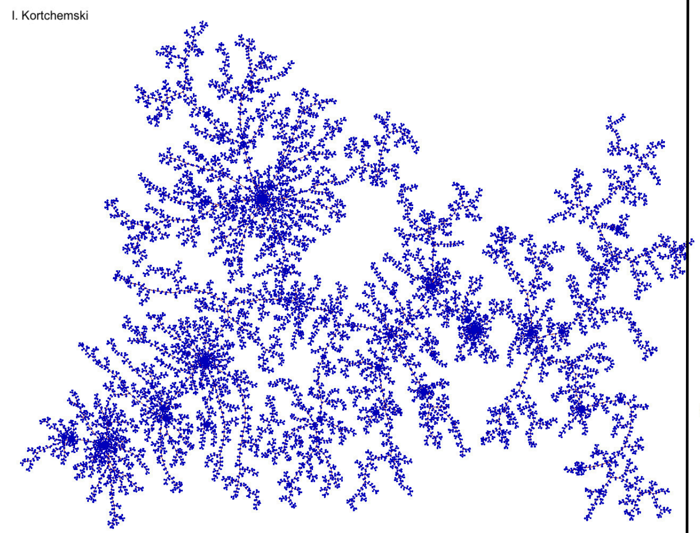
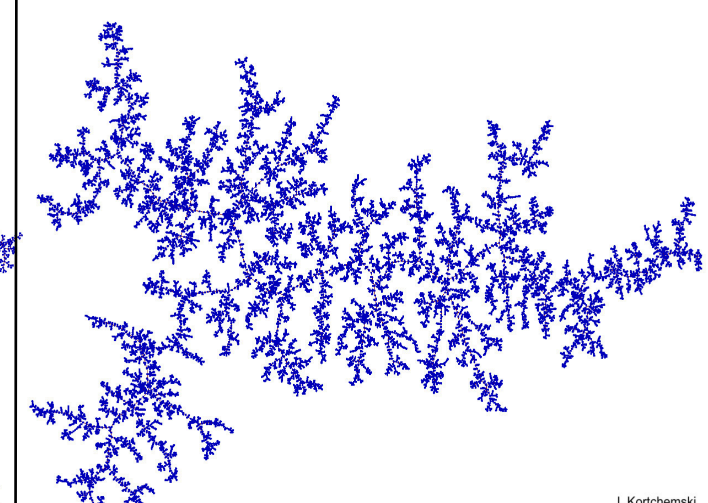
$Z_{n,u}$ = normalisation.

Inspired by [Bonzom 2016].

- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

Given u, asymptotic behaviour when $n \to \infty$?

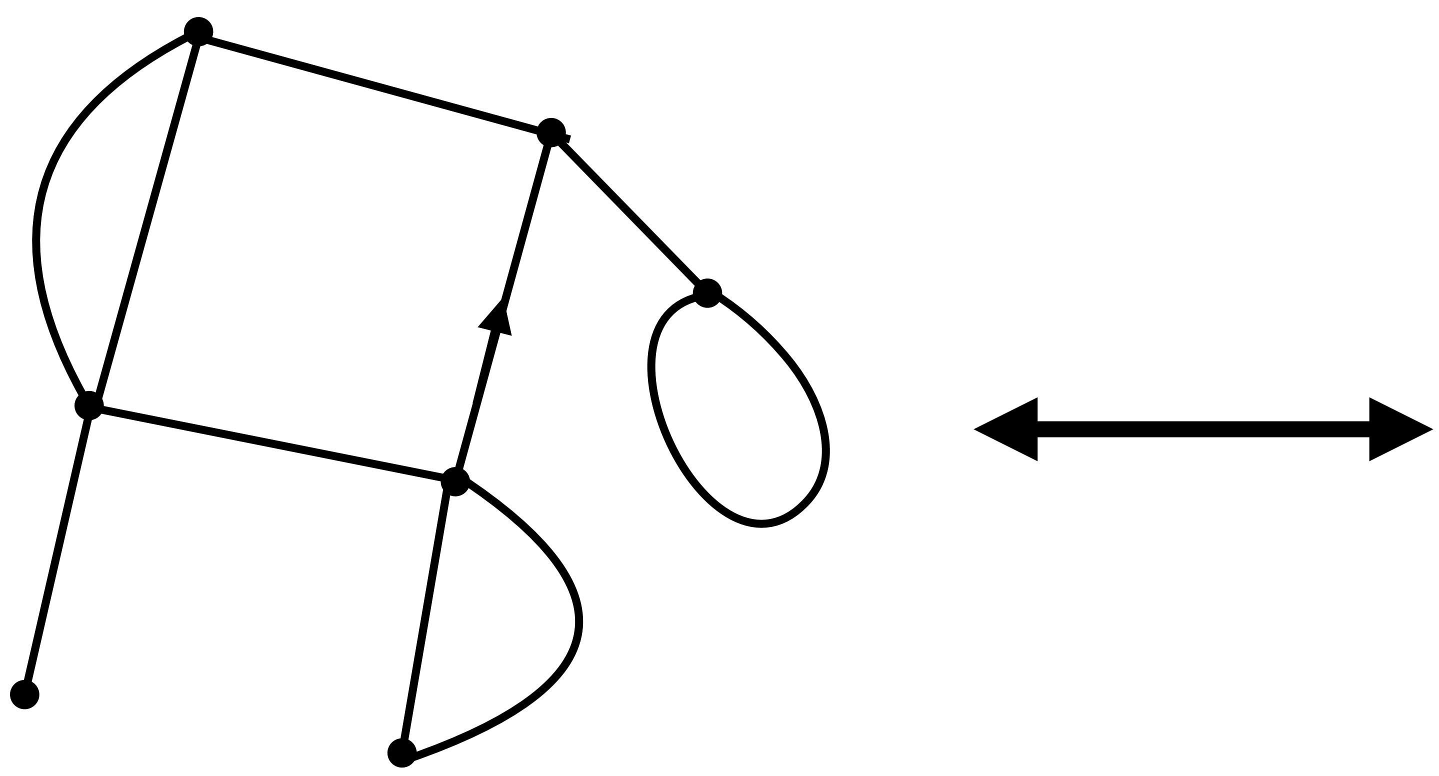
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration	$\rho(u)^n n^{-5/2}$	$\rho(u)^n n^{-5/3}$	$\rho(u)^n n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ Stufler 2020	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n (up to constant factors)	$\frac{1}{n^{1/4}} M_n \to \mathcal{S}_e$  J. Bettinelli Assuming the convergence of 2-connected maps towards the brownian sphere	$\frac{1}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ I. Kortchemski 	$\frac{1}{n^{1/2}} M_n \to \mathcal{T}_e$ Stufler 2020  I. Kortchemski

Decomposition of a map into blocks

Maps are 2-connected maps of maps (Tutte 1963) : $M(z) = B(zM^2(z))$

⇒ Underlying block tree structure, made explicit by Addario-Berry (2019).



With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

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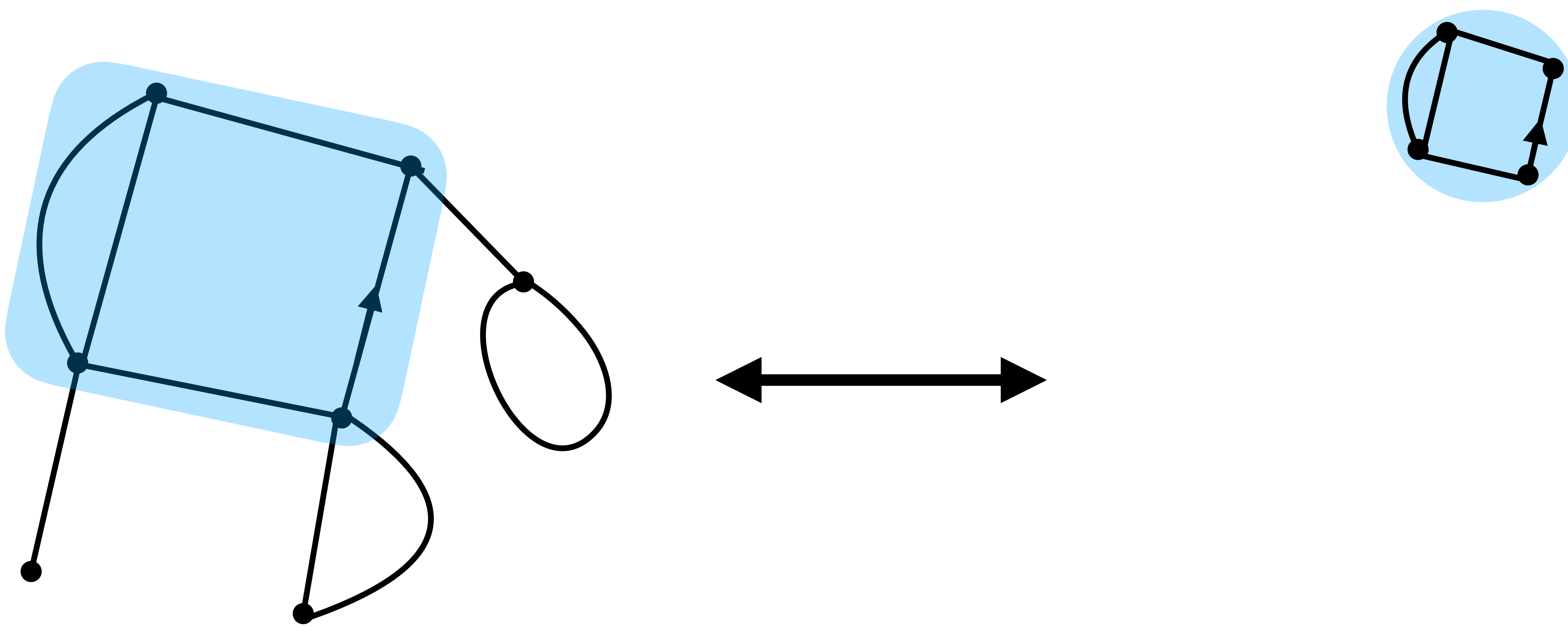

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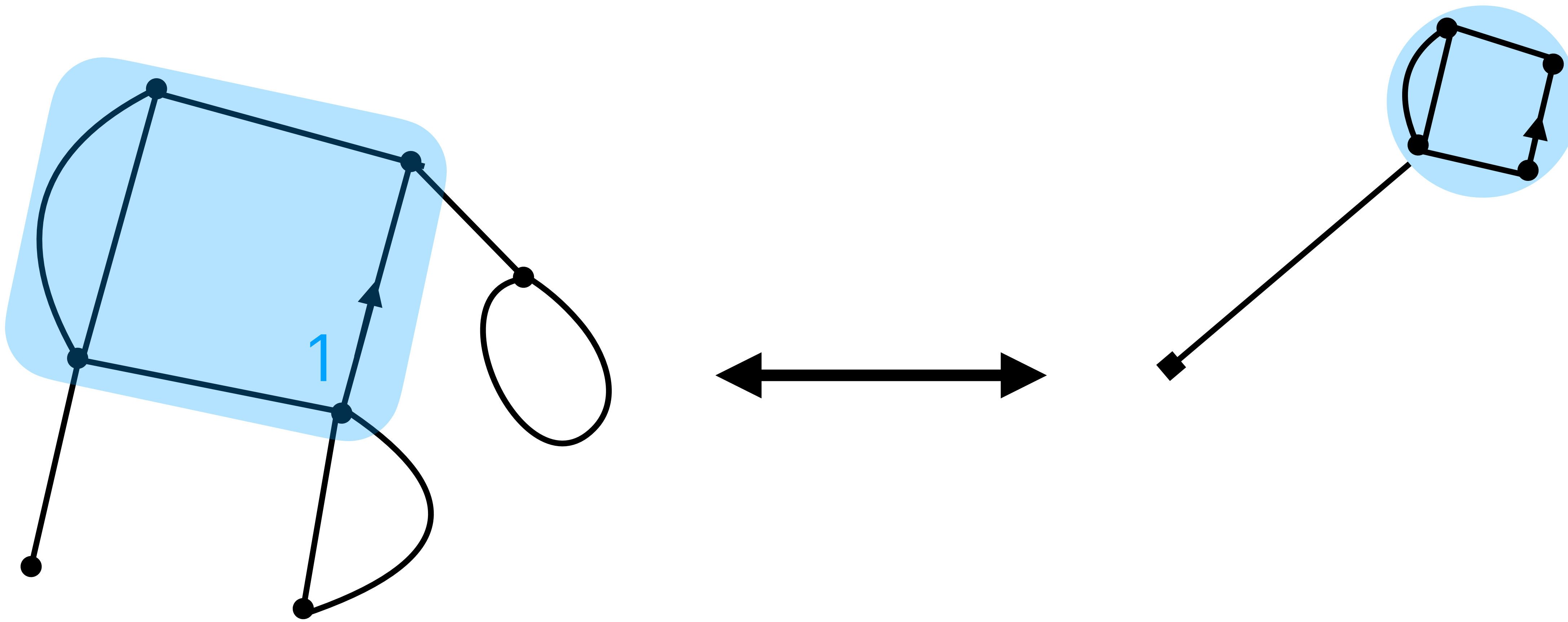

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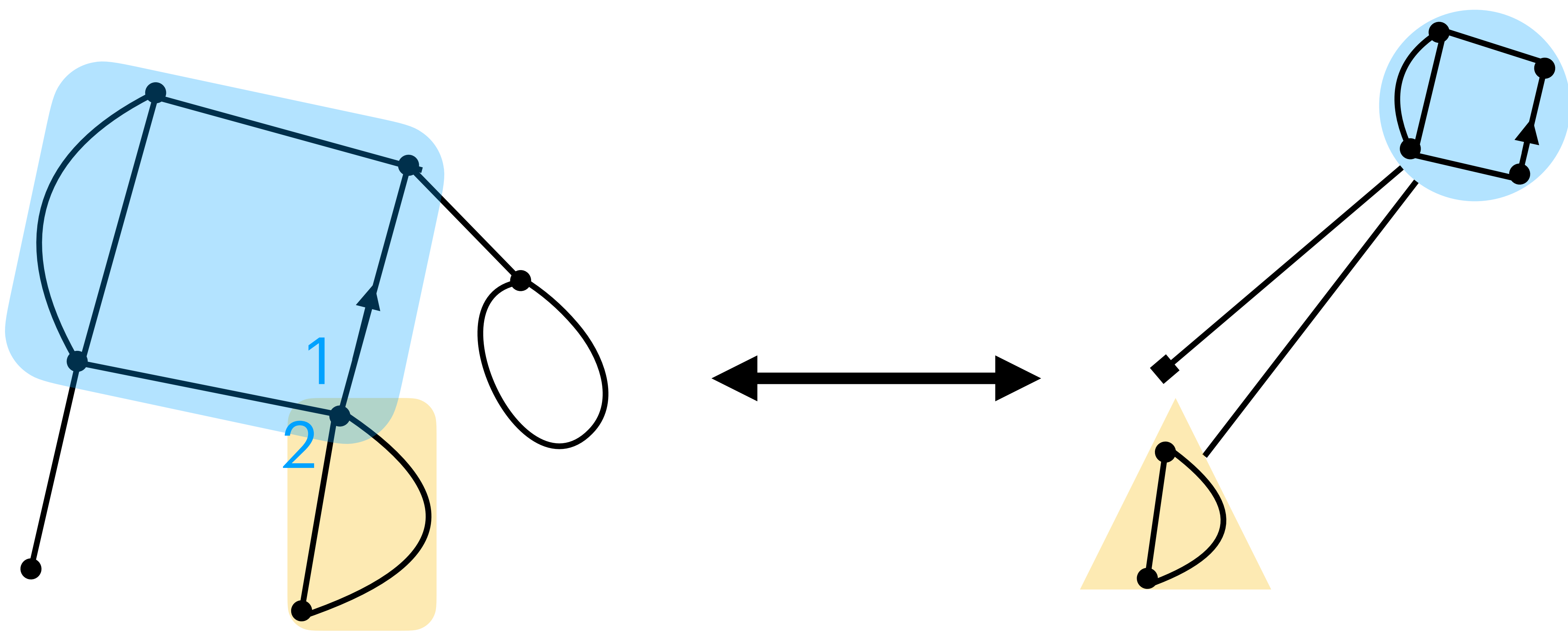

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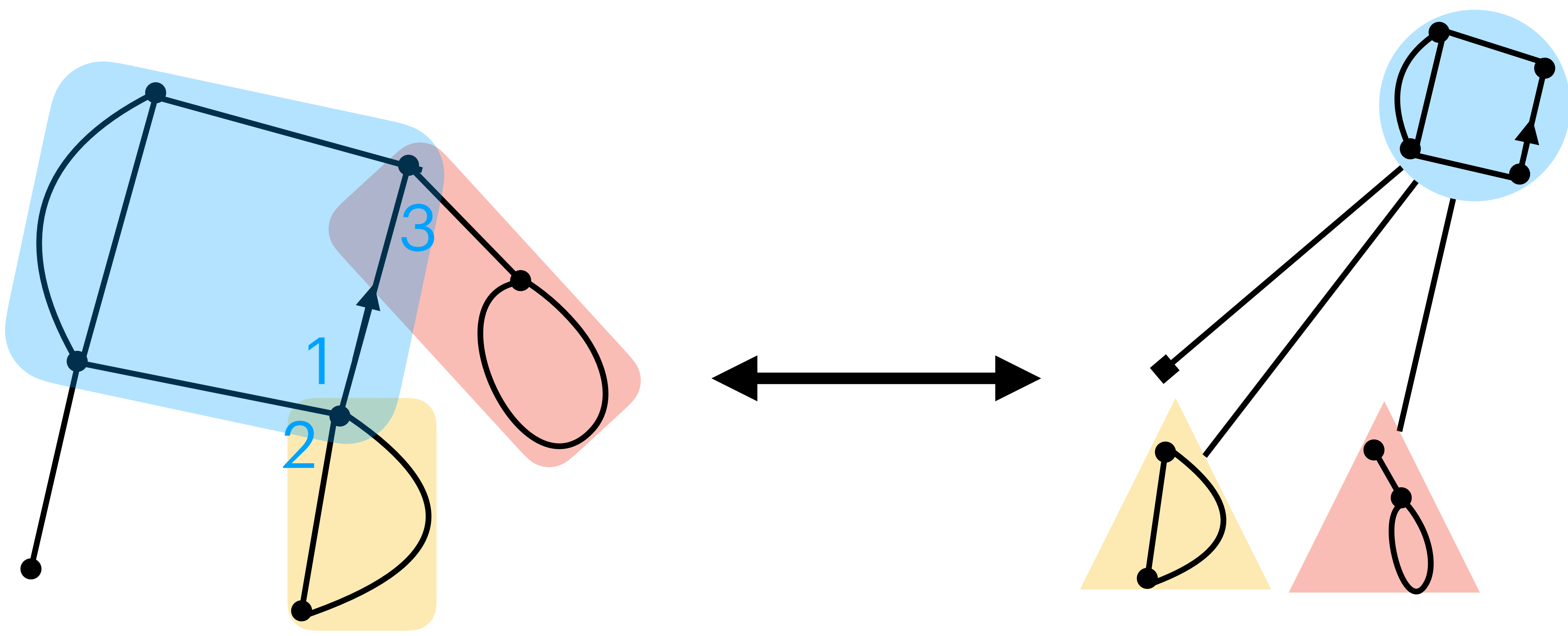

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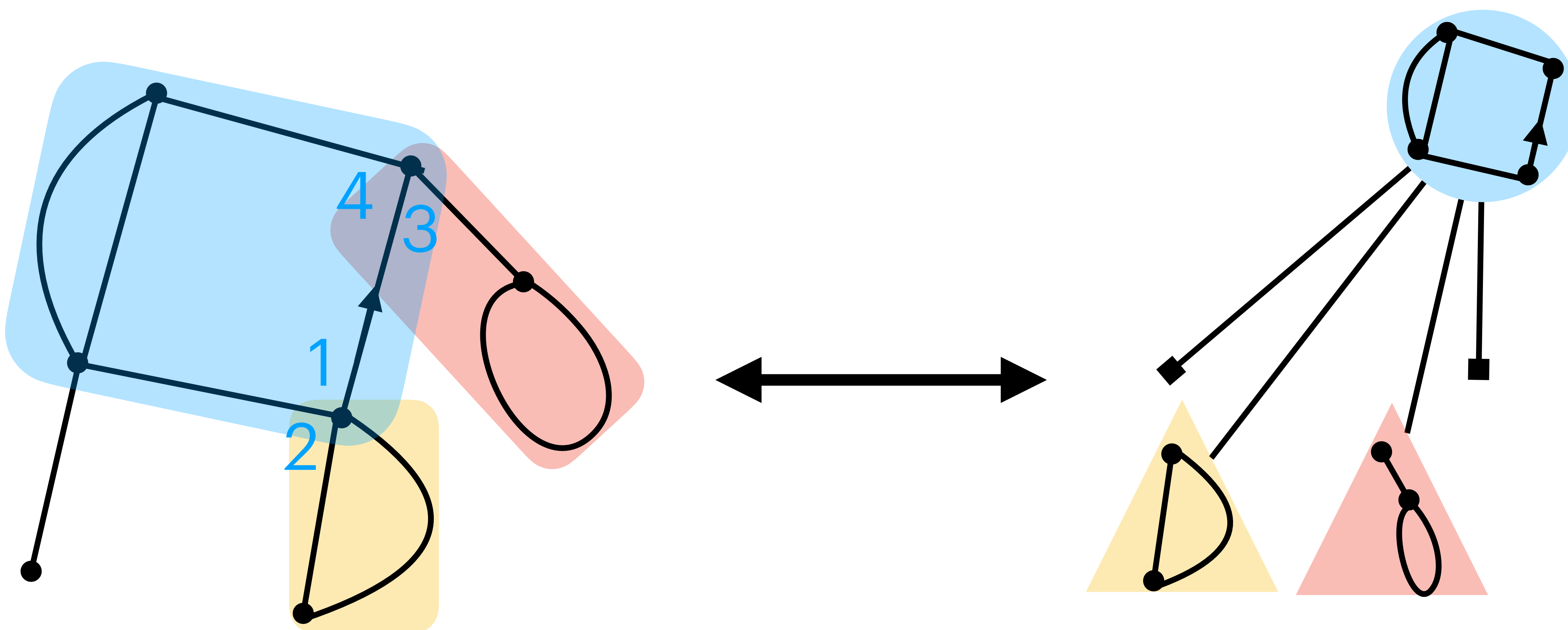

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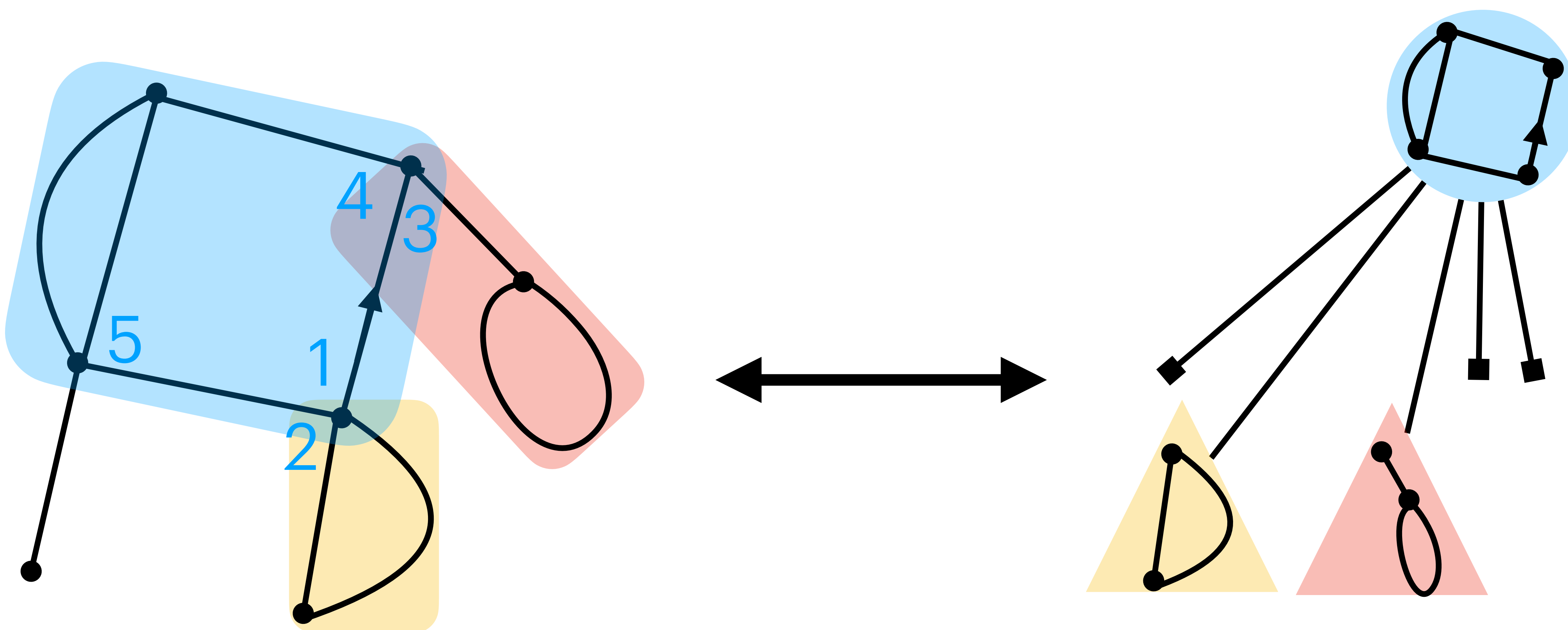

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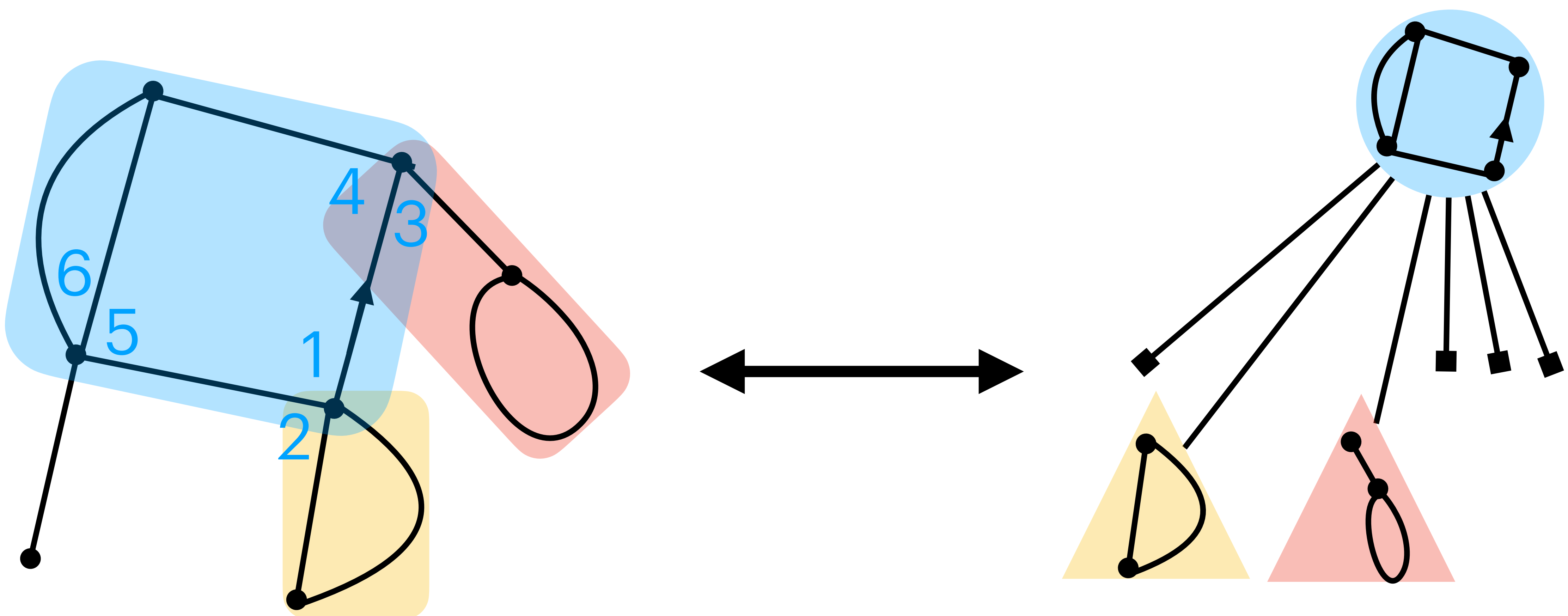

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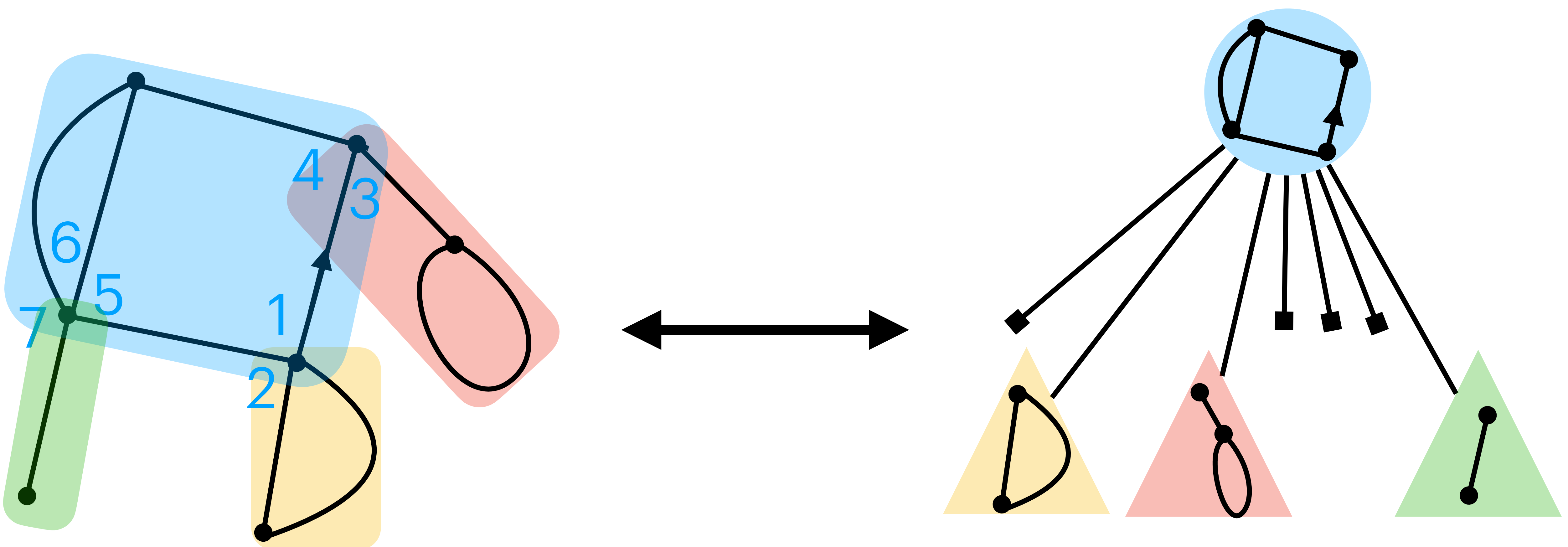

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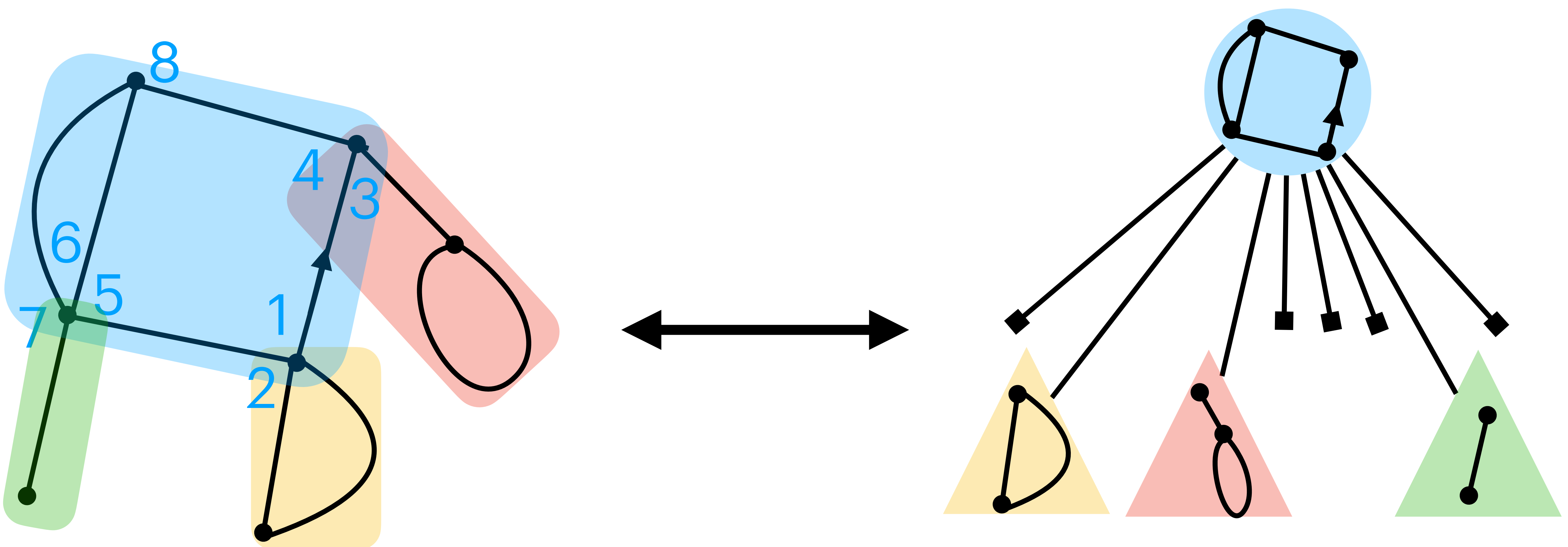

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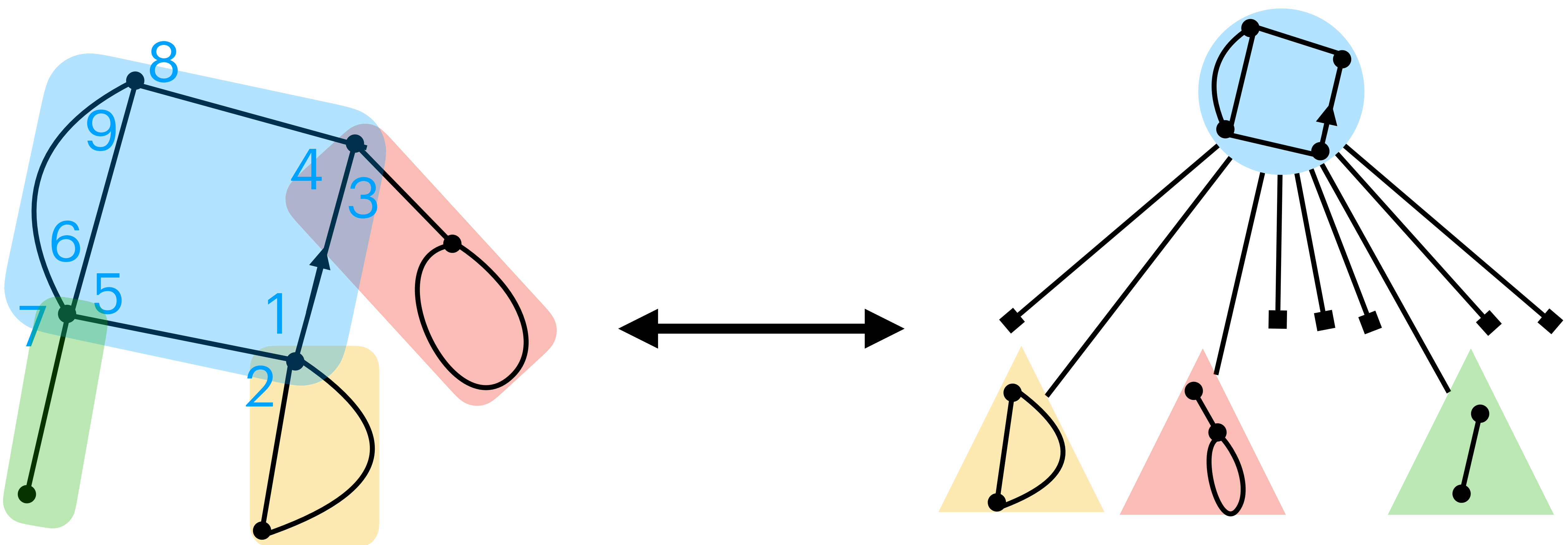

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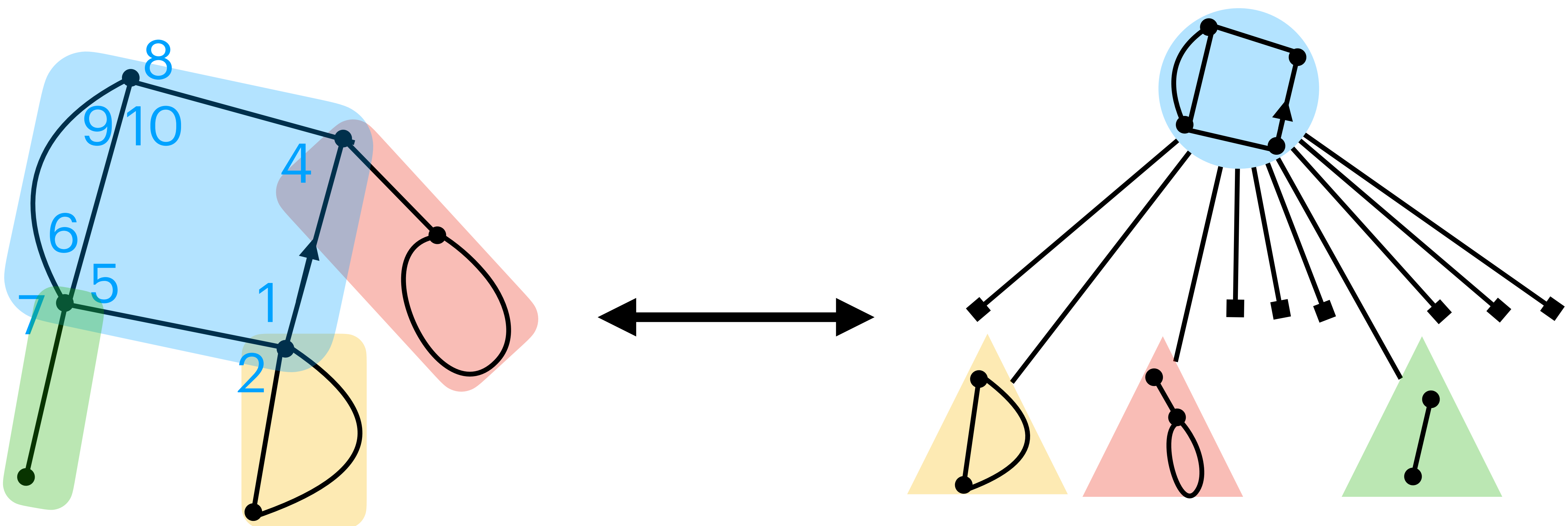

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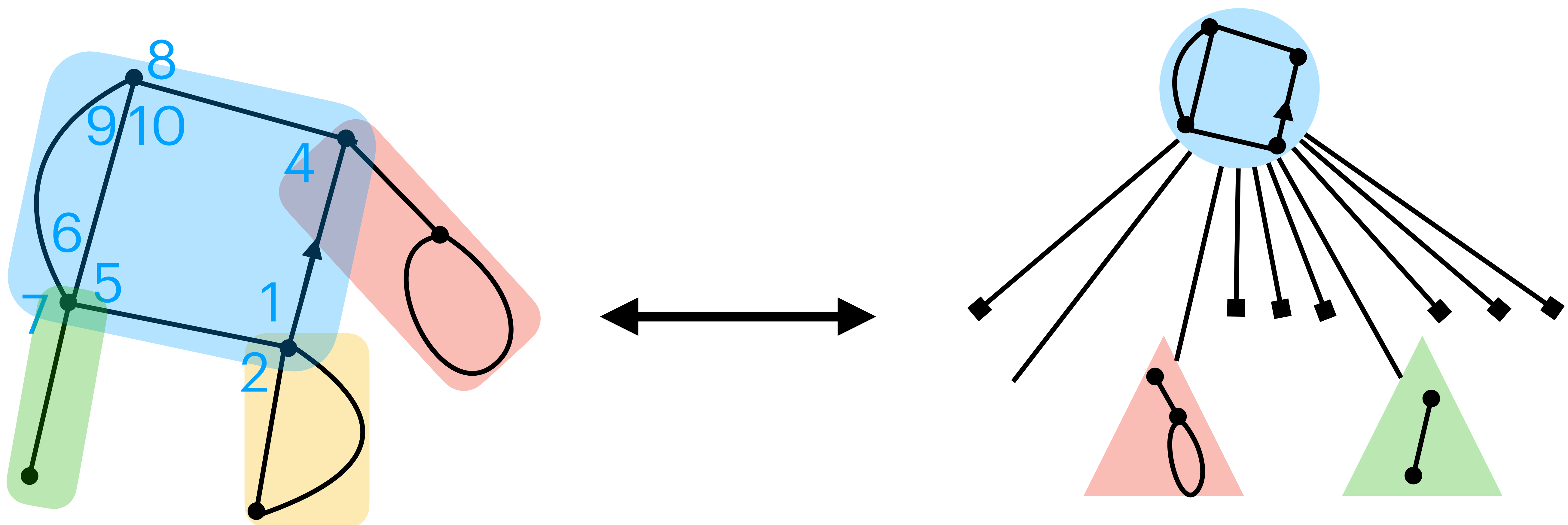

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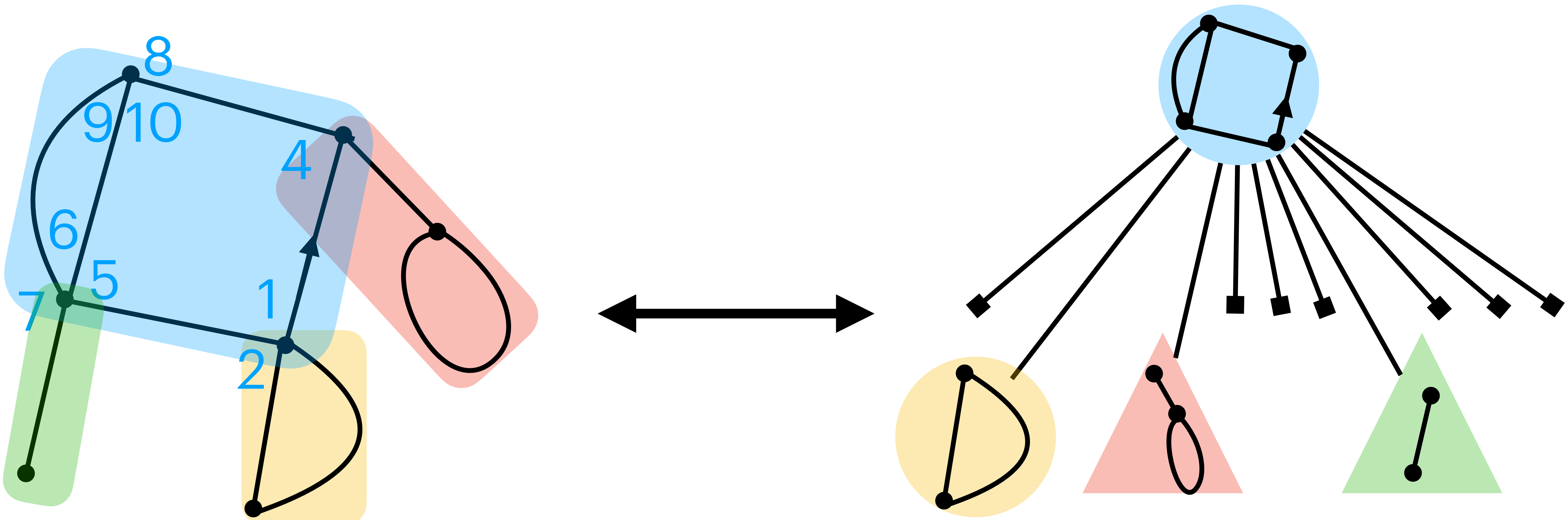

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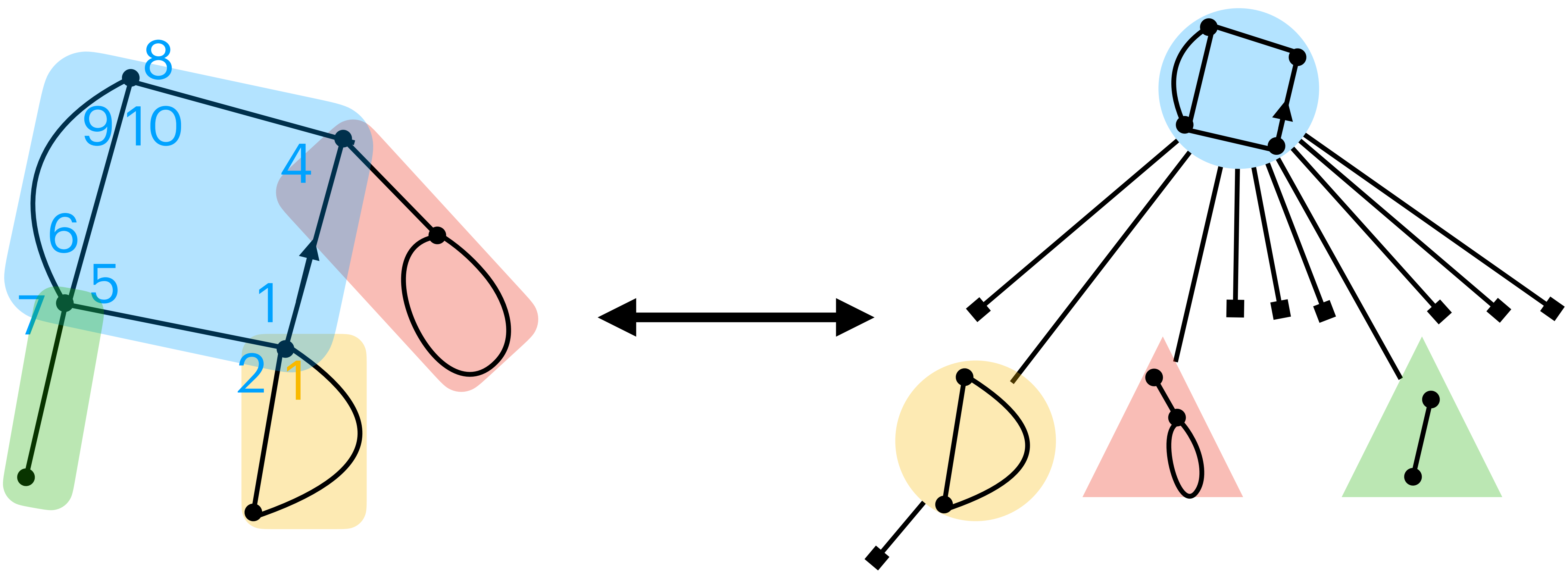

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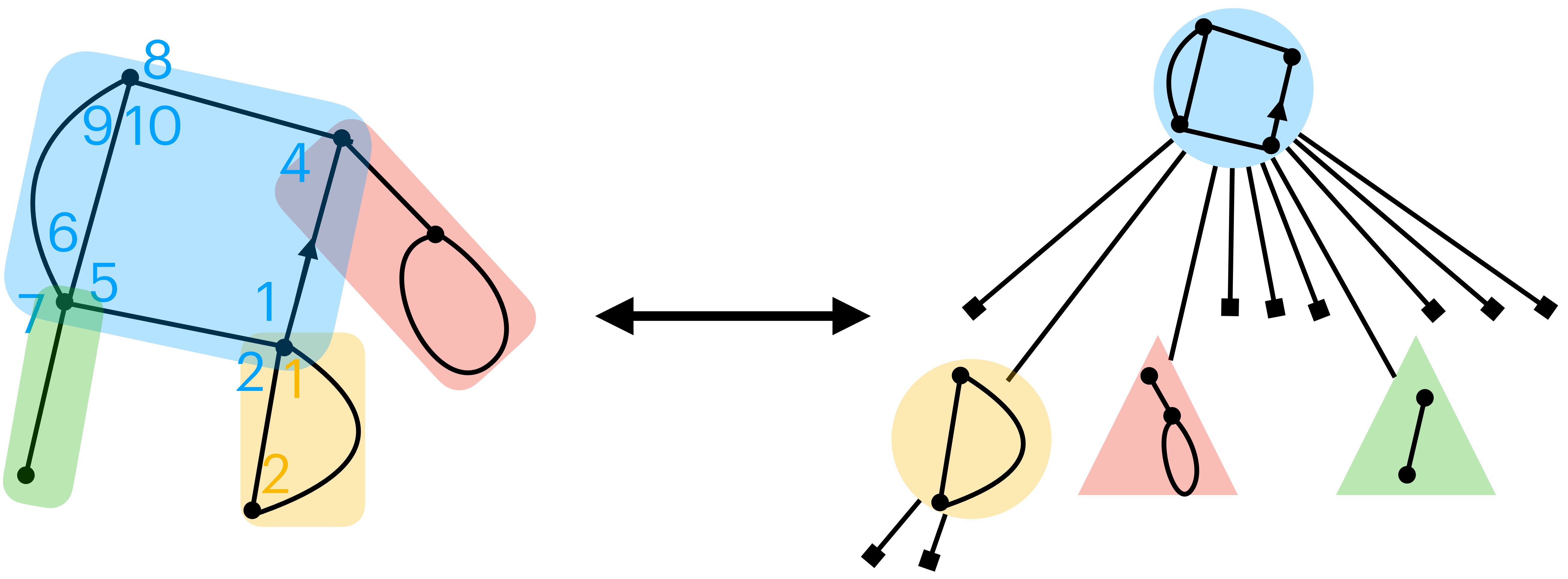

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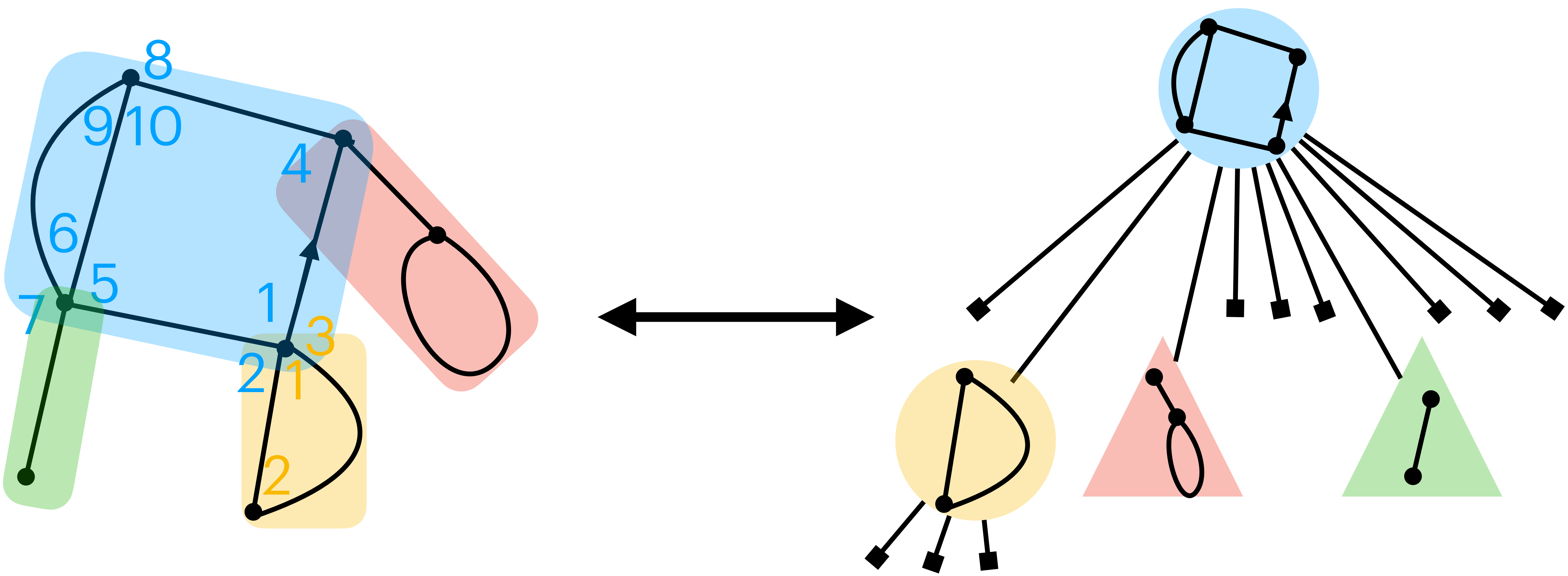

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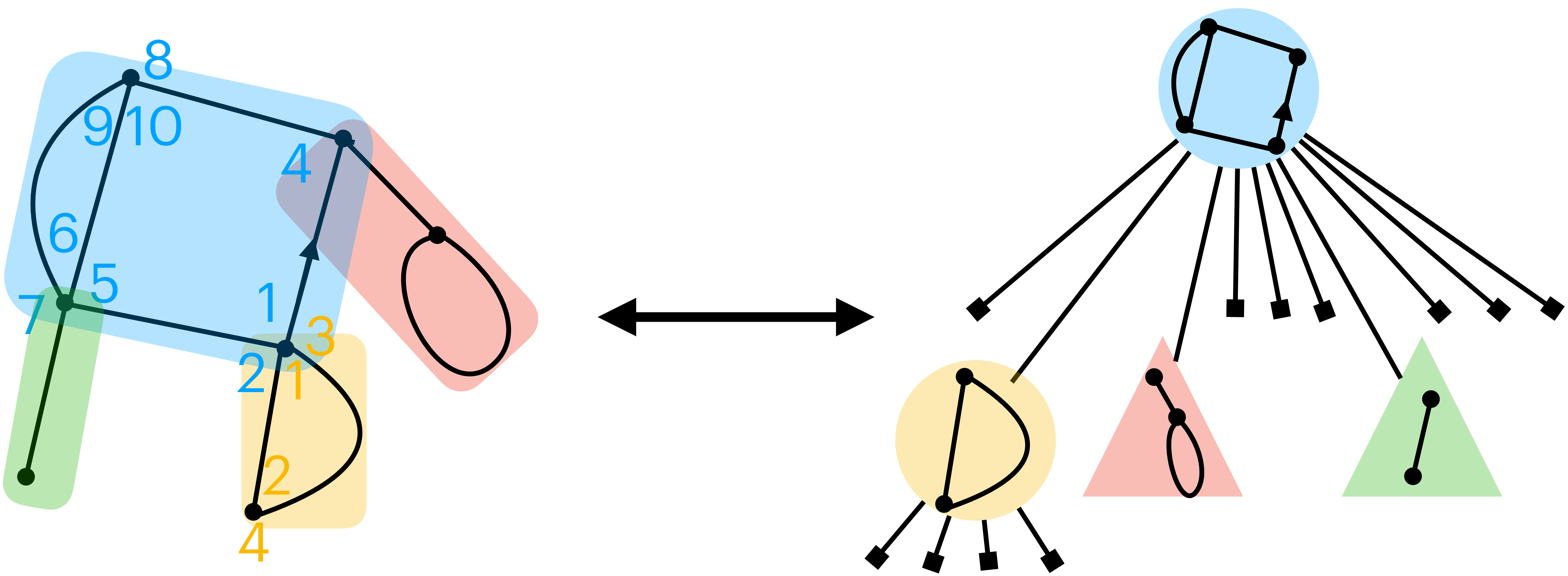

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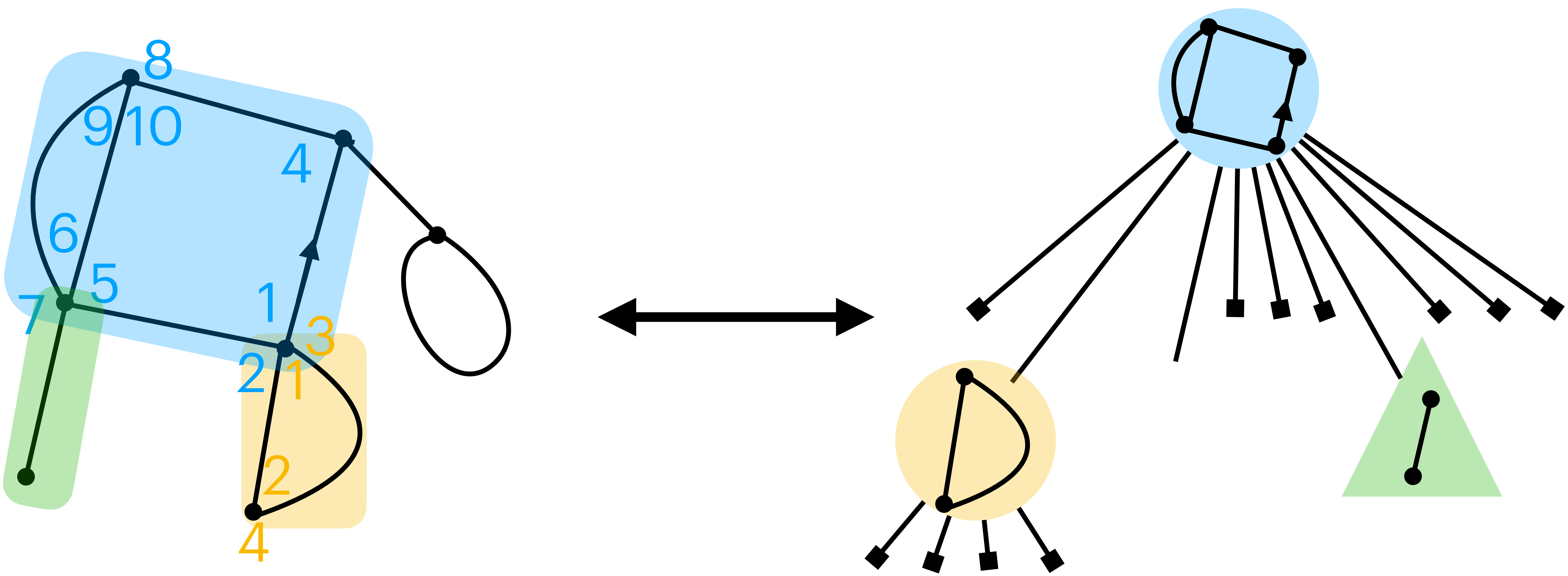

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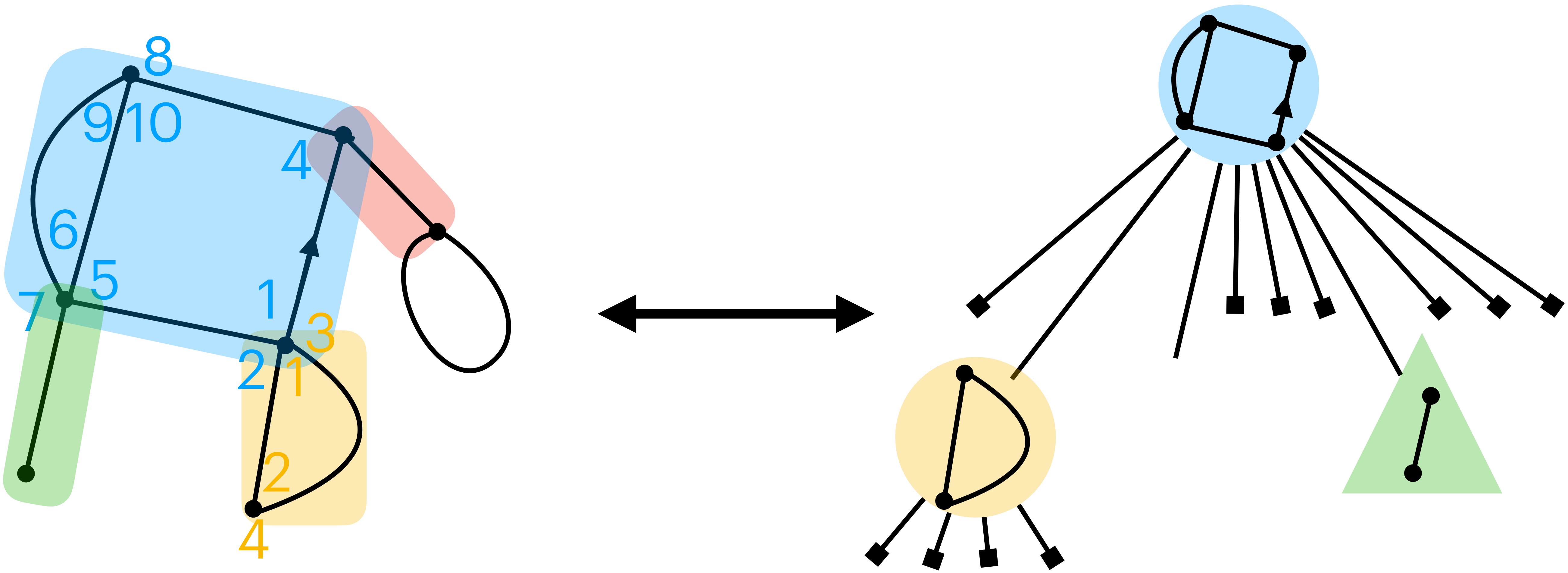

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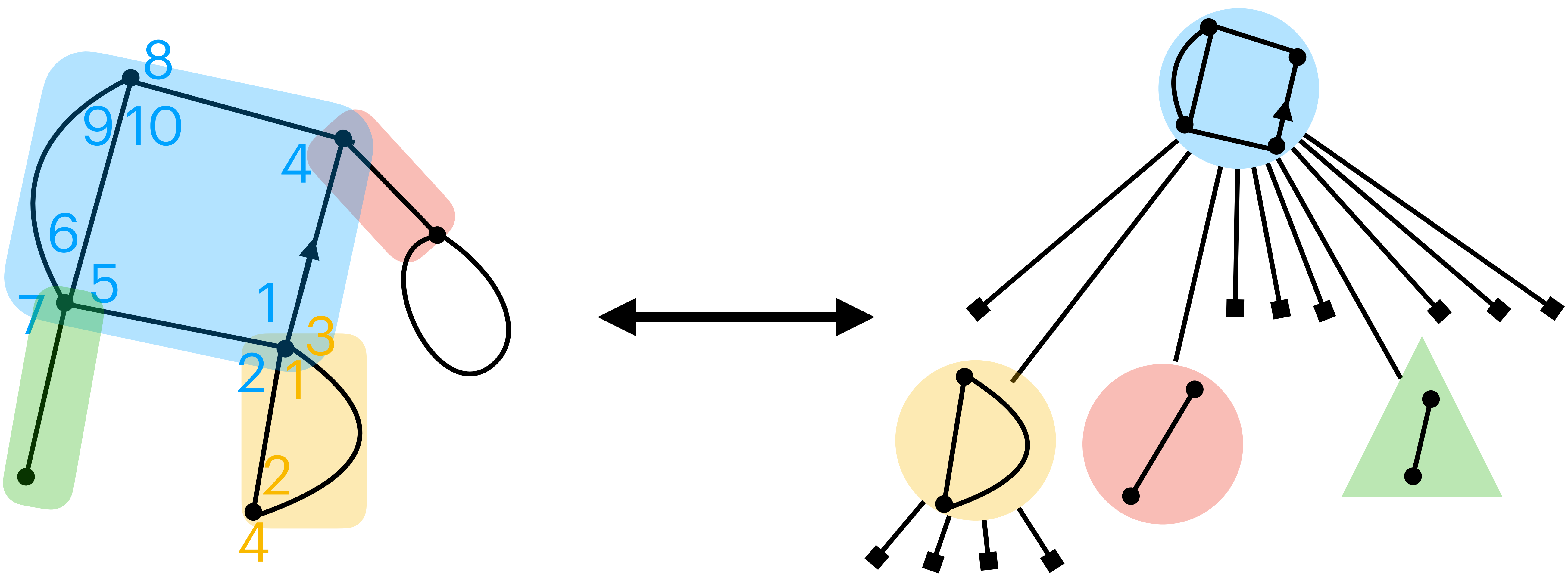

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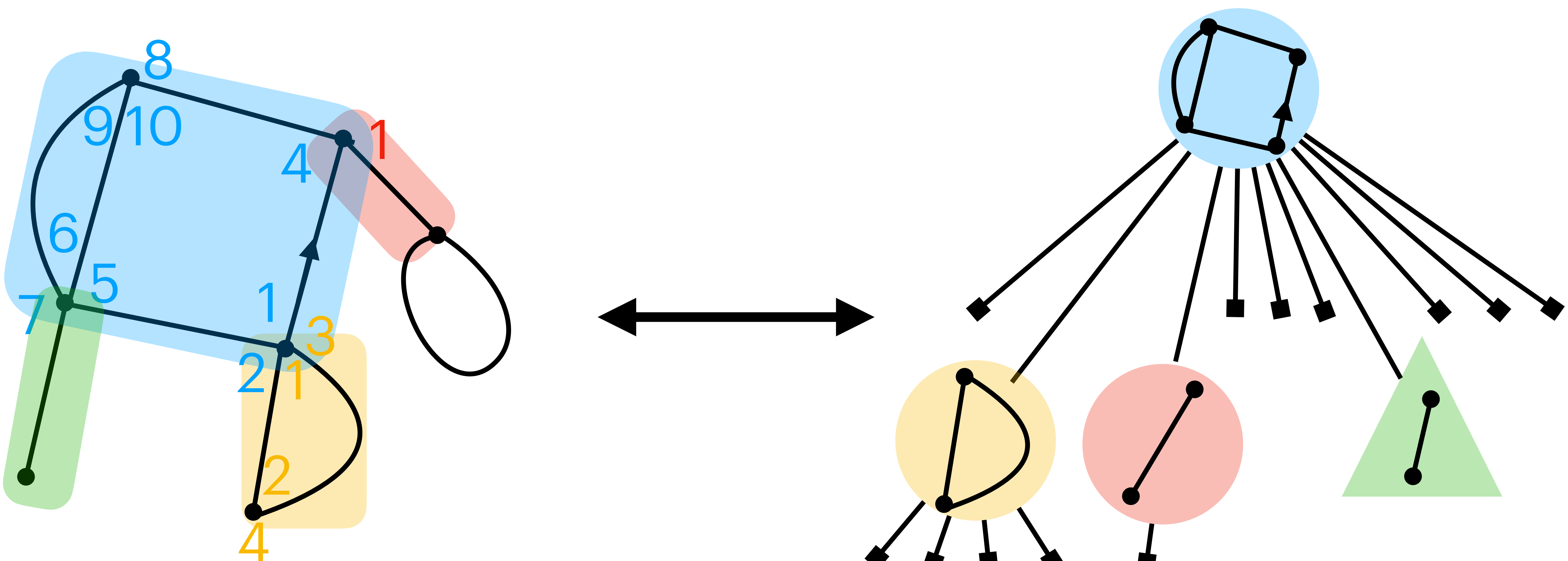

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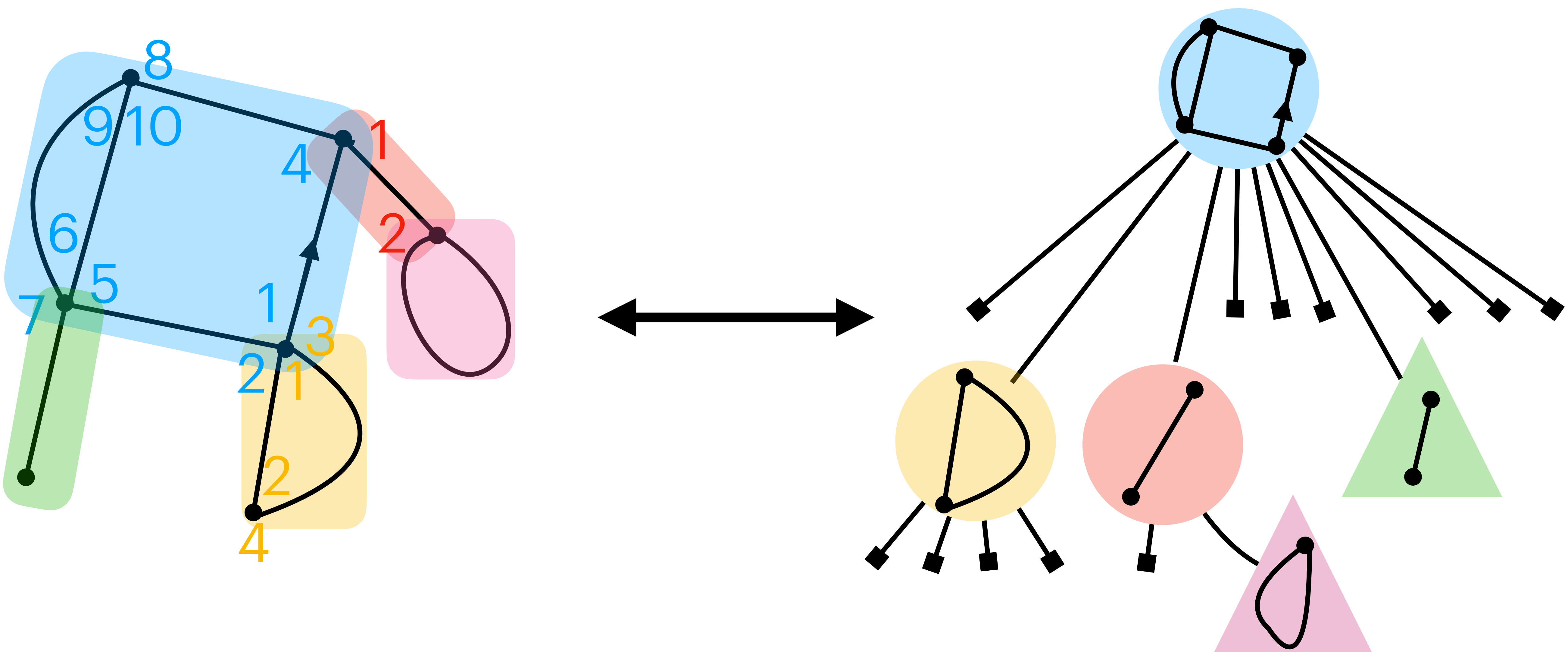

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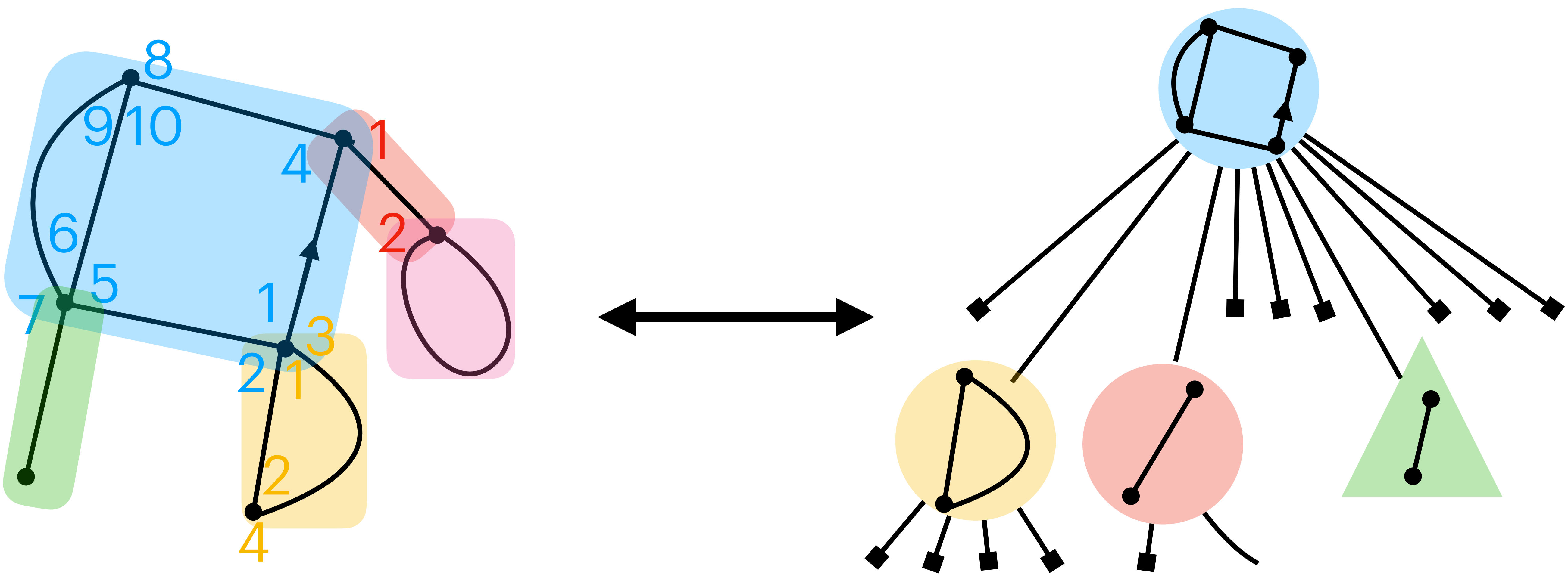

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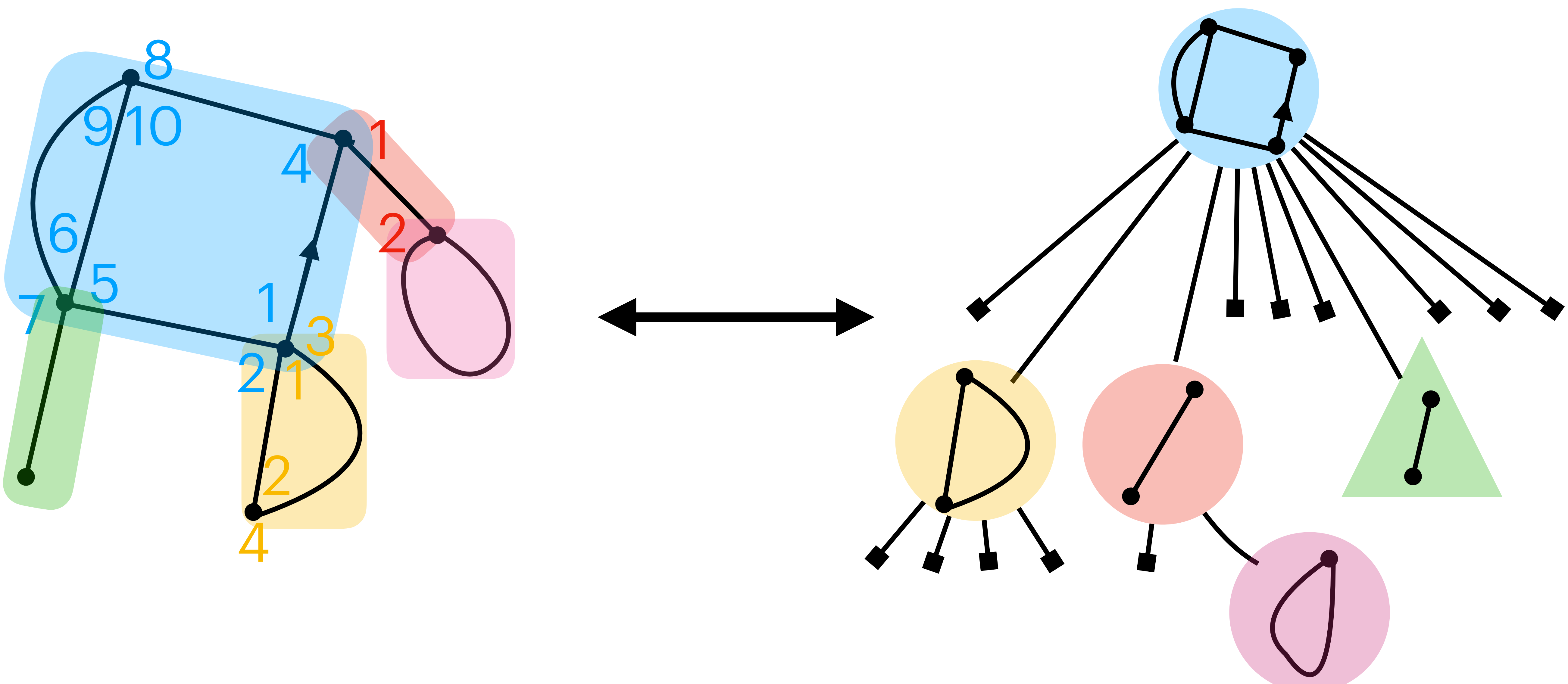

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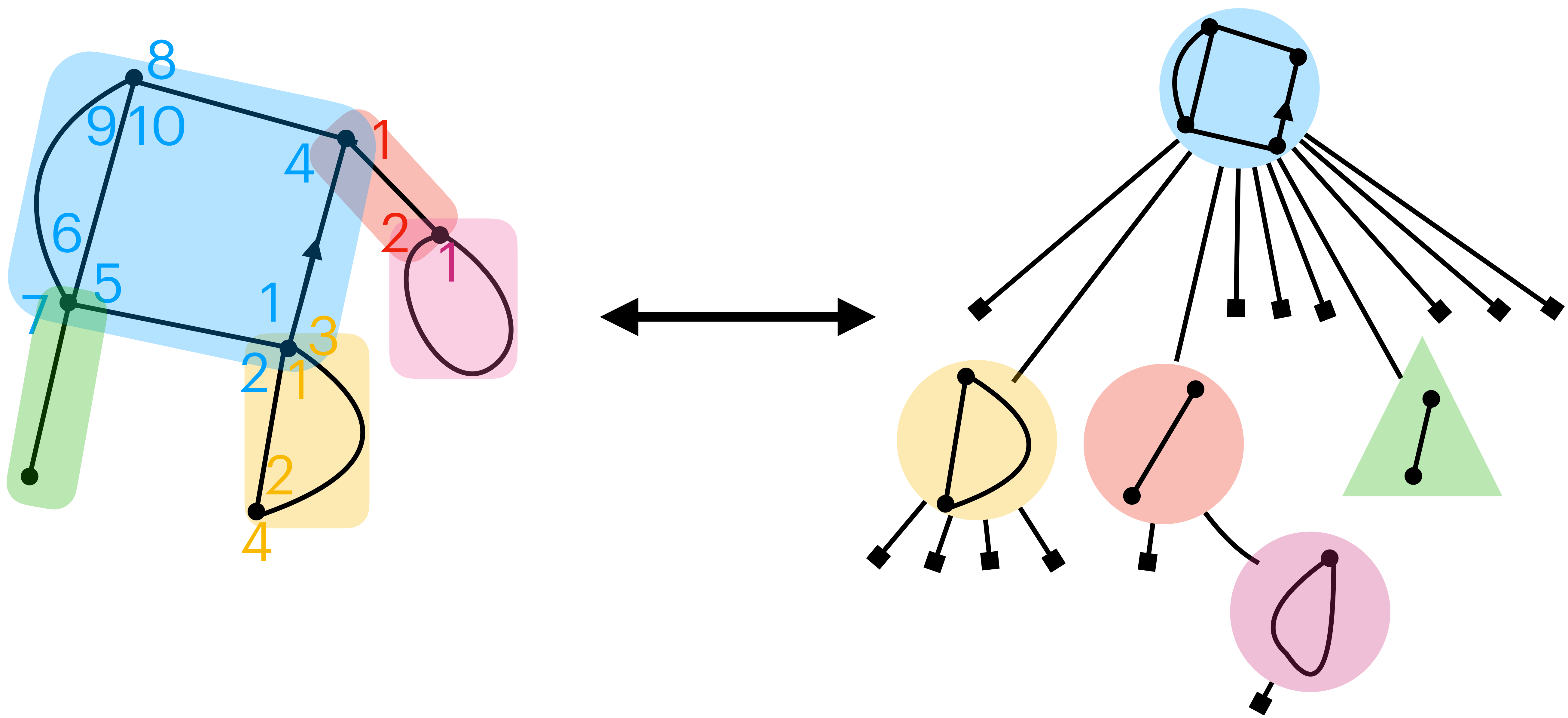

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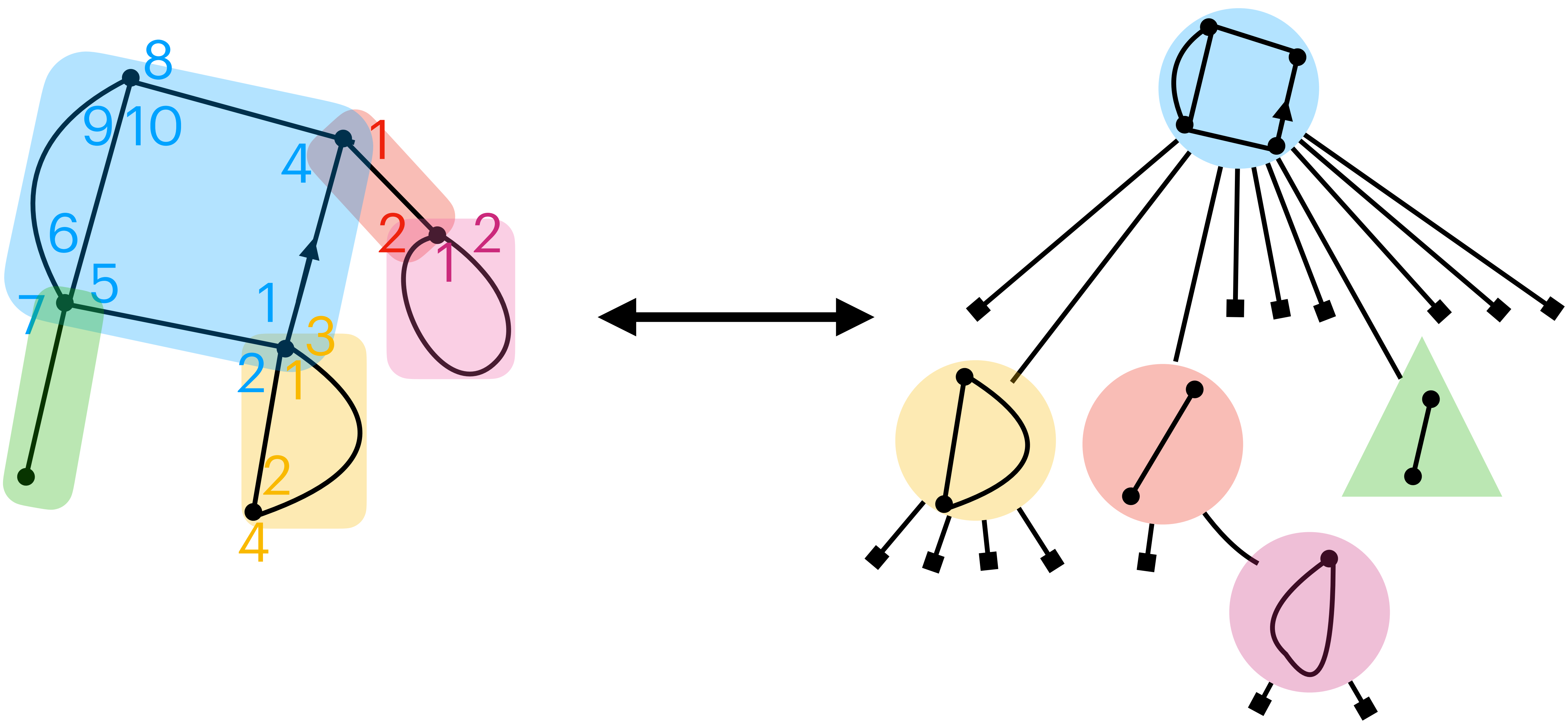

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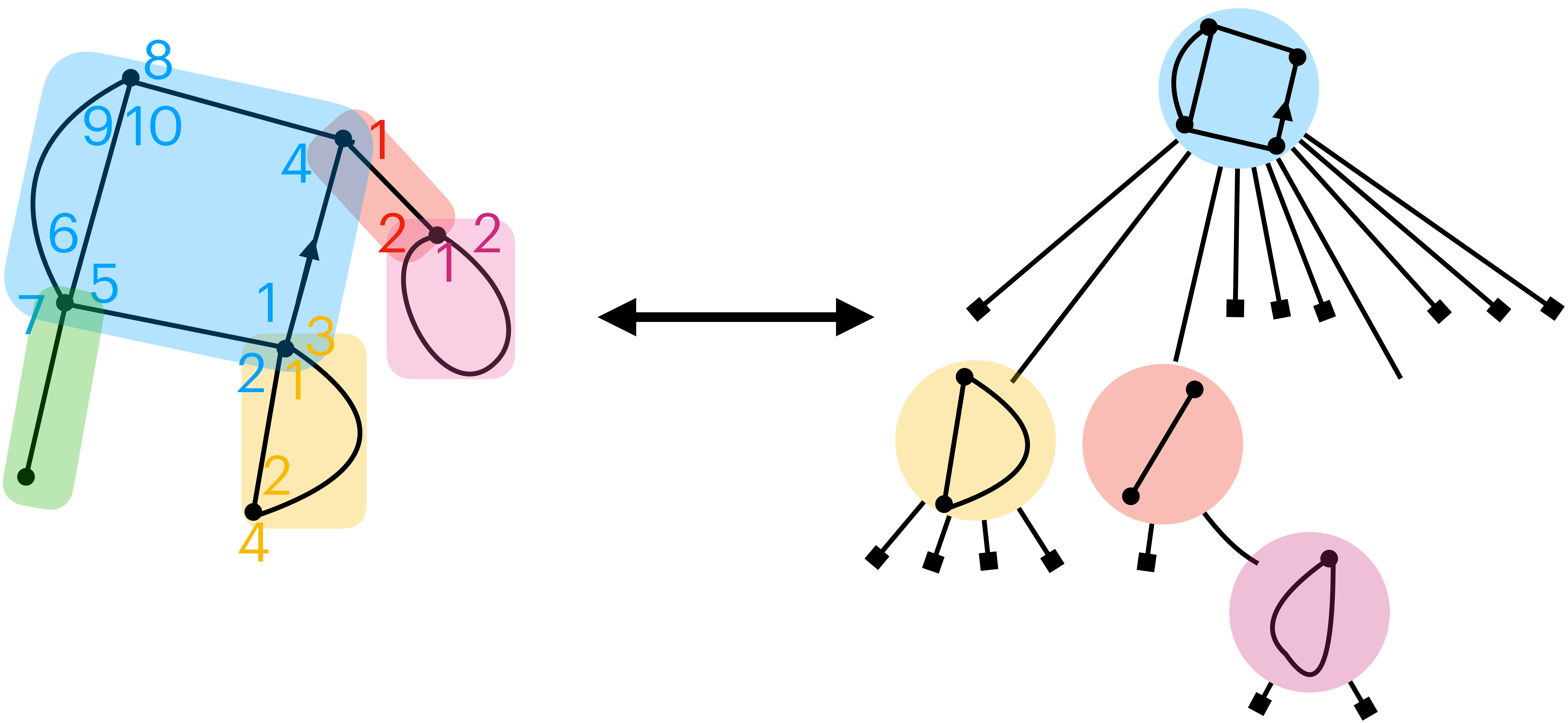
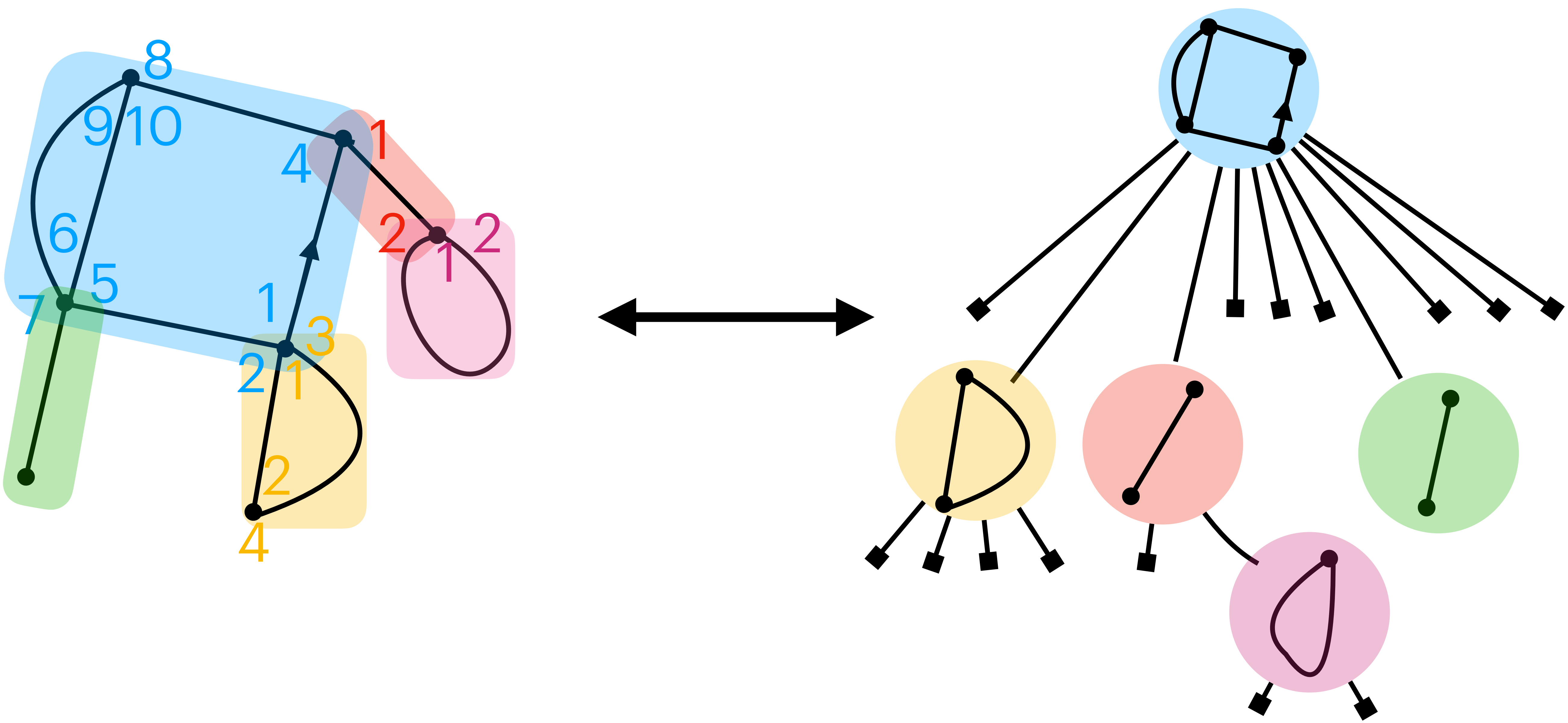


With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Maps are 2-connected maps of maps (Tutte 1963) : $M(z) = B(zM^2(z))$

⇒ Underlying block tree structure, made explicit by Addario-Berry (2019).



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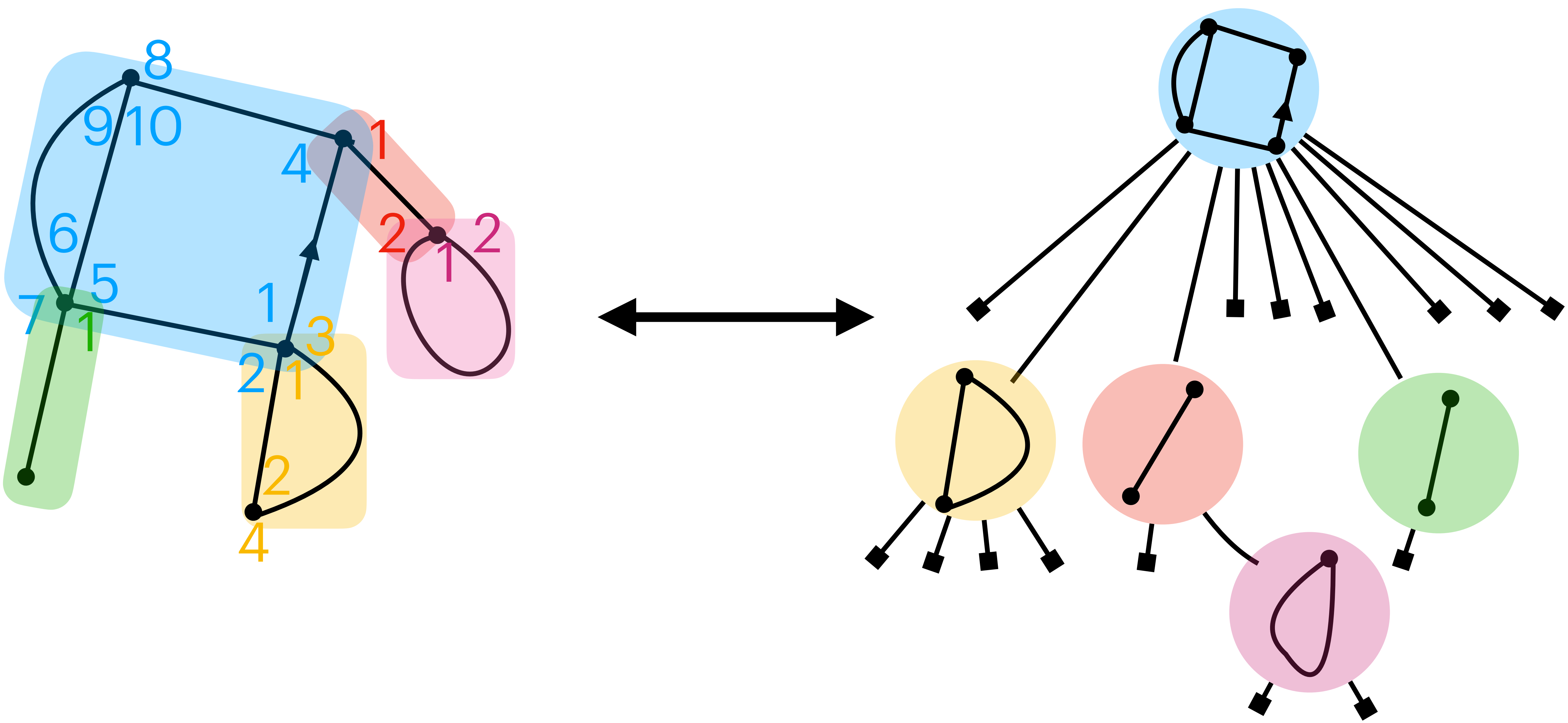

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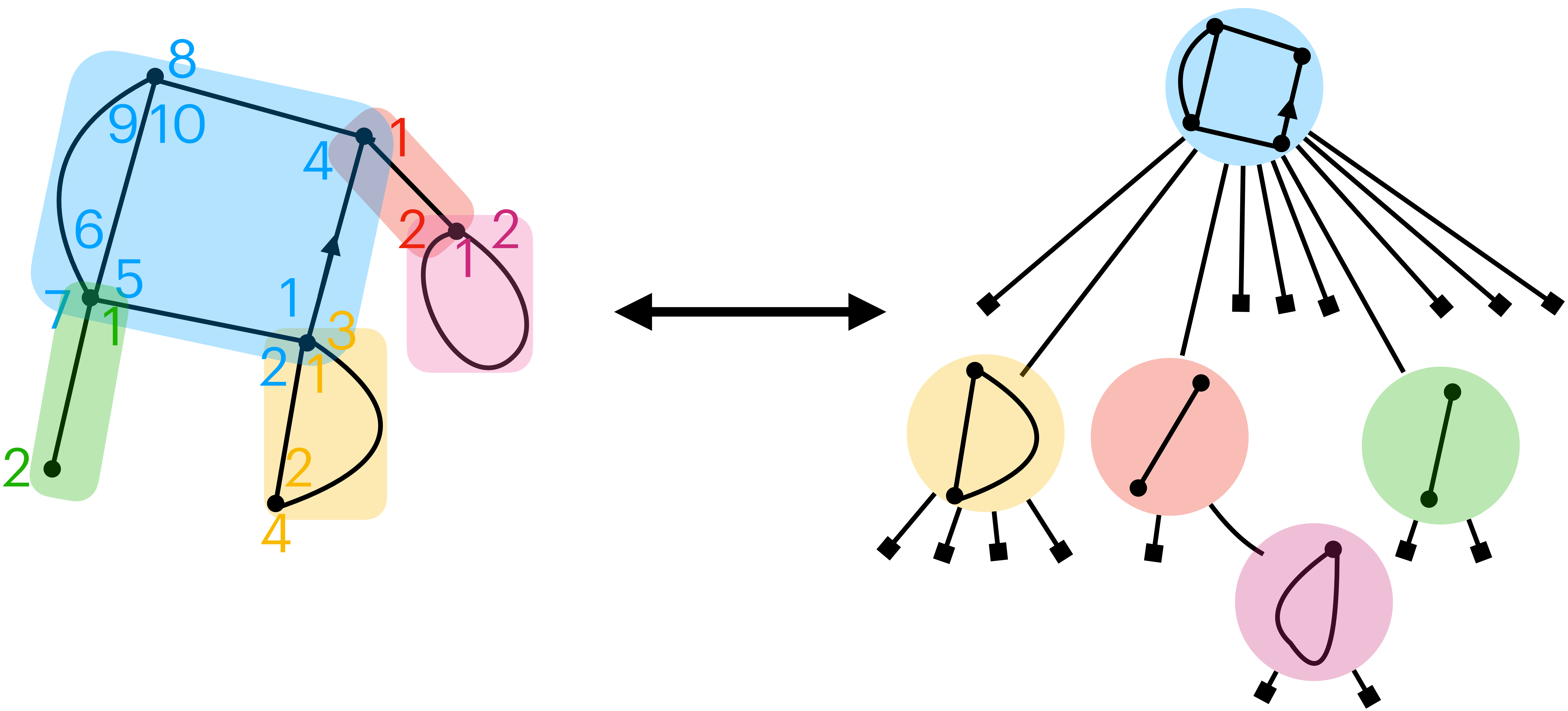
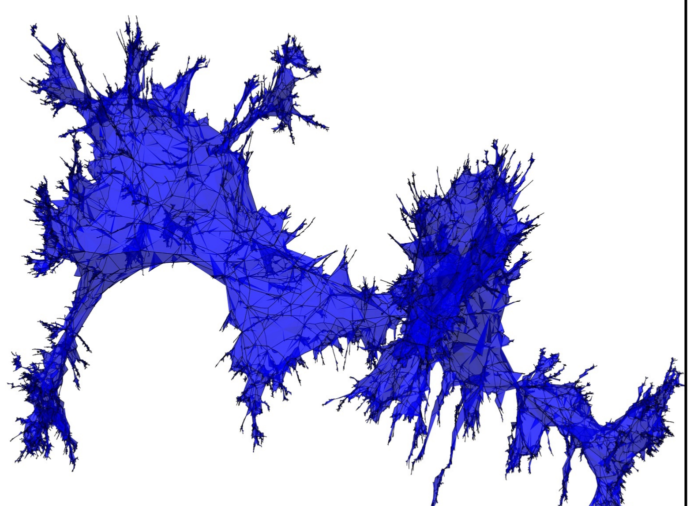
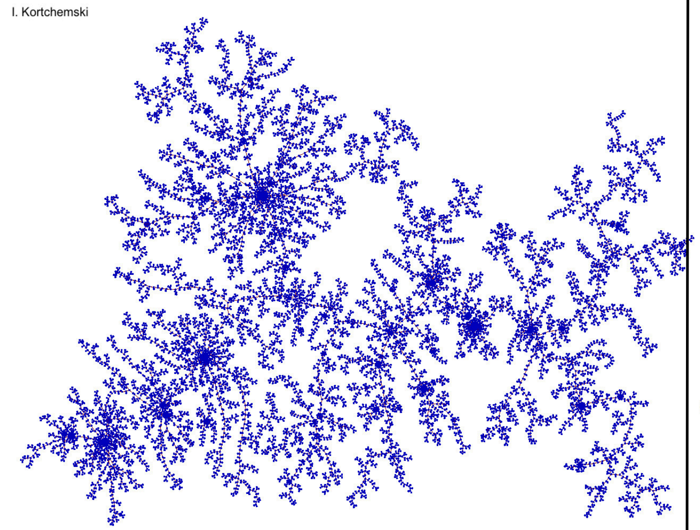
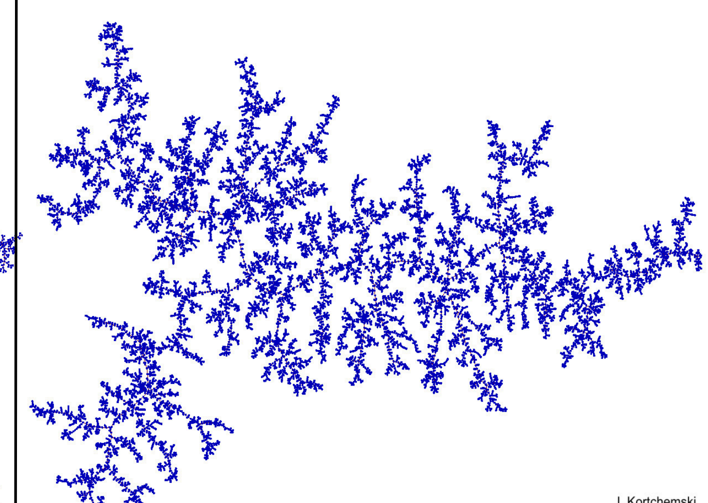
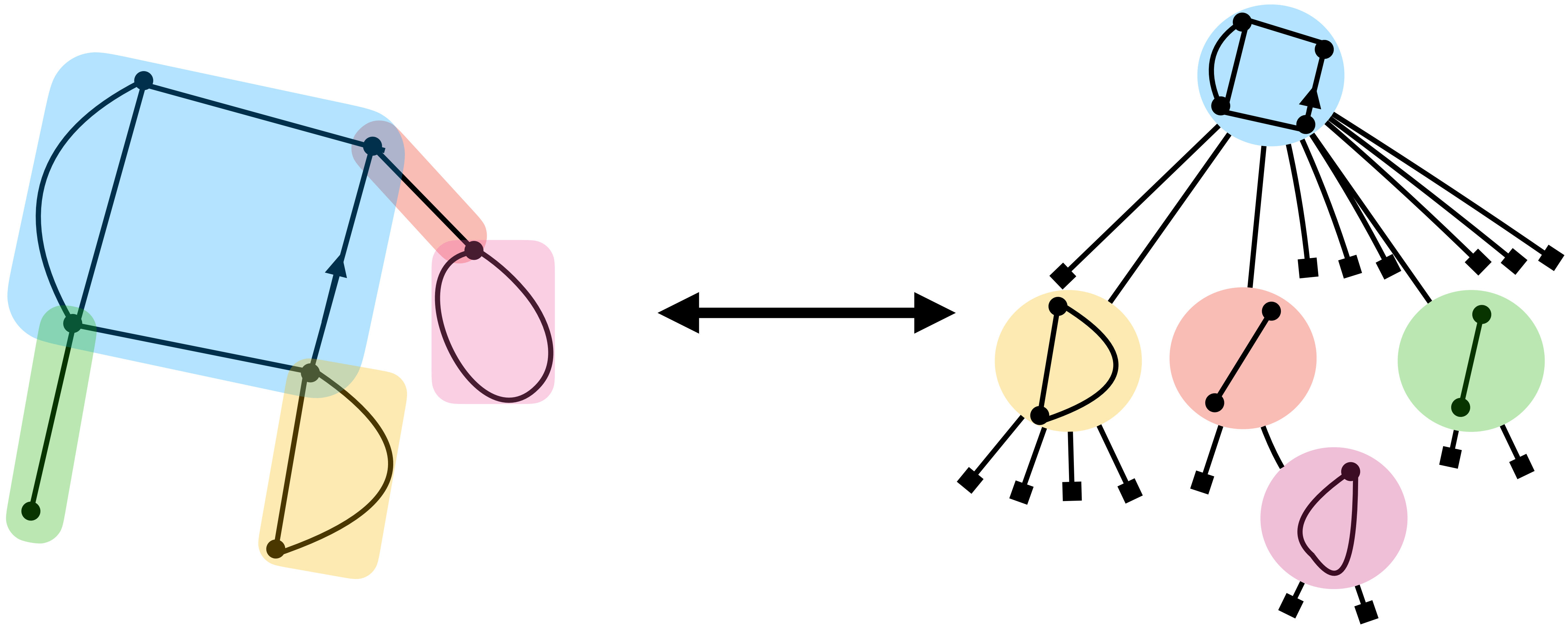


With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration	$\rho(u)^n n^{-5/2}$	$\rho(u)^n n^{-5/3}$	$\rho(u)^n n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ Stufler 2020	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n (up to constant factors)	$\frac{1}{n^{1/4}} M_n \to \mathcal{S}_e$  J. Bettinelli Assuming the convergence of 2-connected maps towards the brownian sphere	$\frac{1}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ I. Kortchemski 	$\frac{1}{n^{1/2}} M_n \to \mathcal{T}_e$ Stufler 2020  I. Kortchemski

Decomposition of a map into blocks: properties



- Internal node (with k children) of $T_{\mathfrak{m}}$ $\leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$;
- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$.

T_{M_n} gives the block sizes of a random map M_n.

Galton-Watson Trees

μ-Galton-Watson tree : random tree where the degree of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

Exemple $\mu(\{0\}) = 1/3,\ \mu(\{1\}) = 1/2,\ \mu(\{2\}) = 1/6$

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- 1 $1/2$

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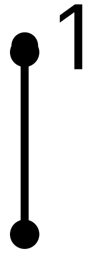

$1/2$

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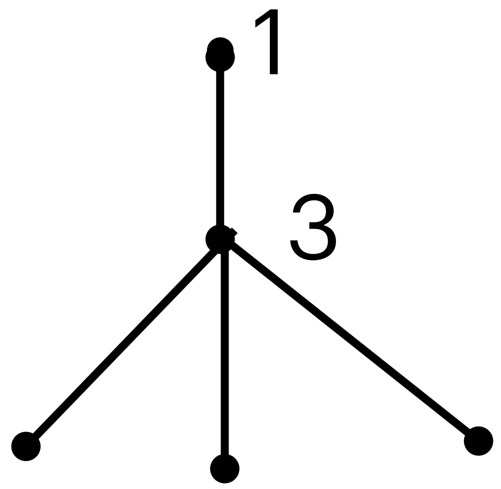

$1/2 \times 1/6$

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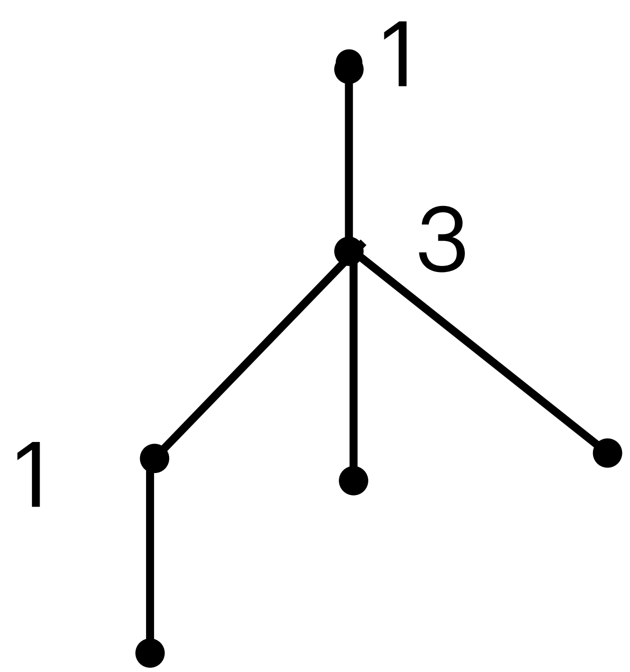

$1/2 \times 1/6 \times 1/2$

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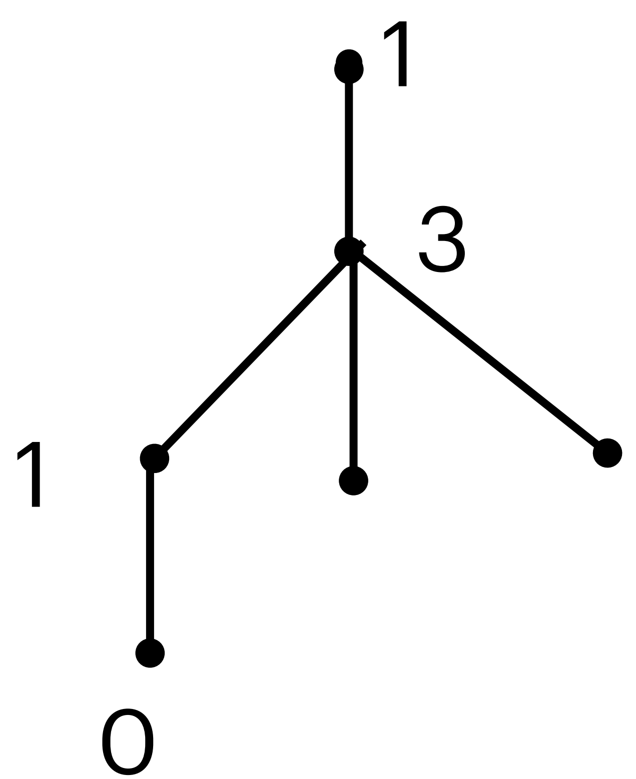

$1/2 \times 1/6 \times 1/2 \times 1/3$

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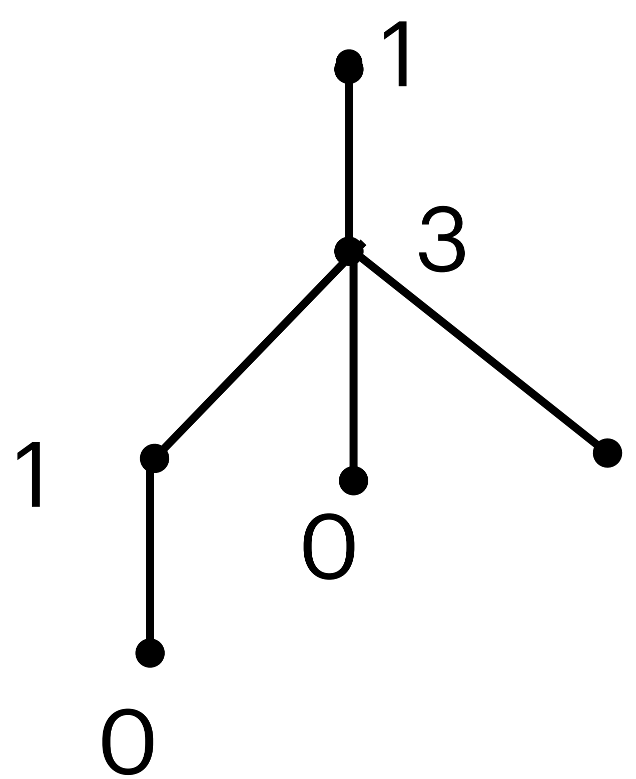

$1/2 \times 1/6 \times 1/2 \times 1/3 \times 1/3$

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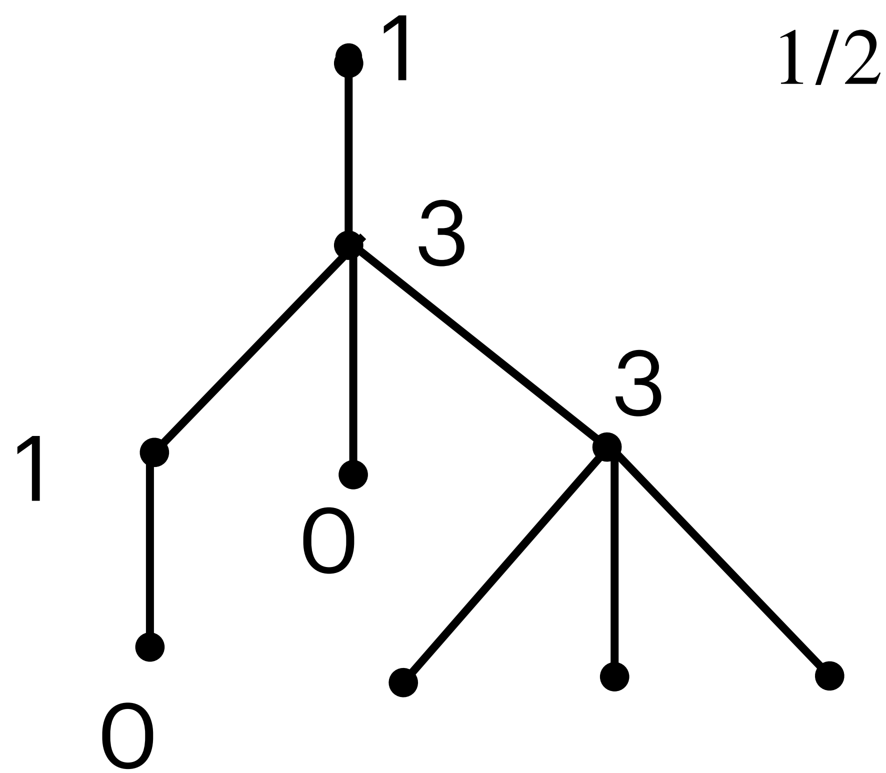

$1/2 \times 1/6 \times 1/2 \times 1/3 \times 1/3 \times 1/6$

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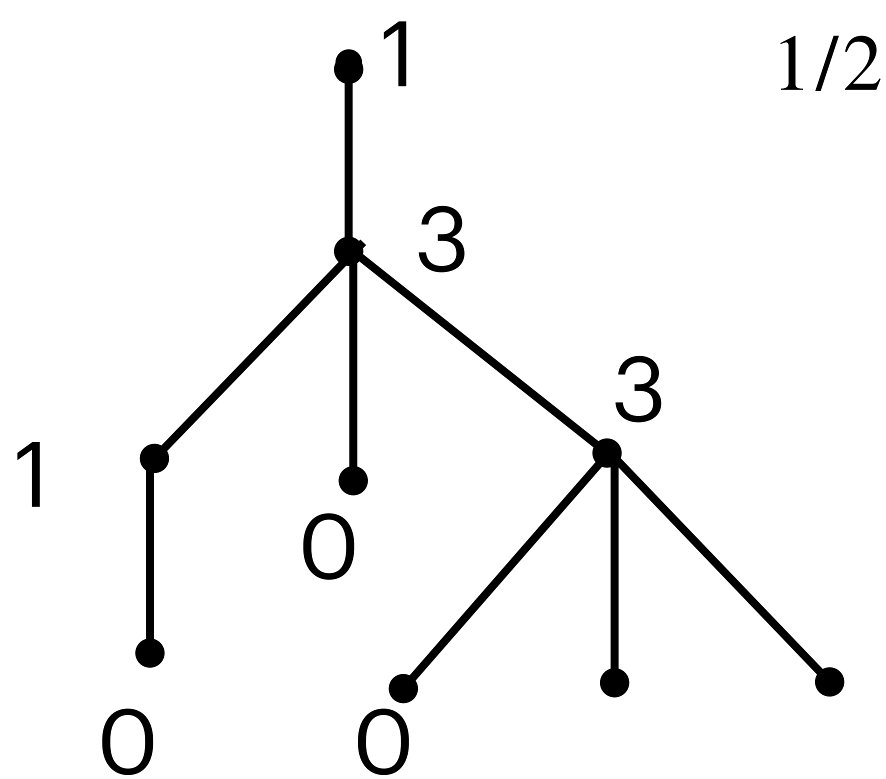

$1/2 \times 1/6 \times 1/2 \times 1/3 \times 1/3 \times 1/6 \times 1/3$

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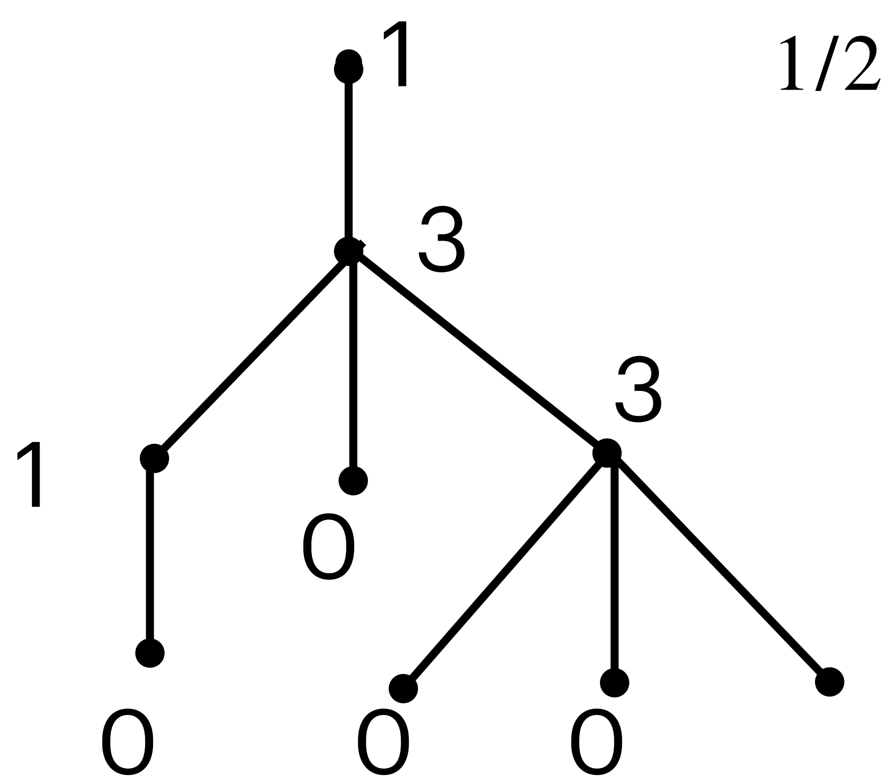

$$1/2 \times 1/6 \times 1/2 \times 1/3 \times 1/3 \times 1/6 \times 1/3 \times 1/3$$

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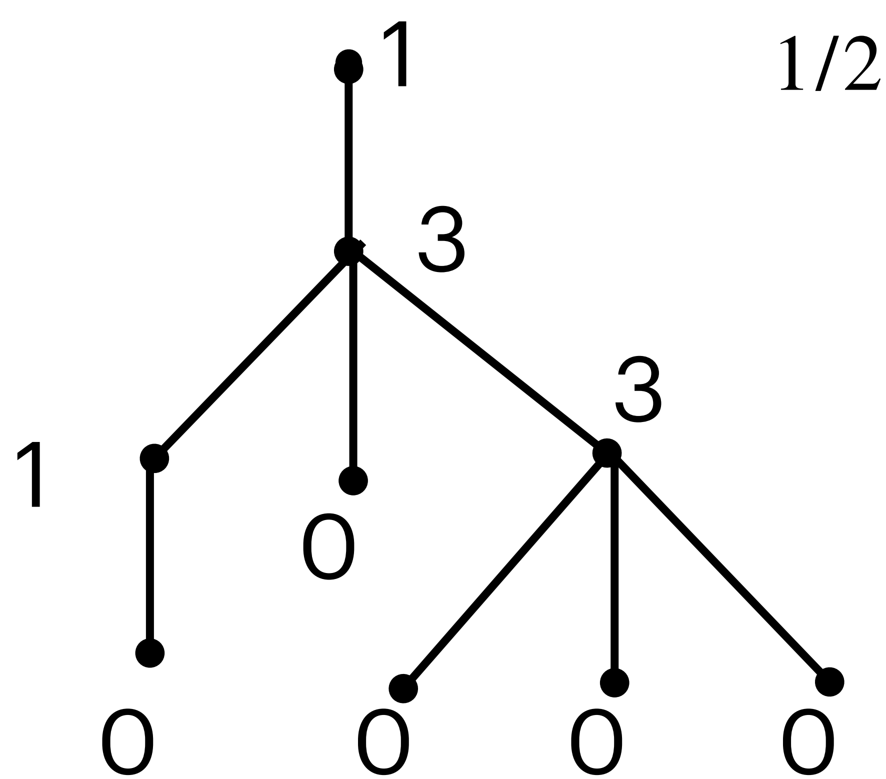

$1/2 \times 1/6 \times 1/2 \times 1/3 \times 1/3 \times 1/6 \times 1/3 \times 1/3 \times 1/3$

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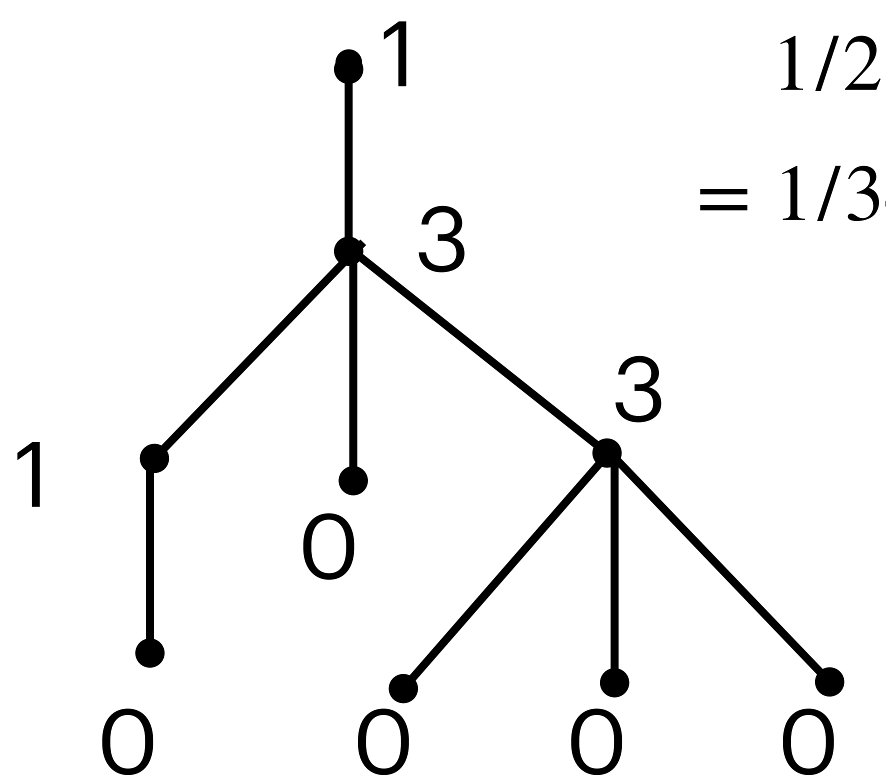

$$1/2 \times 1/6 \times 1/2 \times 1/3 \times 1/3 \times 1/6 \times 1/3 \times 1/3 \times 1/3$$

$$= 1/34992$$

Galton-Watson Trees

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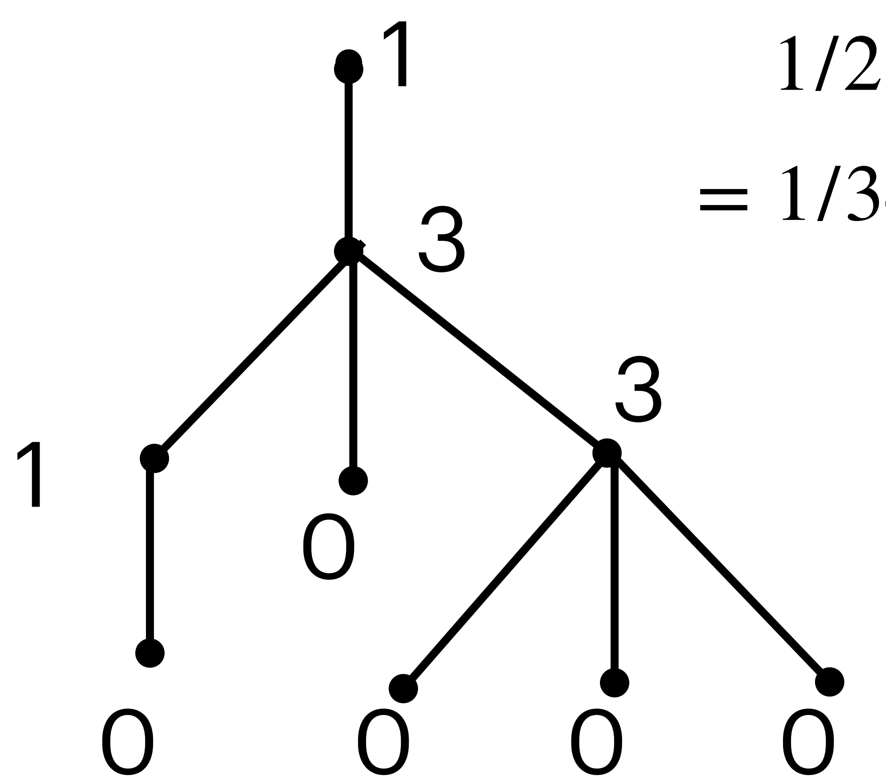
Exemple $\mu(\{0\}) = 1/3,\ \mu(\{1\}) = 1/2,\ \mu(\{2\}) = 1/6$



$1/2 \times 1/6 \times 1/2 \times 1/3 \times 1/3 \times 1/6 \times 1/3 \times 1/3 \times 1/3$

$= 1/34992$ = probability to draw this tree

Galton-Watson tree for map blocks

$$\mu^{y,u}(\{2j\}) = \frac{B_j y^j u^{\mathbf{1}_{j\neq 0}}}{uB(y) + 1 - u}$$

$u > 0$

$y \in (0,4/27]$

$B(y) = \sum_{j\in\mathbb{N}} B_j y^j$ generating function of 2-connected maps enumerated by edges.

Galton-Watson tree for map blocks

$$\mu^{y,u}(\{2j\}) = \frac{B_j y^j u^{\mathbf{1}_{j\neq 0}}}{uB(y) + 1 - u}$$

$u > 0$

$y \in (0,4/27]$

$B(y) = \sum_{j\in\mathbb{N}} B_j y^j$ generating function of 2-connected maps enumerated by edges.

$T_n^{y,u}$ = Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$.

T_n^u = tree of blocks of a map drawn according to $\mathbb{P}_{n,u}$.

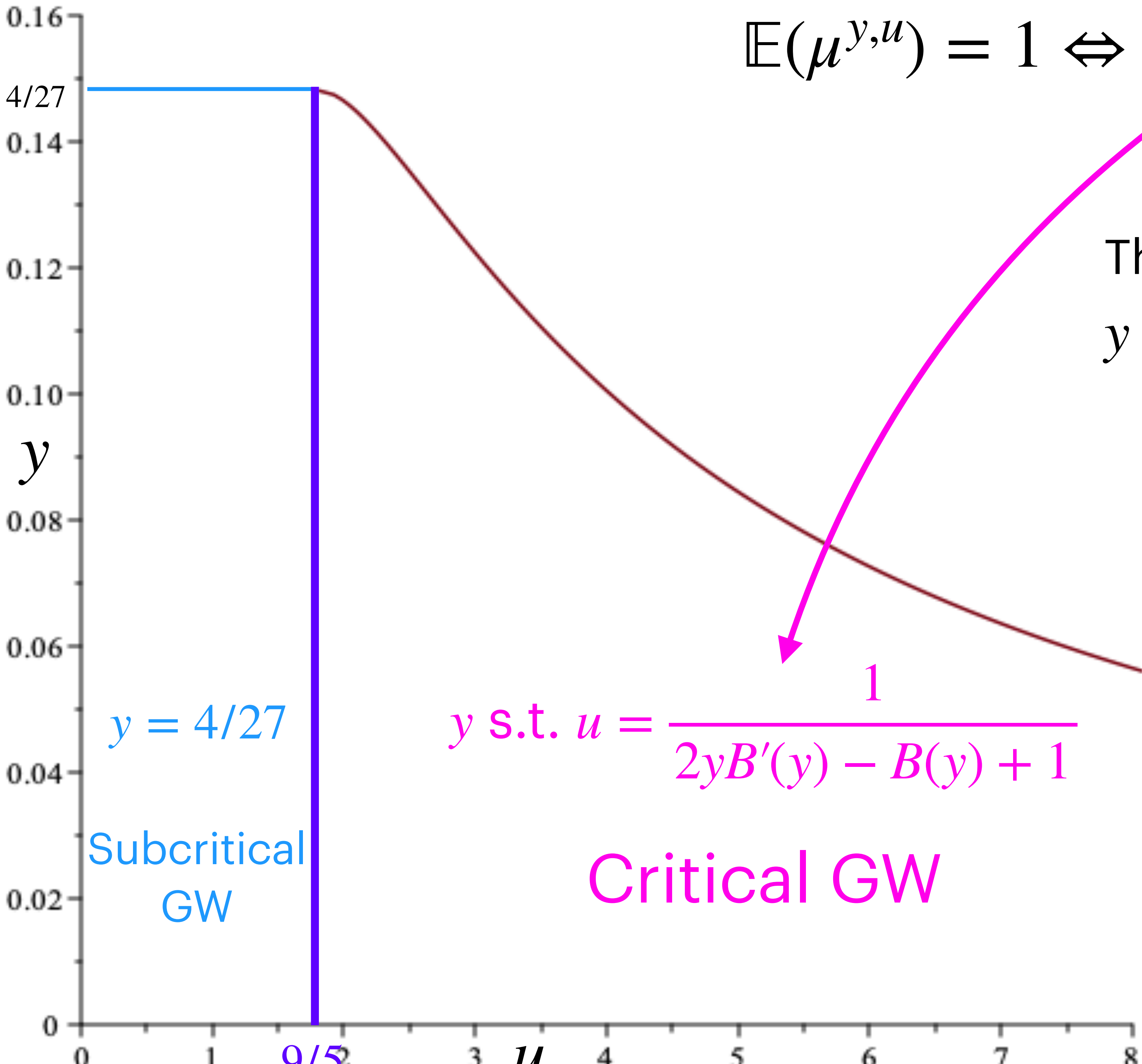
$T_n^{y,u}$ and T_n^u have the same law.

=> We study $T_n^{y,u}$ for a well chosen y

Critical point

When is $T_n^{y,u}$ critical? (= $\mathbb{E}(\mu) = 1$?)

$$\mathbb{E}(\mu^{y,u}) = 1 \Leftrightarrow u = \frac{1}{2yB'(y) - B(y) + 1}$$



This describes $[9/5, +\infty)$ when y describes $(0, \rho_B = 4/27]$.

$u_C = 9/5$

"Map regime"

"Tree regime"

II. Largest blocks

Properties of T_n^u

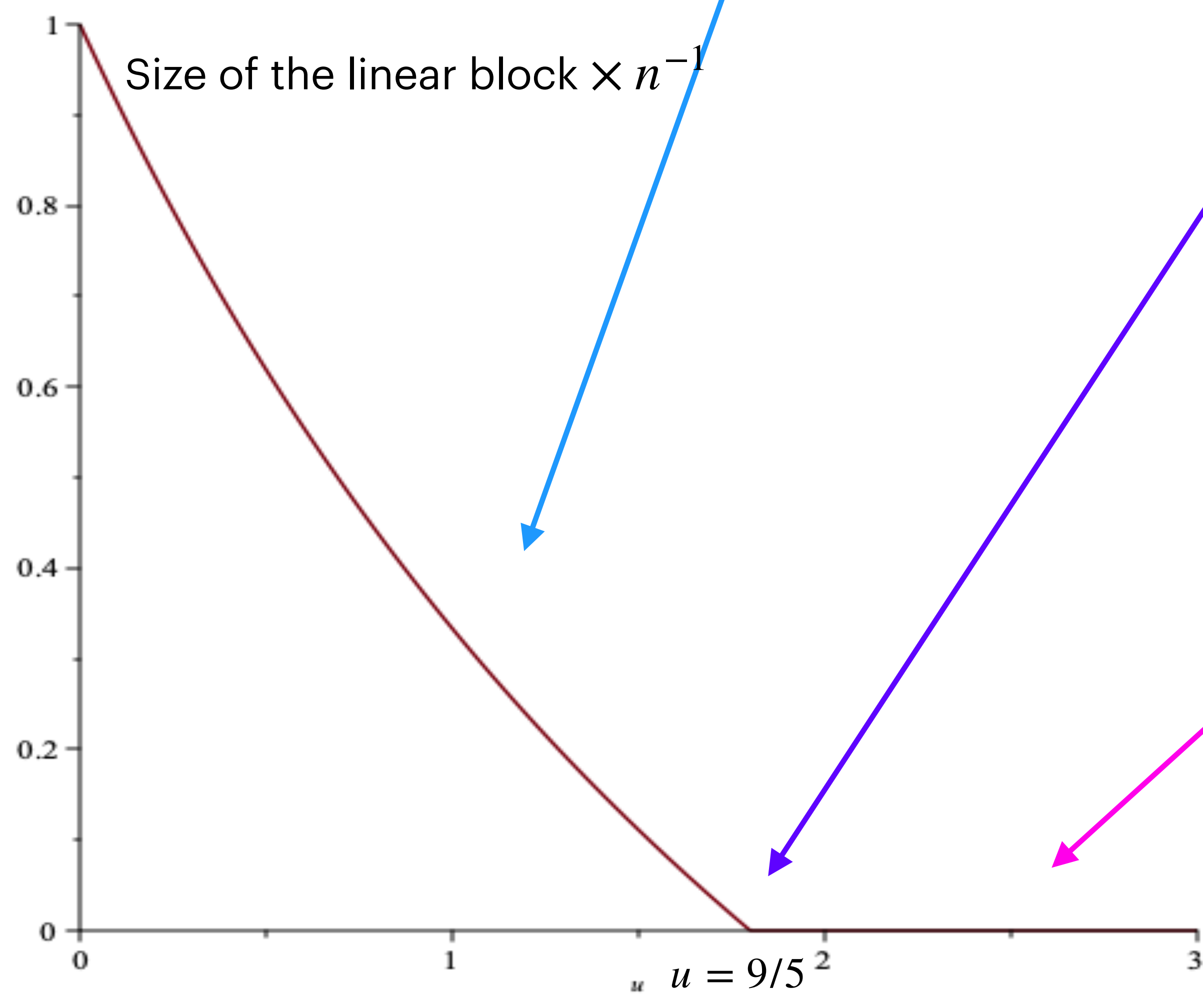
	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
Variance	∞		$< \infty$
Galton-Watson tree	subcritical	critical	

Tool: Janson 2012 = extensive study of the degrees in Galton-Watson trees

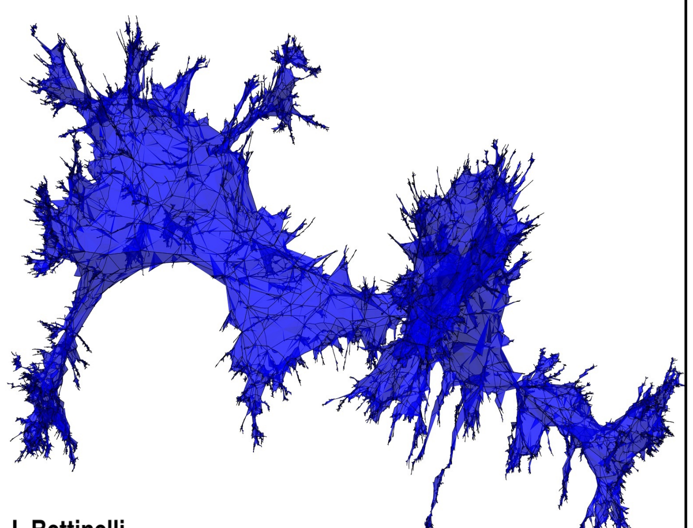
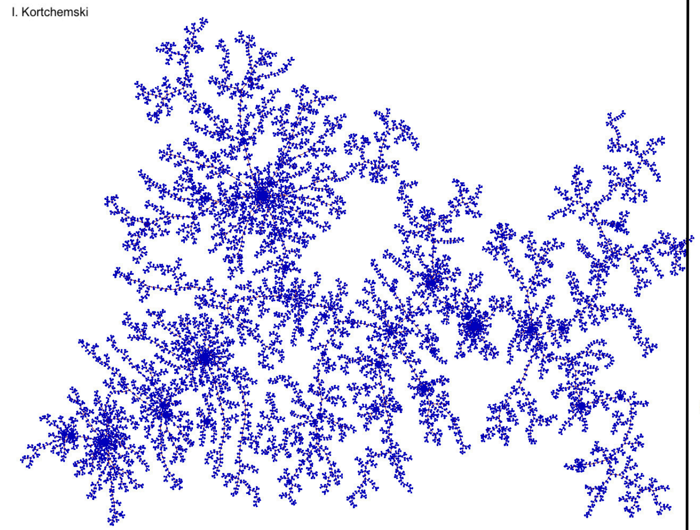
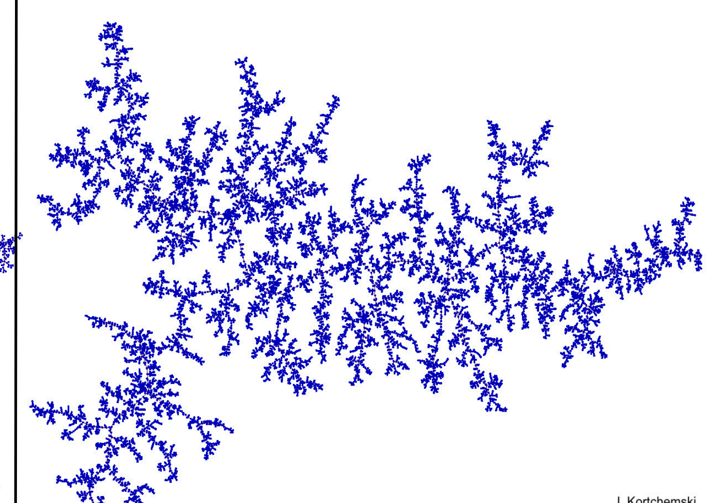
Properties on trees give properties of maps.

Size $L_{n,k}$ of the k-th largest block

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ Stufler 2020	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$ Stufler 2020		



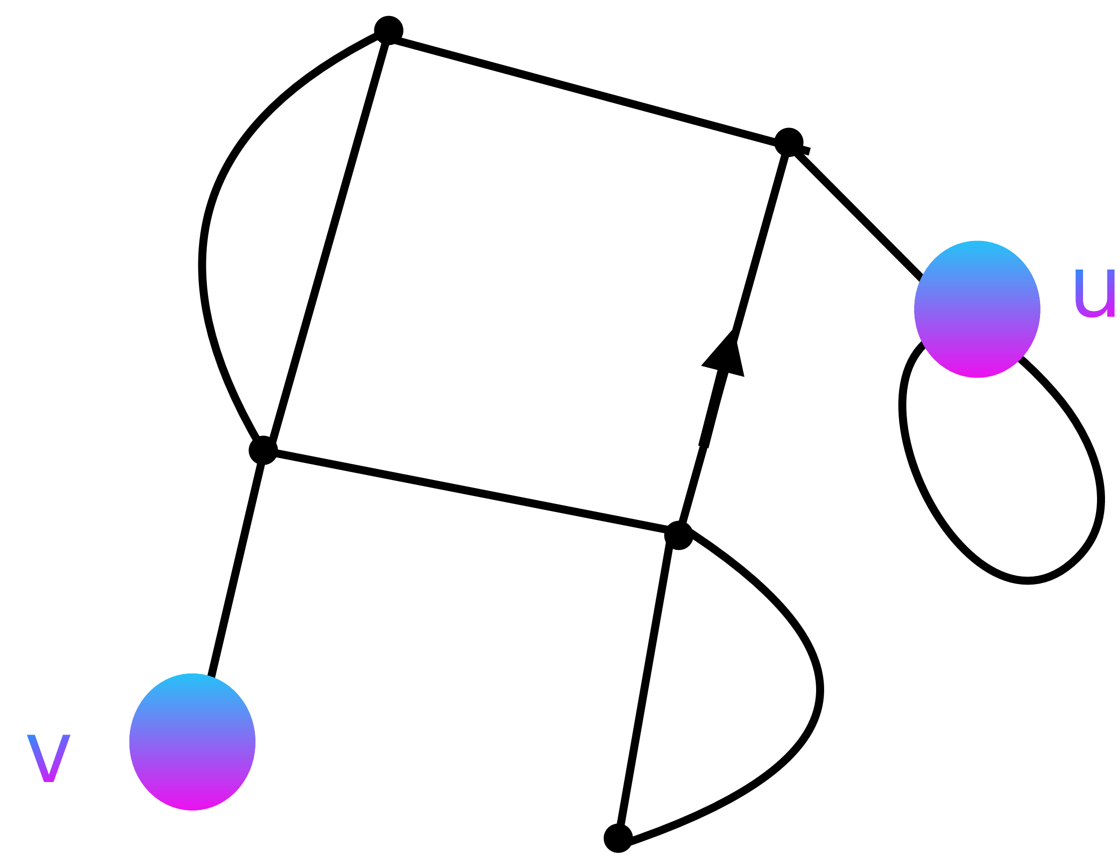
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration	$\rho(u)^n n^{-5/2}$	$\rho(u)^n n^{-5/3}$	$\rho(u)^n n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ Stufler 2020	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n (up to constant factors)	$\frac{1}{n^{1/4}} M_n \to \mathcal{S}_e$  J. Bettinelli Assuming the convergence of 2-connected maps towards the brownian sphere	$\frac{1}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ I. Kortchemski 	$\frac{1}{n^{1/2}} M_n \to \mathcal{T}_e$ Stufler 2020  I. Kortchemski

III. Scaling limits

Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.



$d(u, v) = 4$

$M_n \hookrightarrow \mathbb{P}_{n,u}$

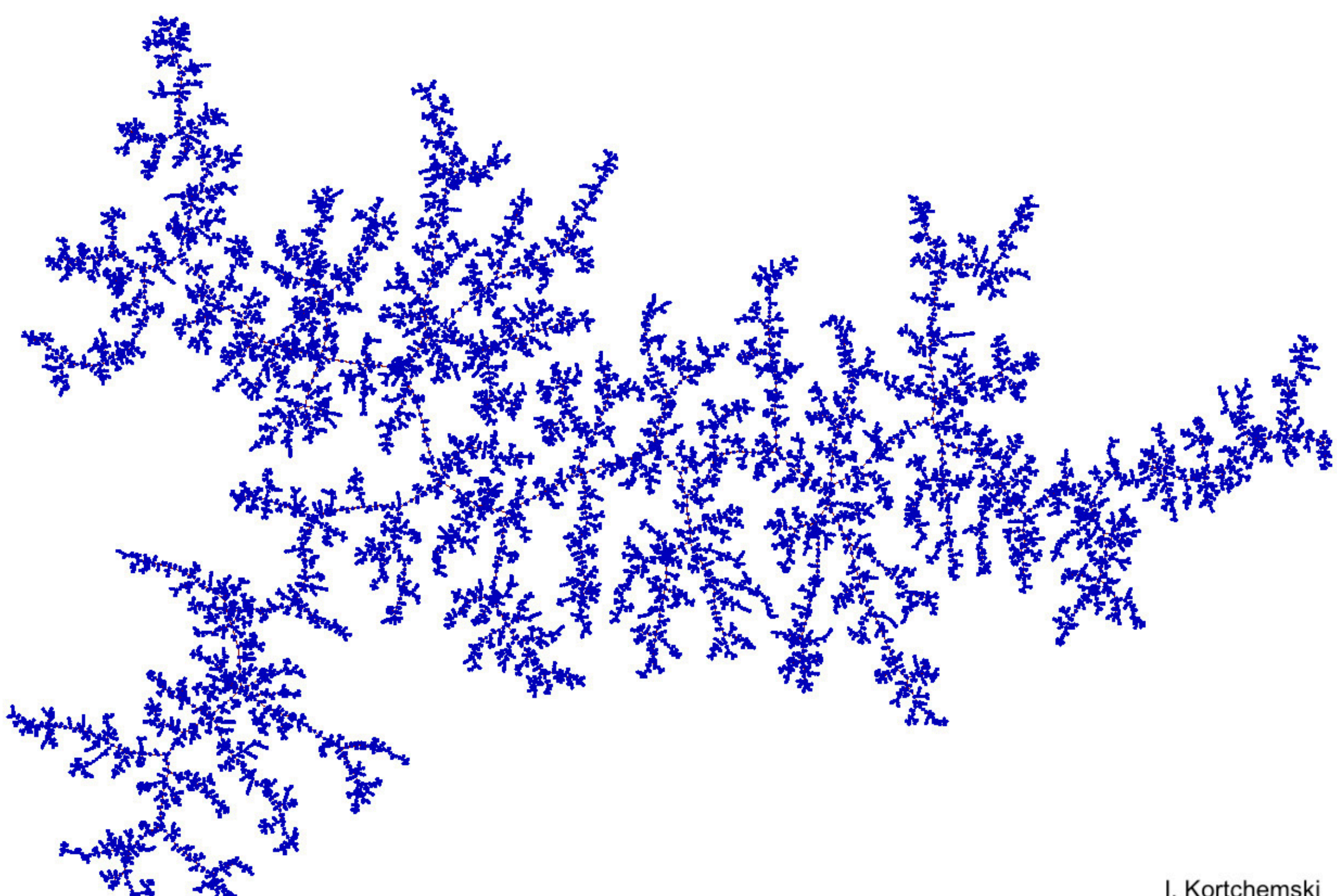
What is the limit of the sequence of metric spaces $((M_n,\ d/n^{?}))_{n\in\mathbb{N}}$?

(Convergence for Gromov-Hausdorff metric)

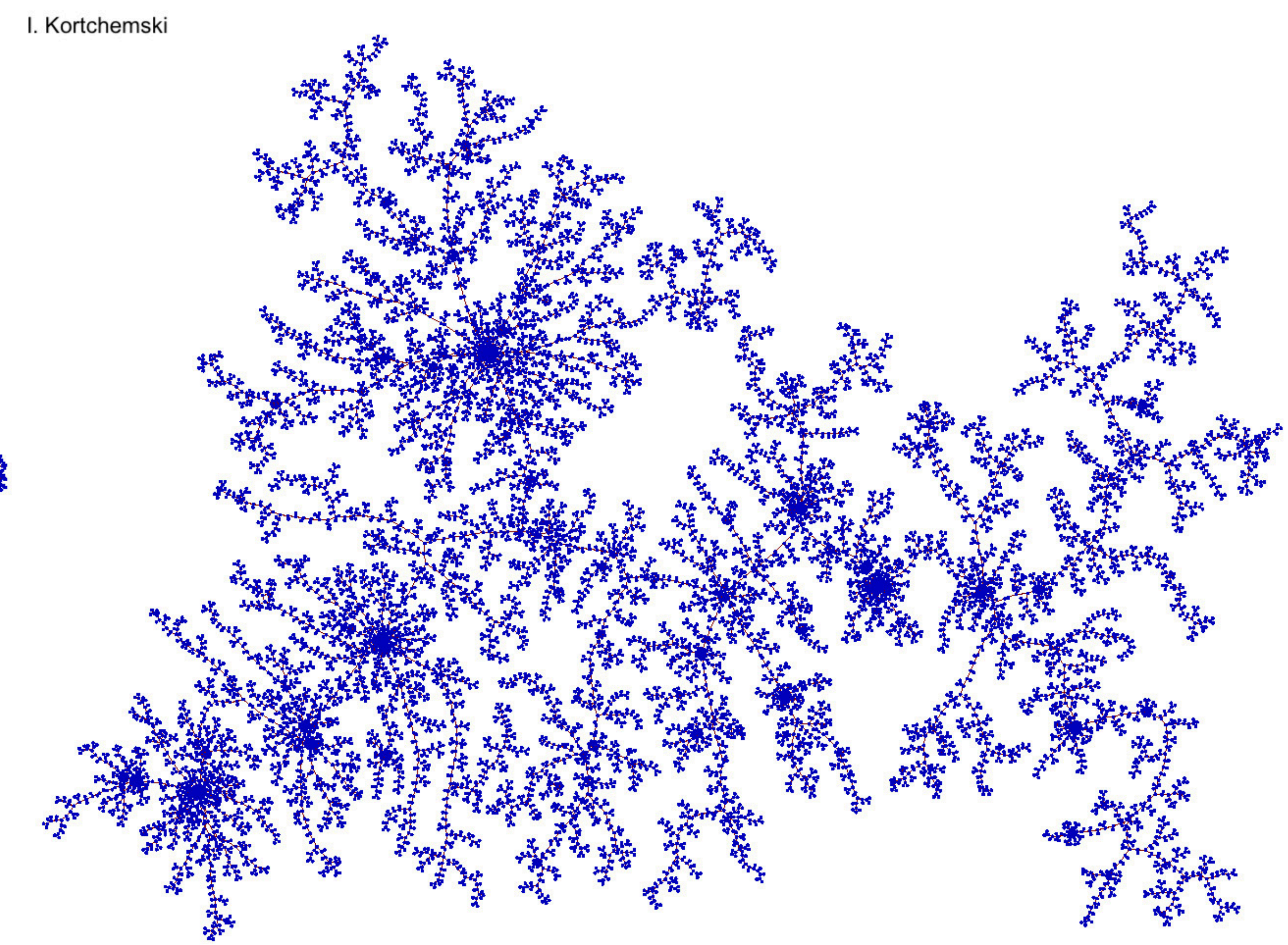
Scaling limits of Galton-Watson trees

Scaling limit of critical Galton-Watson trees

- with finite variance : Brownian tree, rescaling in $n^{-1/2}$ [Aldous 1993, Le Gall 2006];
- with infinite variance and nice tails: Stable tree [Duquesne 2003].



Brownian tree $\mathscr{T}_e$



Stable tree 3/2 $\mathscr{T}_{3/2}$

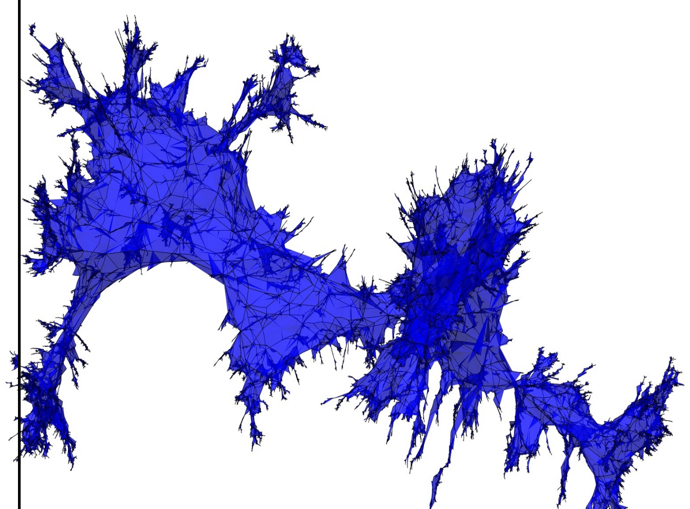
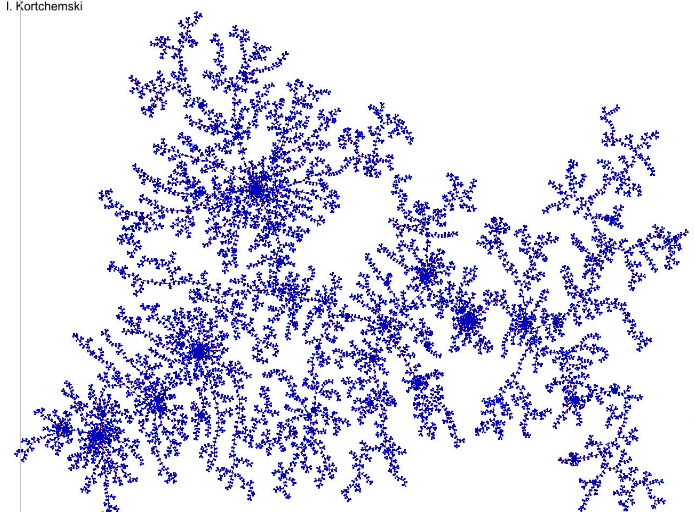
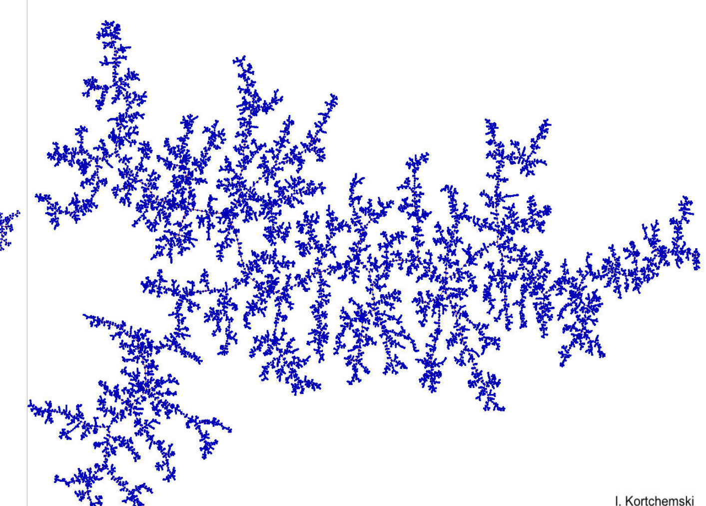
Scaling limits of block trees

Scaling limit of critical Galton-Watson trees

- with finite variance : Brownian tree, rescaling in $n^{-1/2}$ [Aldous 1993, Le Gall 2006];
- with infinite variance and nice tails: Stable tree [Duquesne 2003].

	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
Variance	∞		$< \infty$
Galton-Watson tree	subcritical	critical	
Scaling limit of T_{M_n} (up to constant factors)		$\frac{1}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}$	$\frac{1}{n^{1/2}} T_{M_n} \to \mathcal{T}_e$

Scaling limits of maps

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Scaling limit of T_{M_n}		$\frac{1}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}$	$\frac{1}{n^{1/2}} T_{M_n} \to \mathcal{T}_e$
Scaling limit of M_n	$\frac{1}{n^{1/4}} M_n \to \mathcal{S}_e$  J. Bettinelli Assuming the convergence of 2-connected maps towards the brownian sphere	$\frac{1}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ I. Kortchemski 	$\frac{1}{n^{1/2}} M_n \to \mathcal{T}_e$ Stufler 2020 

Up to constant factors

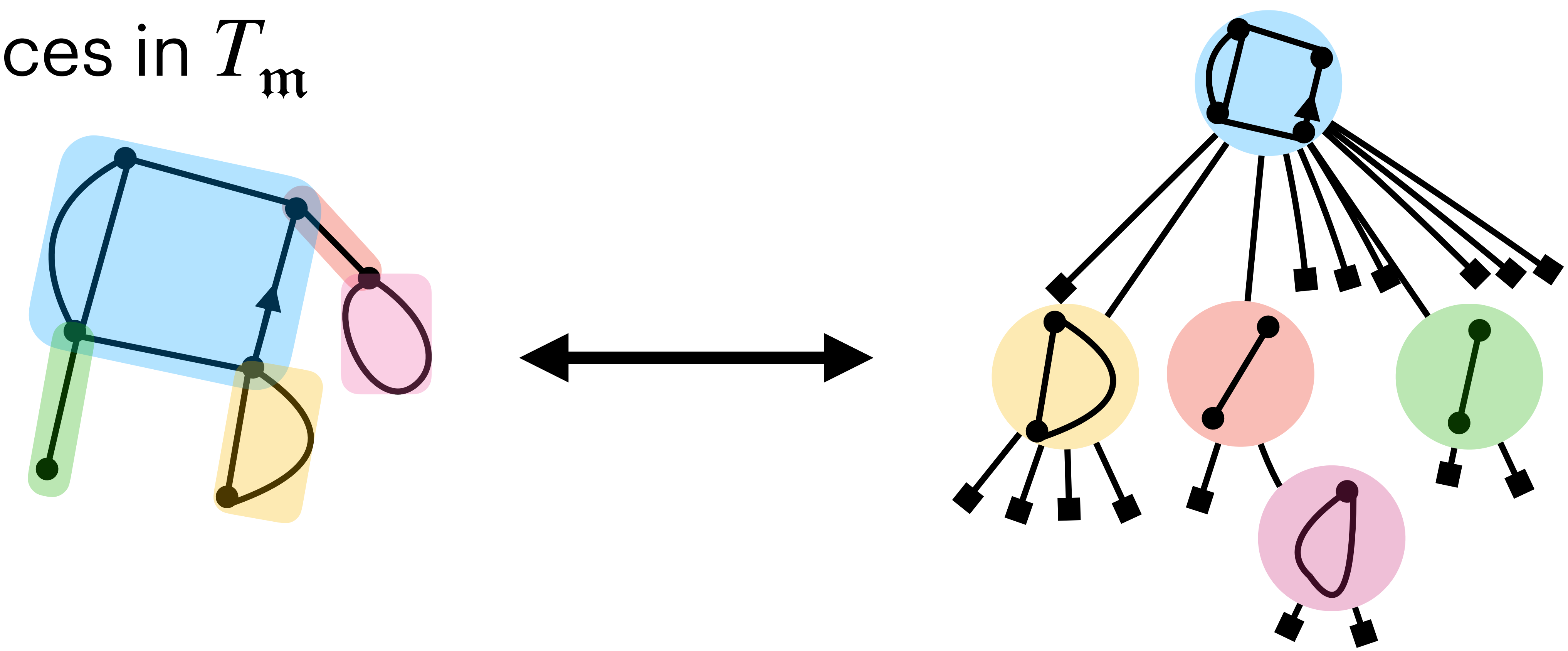
See [Addario-Berry, Wen 2019] for a similar result and method

Brownian sphere = map

(Very) rough idea of the proofs

Supercritical and critical cases

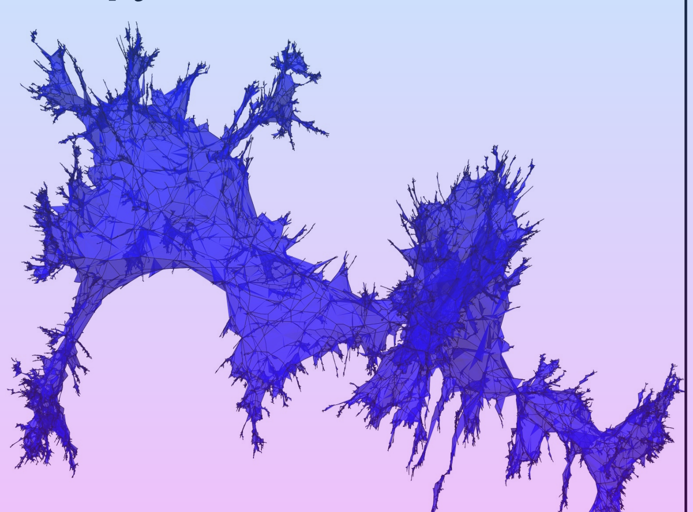
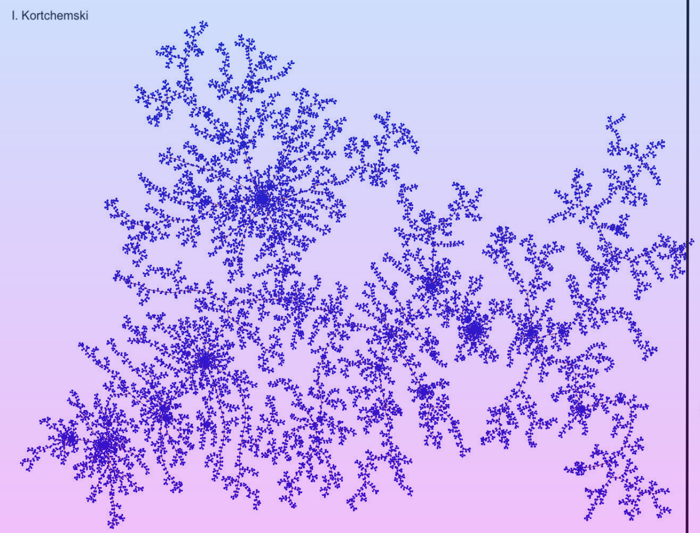
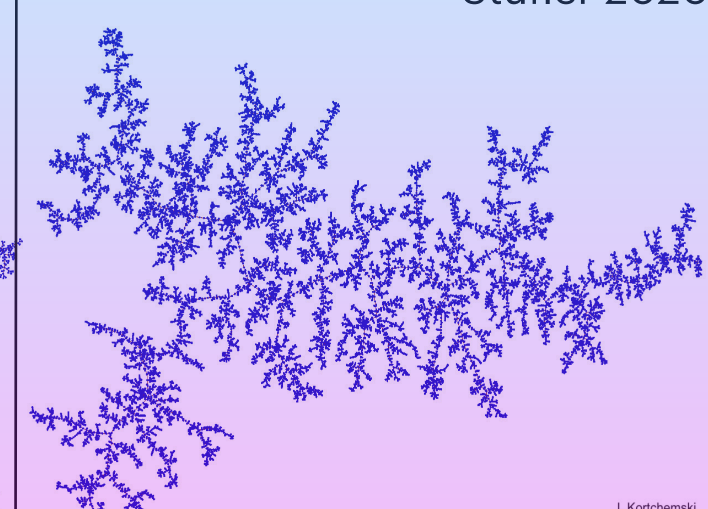
- Difficult part = show that distances in $\mathfrak{m}$ behave like distances in $T_{\mathfrak{m}}$



Subcritical case

- One big block and decorations = scaling limit is the scaling limit of 2-connected maps = brownian sphere (admitted)

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration	$\rho(u)^n n^{-5/2}$	$\rho(u)^n n^{-5/3}$	$\rho(u)^n n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ Stufler 2020	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n (up to constant factors)	$\frac{1}{n^{1/4}} M_n \to \mathcal{S}_e$  Assuming the convergence of 2-connected maps towards the brownian sphere	$\frac{1}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ 	$\frac{1}{n^{1/2}} M_n \to \mathcal{T}_e$ Stufler 2020 

IV. Conclusion

Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n)$, $\varepsilon^3 n \to \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, conjecture $L_{n,1} \sim 2{,}7648\varepsilon^{-2}\ln(\varepsilon^3 n)$ when $\varepsilon^3 n \to \infty$ (analogous to Bollobás's 1984 result for Erdős-Rényi graphs!);
- Results exist for scaling limits in ER graphs (Addario-Berry, Broutin, Goldschmidt 2010), open question in our case.

Is there a critical window? If so, what is its width?

Perspectives

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u_n = 9/5 - \varepsilon(n)$ $\varepsilon^3 n \to \infty$	$u = 9/5$	$u_n = 9/5 + \varepsilon(n)$ $\varepsilon^3 n \to \infty$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$		$\Theta(n^{2/3})$	$\sim 2,7648\varepsilon^{-2}\ln(\varepsilon^3 n)$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$				
Scaling limit of M_n up to constant factors	$\frac{1}{n^{1/4}}M_n \to \mathcal{S}_e$ Admitting the convergence of 2-connected maps towards the brownian map	$\varepsilon(n) = n^{-\alpha}$ $\frac{1}{n^{(1-\alpha)/4}}M_n \to \mathcal{S}_e$	$\frac{1}{n^{1/3}}M_n \to \mathcal{T}_{3/2}$	stable tree ?	$\frac{1}{n^{1/2}}M_n \to \mathcal{T}_e$

Pink = work in progress

Thank you!