

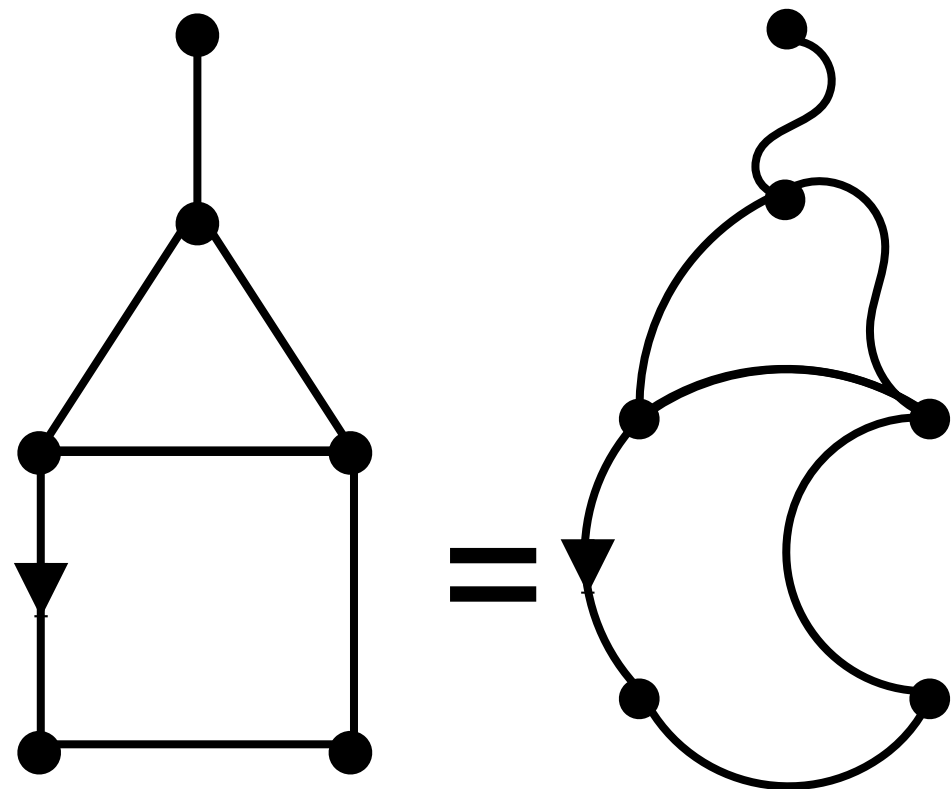
A phase transition in block-weighted random maps

Journée cartes à l'IPhT
24 juin 2022

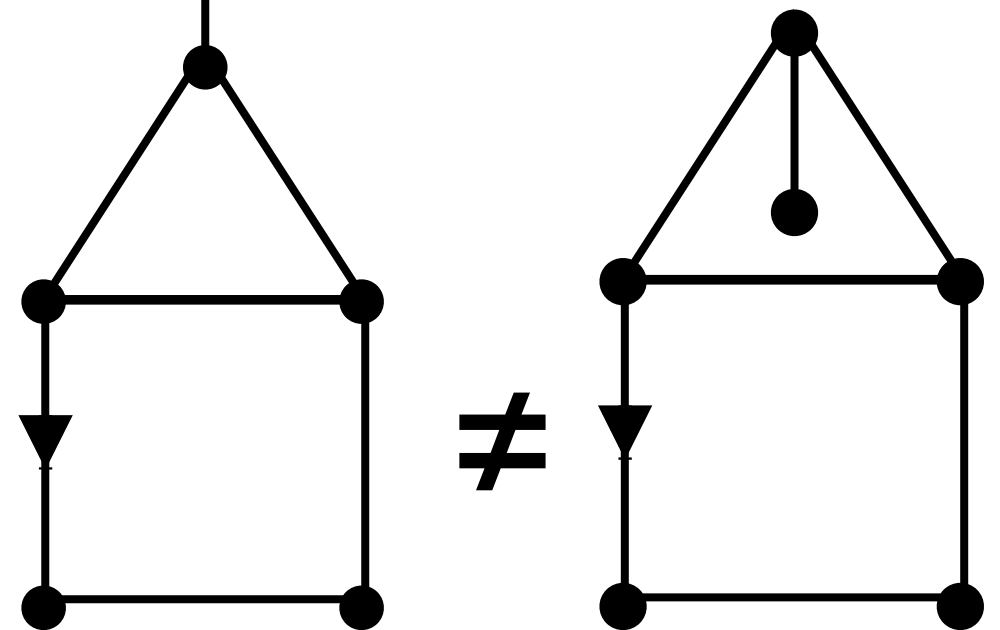
Zéphyr Salvy
William Fleurat

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



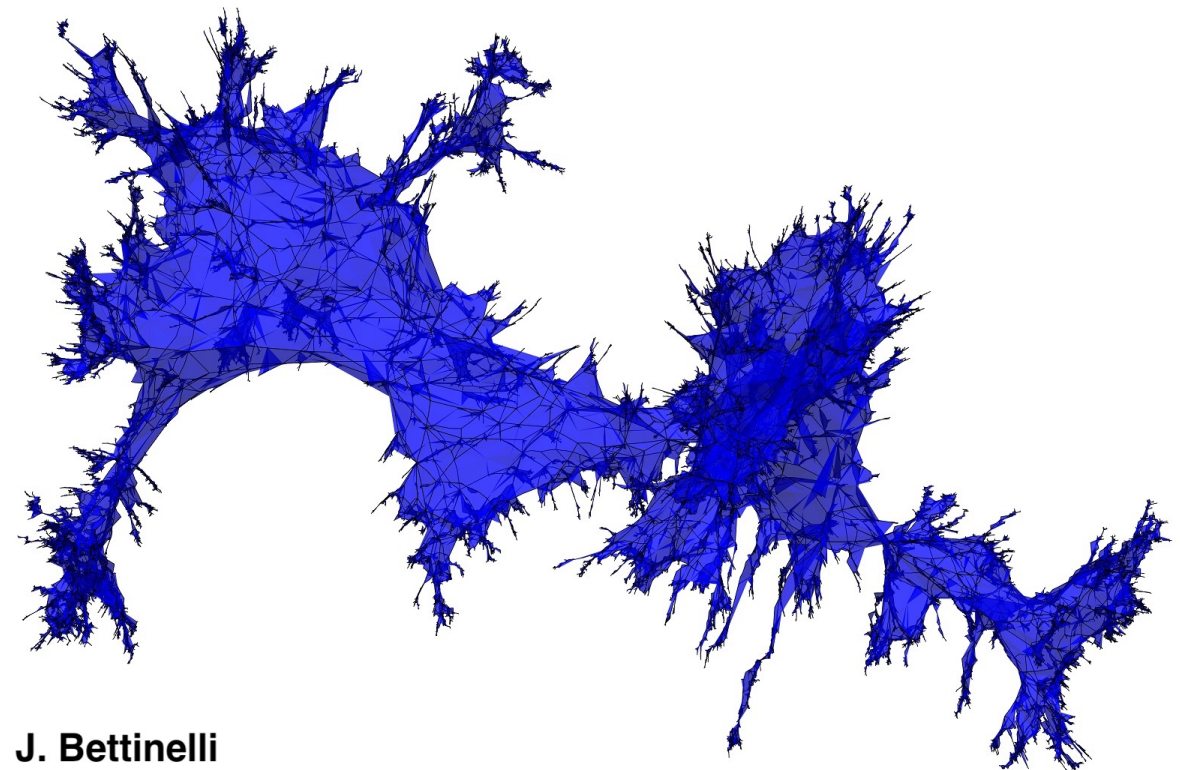
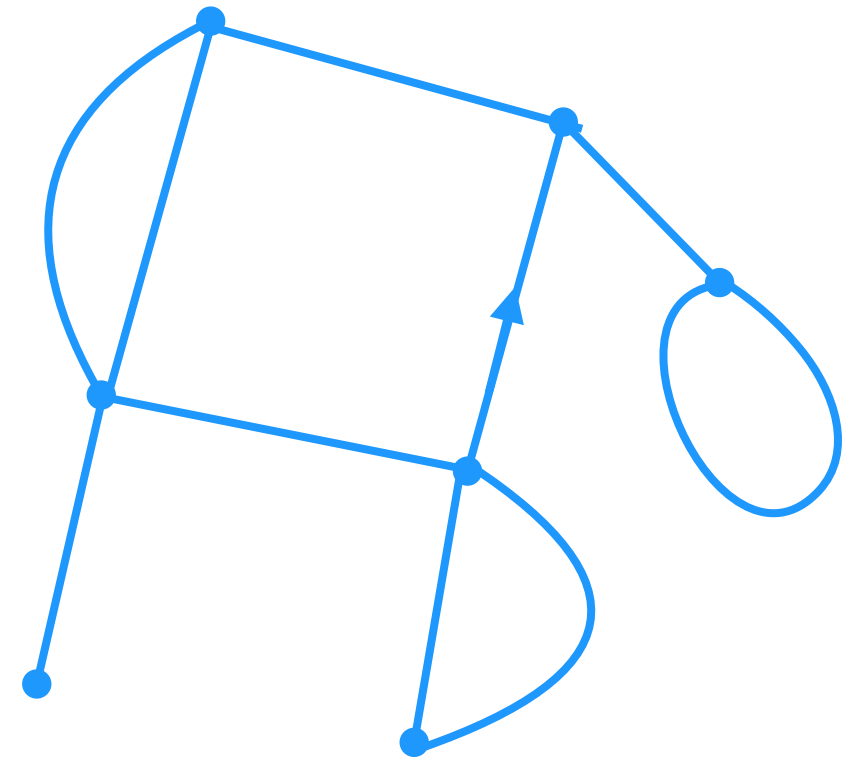
Map = graph + **cyclic order on neighbours**



- **Rooted** planar map = map endowed with a marked oriented edge (represented by an arrow);
- **Size** $|\mathfrak{m}|$ = number of edges;
- **Corner** (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Universality results for planar maps

- Enumeration: $\kappa \rho^{-n} n^{-5/2}$
[Tutte 1963, Drmota, Not, Yu 2020];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Miermont 2013], triangulations & 2q-angulations [Le Gall 2013];
- Universality:
 - Same enumeration;
 - Same scaling limit, e.g. for simple quadrangulations [Addario-Berry Albenque 2017], arbitrary maps [Bettinelli, Jacob Miermont 2014].

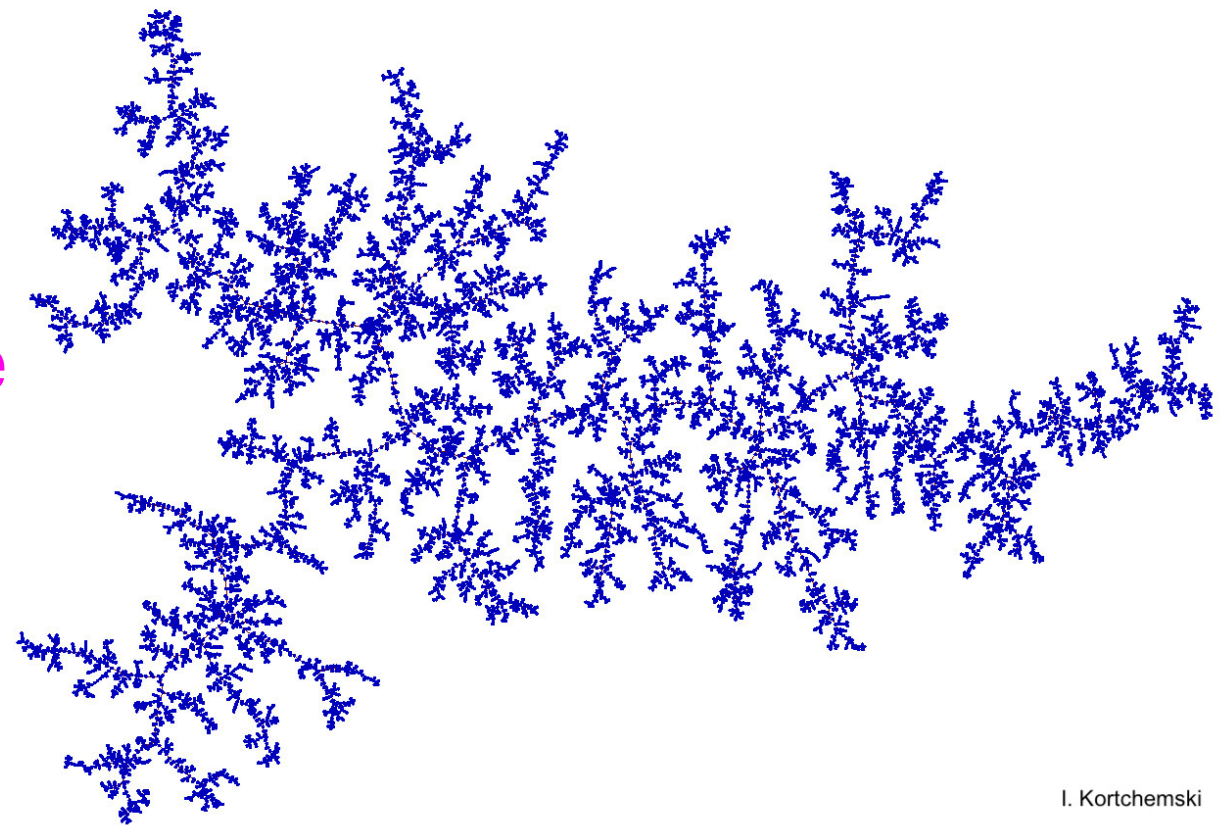
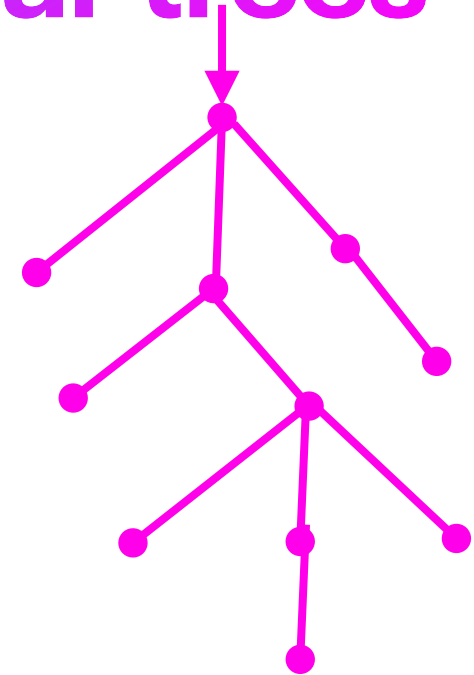


J. Bettinelli

Brownian sphere $\mathcal{S}_e$

Universality results for planar trees

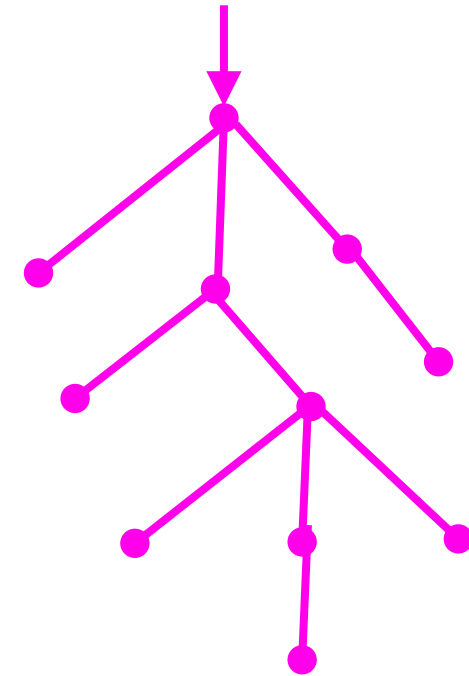
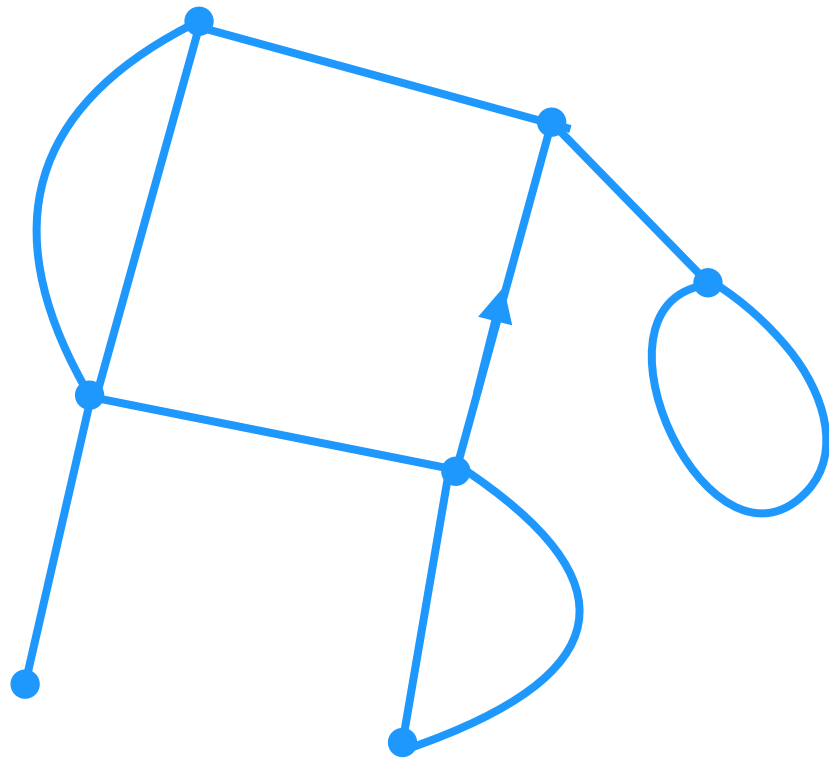
- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$
[Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];
- Universality:
 - Same enumeration,
 - Same scaling limit, even for some classes of **maps**; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size $\gg n^{1/2}$ [Bettinelli 2015].

Brownian tree $\mathcal{T}_e$

I. Kortchemski

Models with (very) constrained boundaries

Motivation



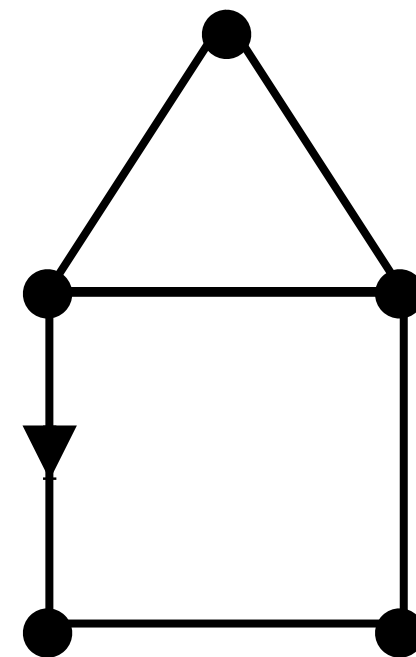
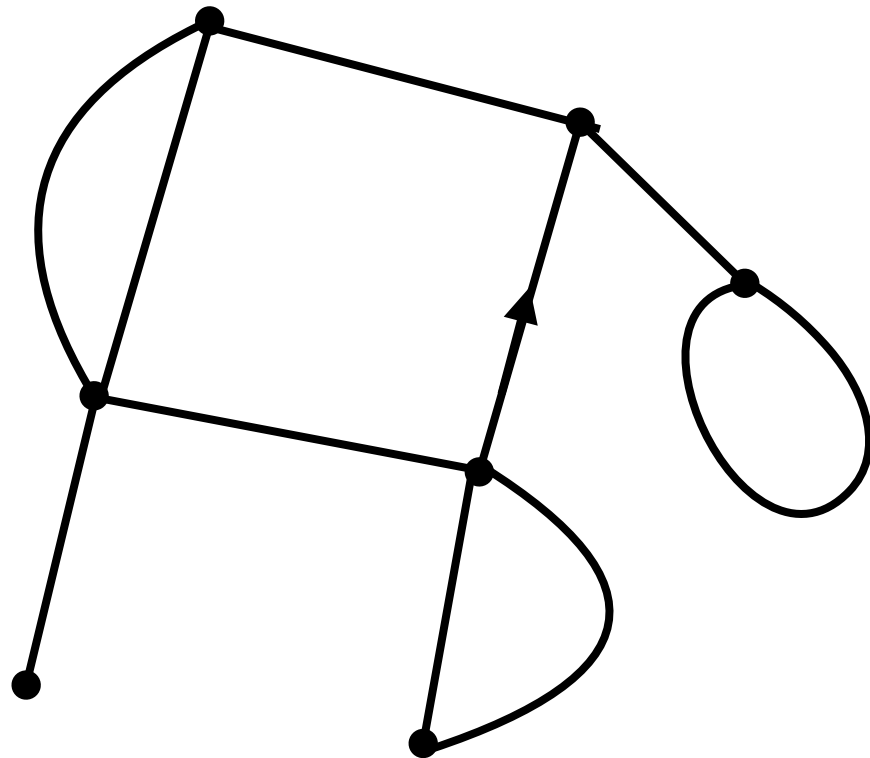
Interpolating model?

2-connectivity

Cut vertex: vertex that when removed disconnects the map

2-connected: no cut vertex (=to be able to disconnect, at least two vertices must be removed)

Block = maximal (for inclusion) 2-connected submap

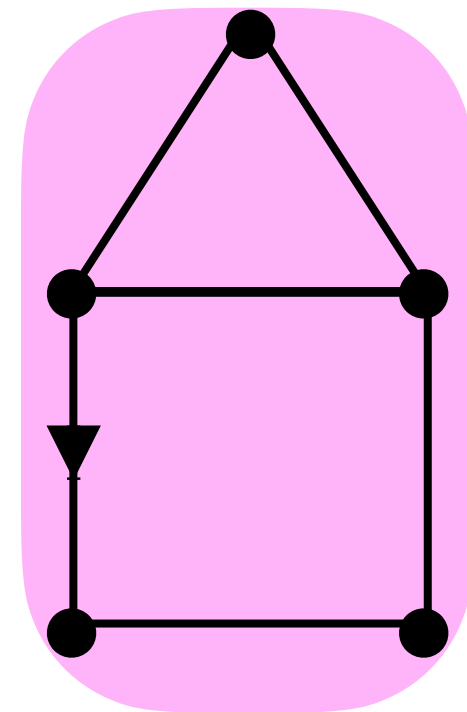
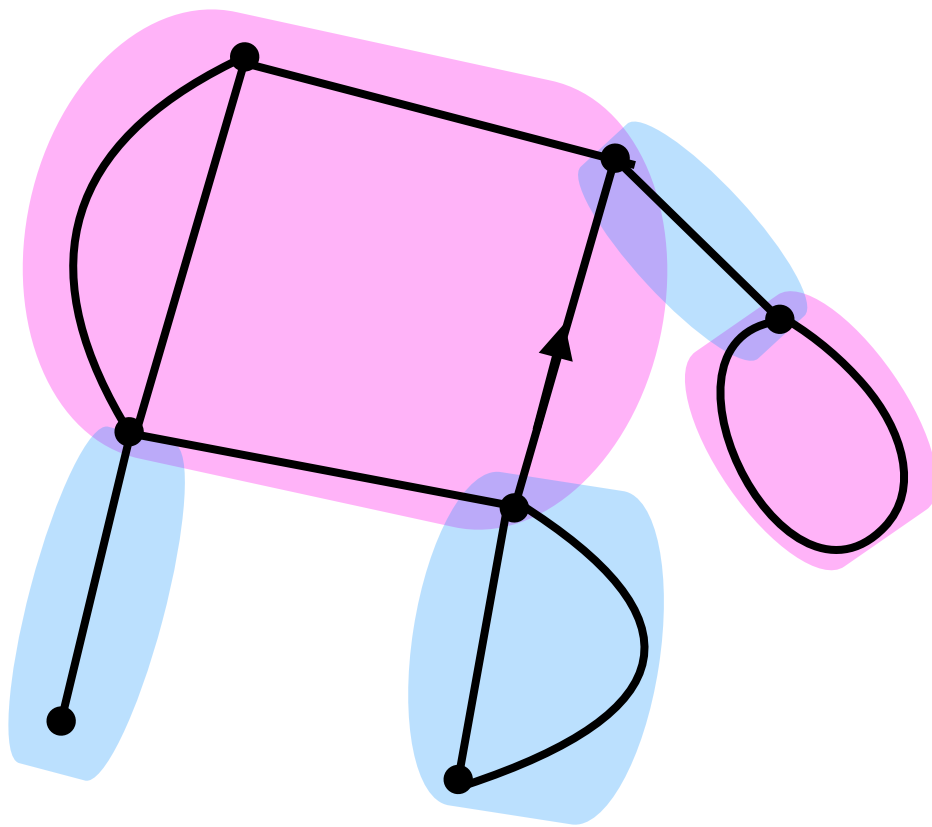


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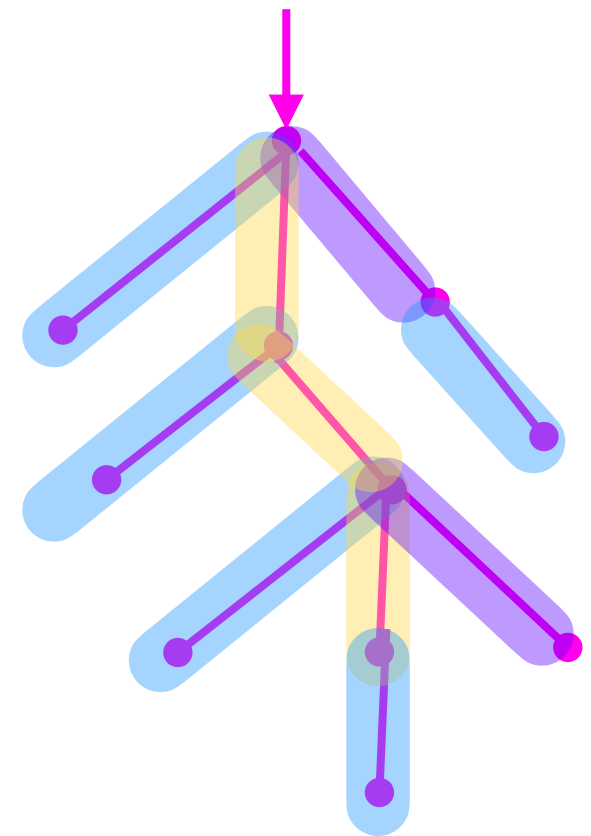
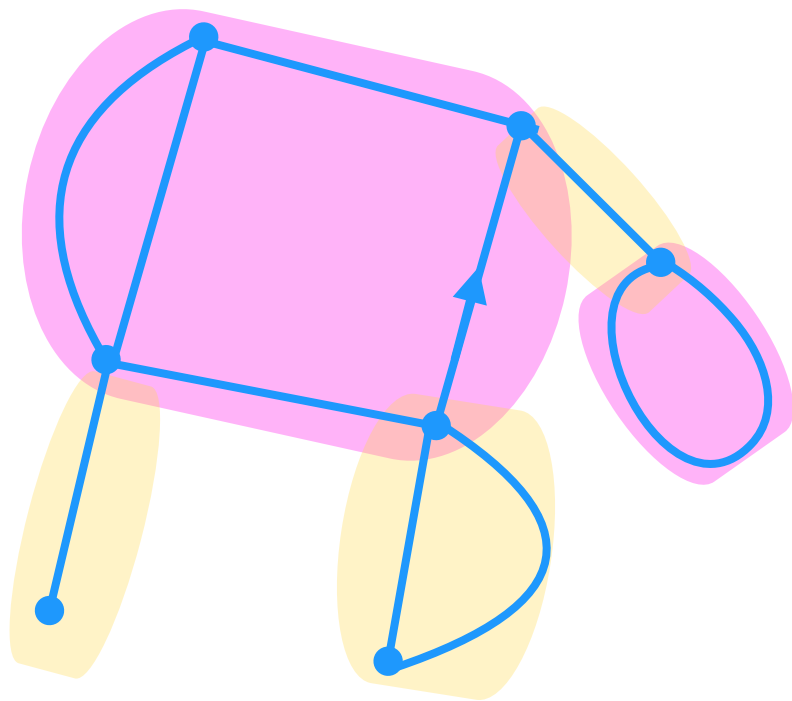
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Motivation



Only small blocks.

Interpolating model?

Outline of the talk

A phase transition in block-weighted random maps

- I. Approach
- II. Largest blocks
- III. Similar model: quadrangulations
- IV. Scaling limits
- V. Perspectives

I. Approach

Model

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0,$
 $\mathcal{M}_n = \{\text{maps of size } n\},$
 $\mathfrak{m} \in \mathcal{M}_n,$
 $Z_{n,u} = \text{normalisation}.$

Inspired by [Bonzom 2016].

- $u = 1$: uniform distribution on maps of size n ;
- $u \rightarrow 0$: minimising the number of blocks (=2-connected maps);
- $u \rightarrow \infty$: maximising the number of blocks (= trees!).

Given u , asymptotic behaviour when $n \rightarrow \infty$?

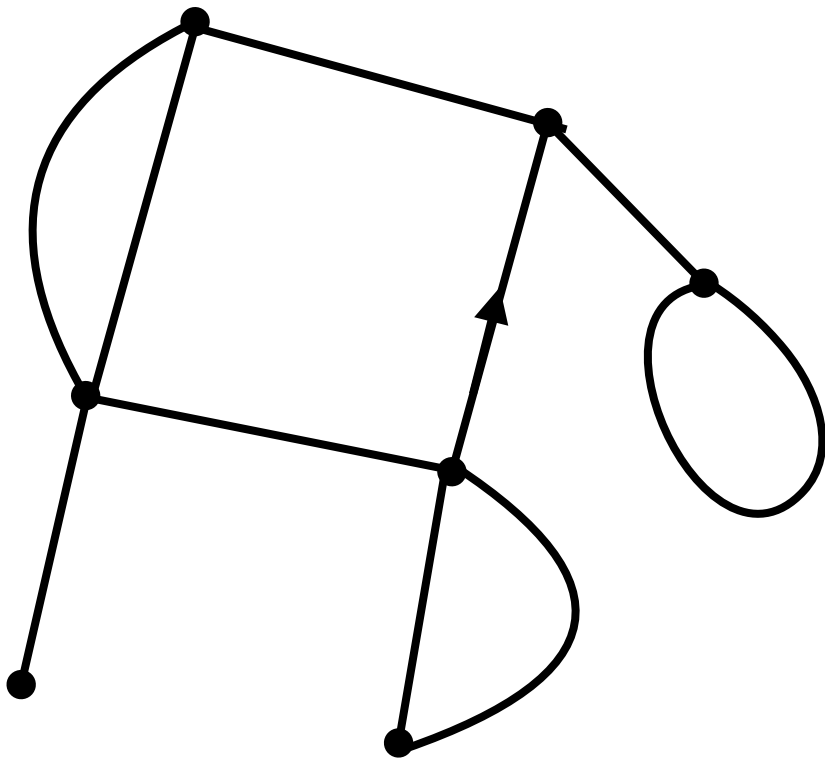
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

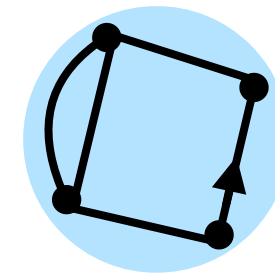
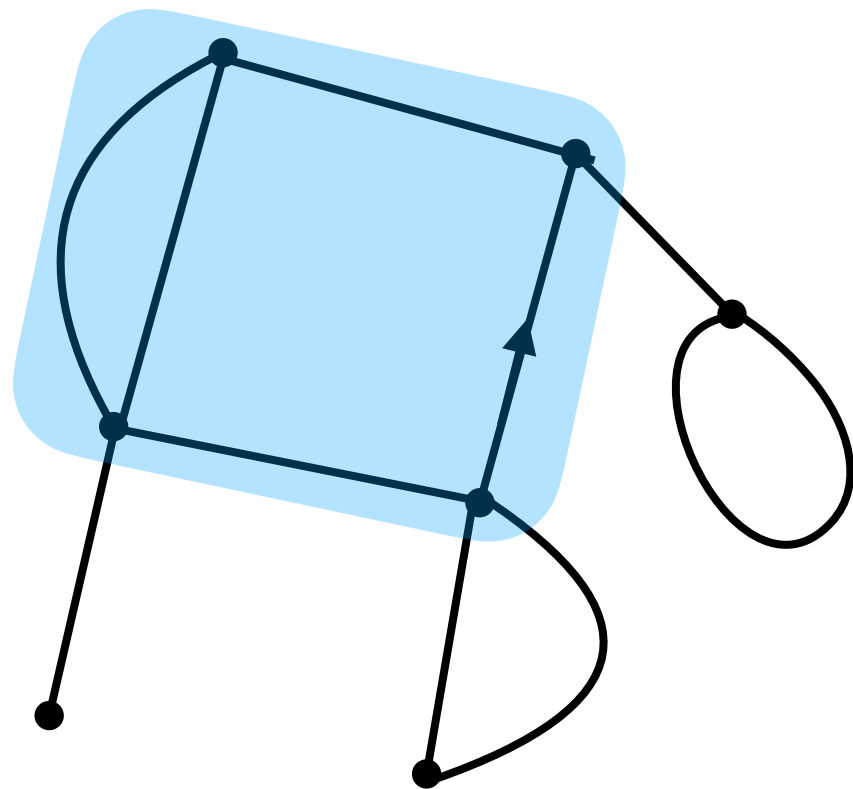
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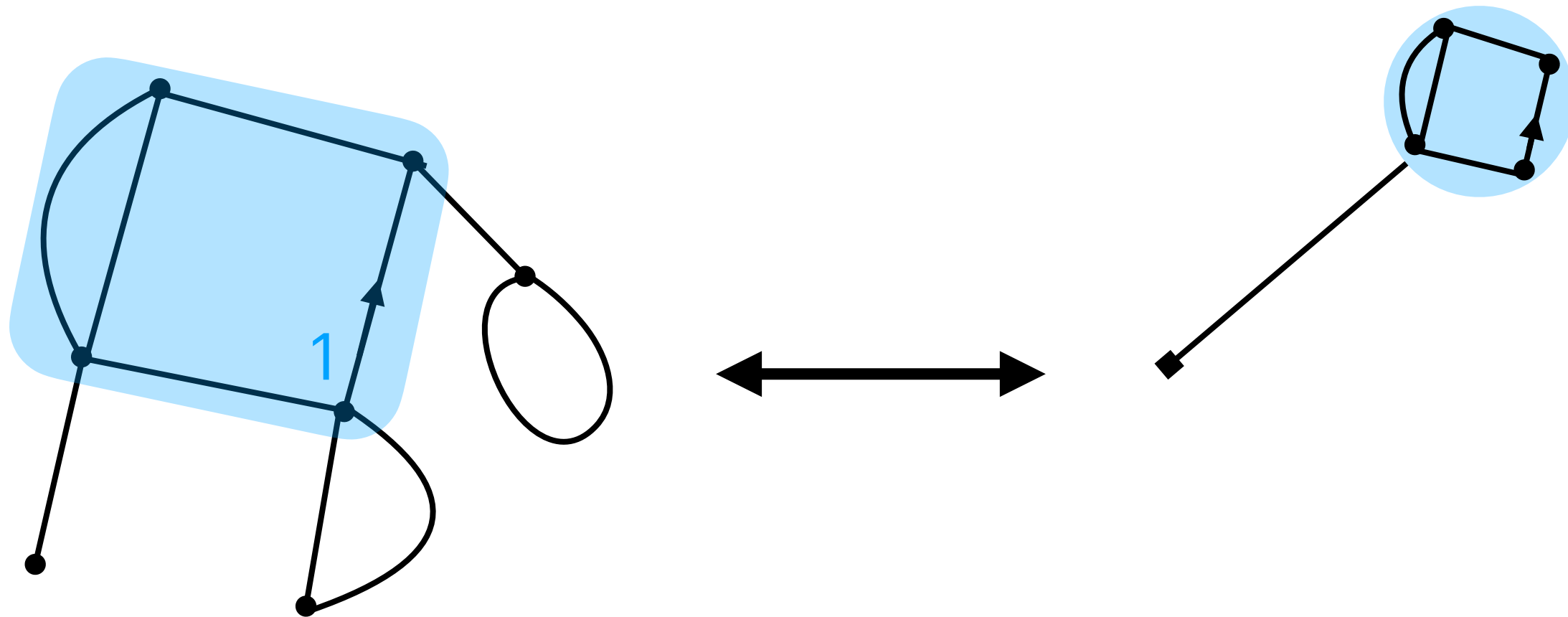
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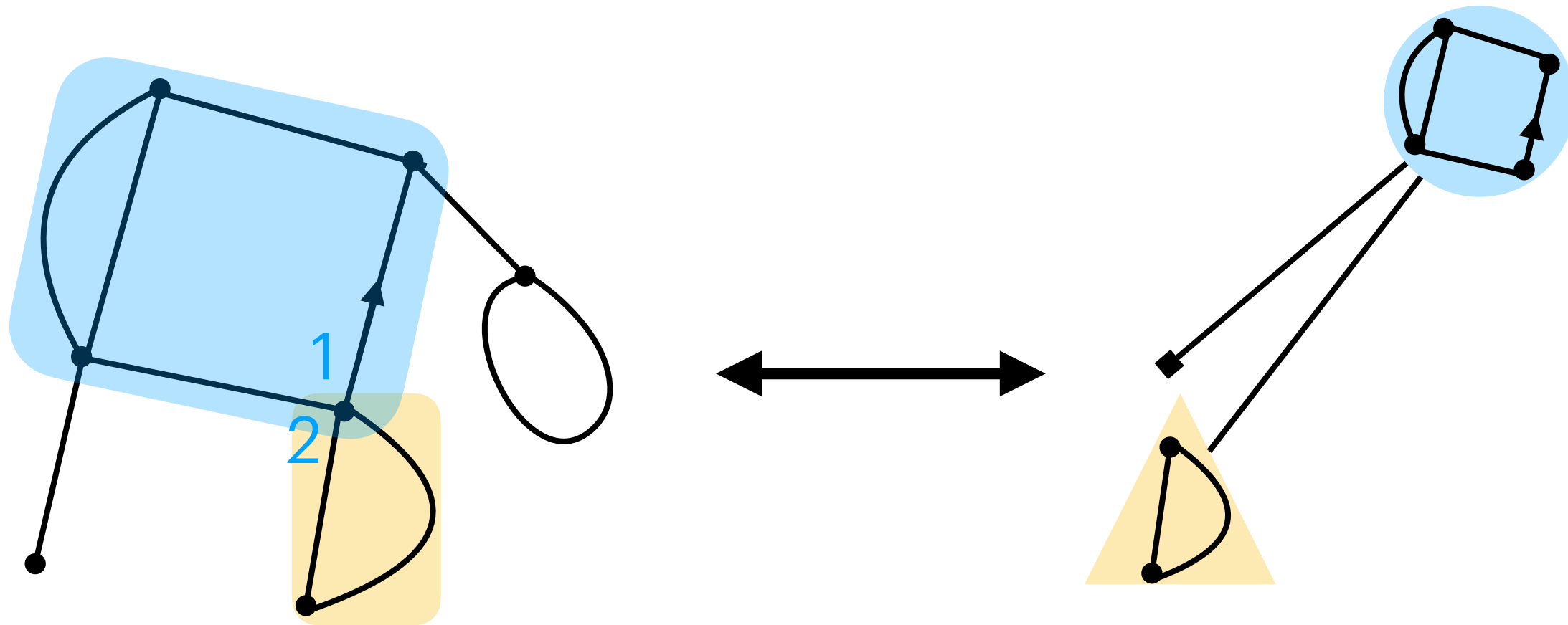
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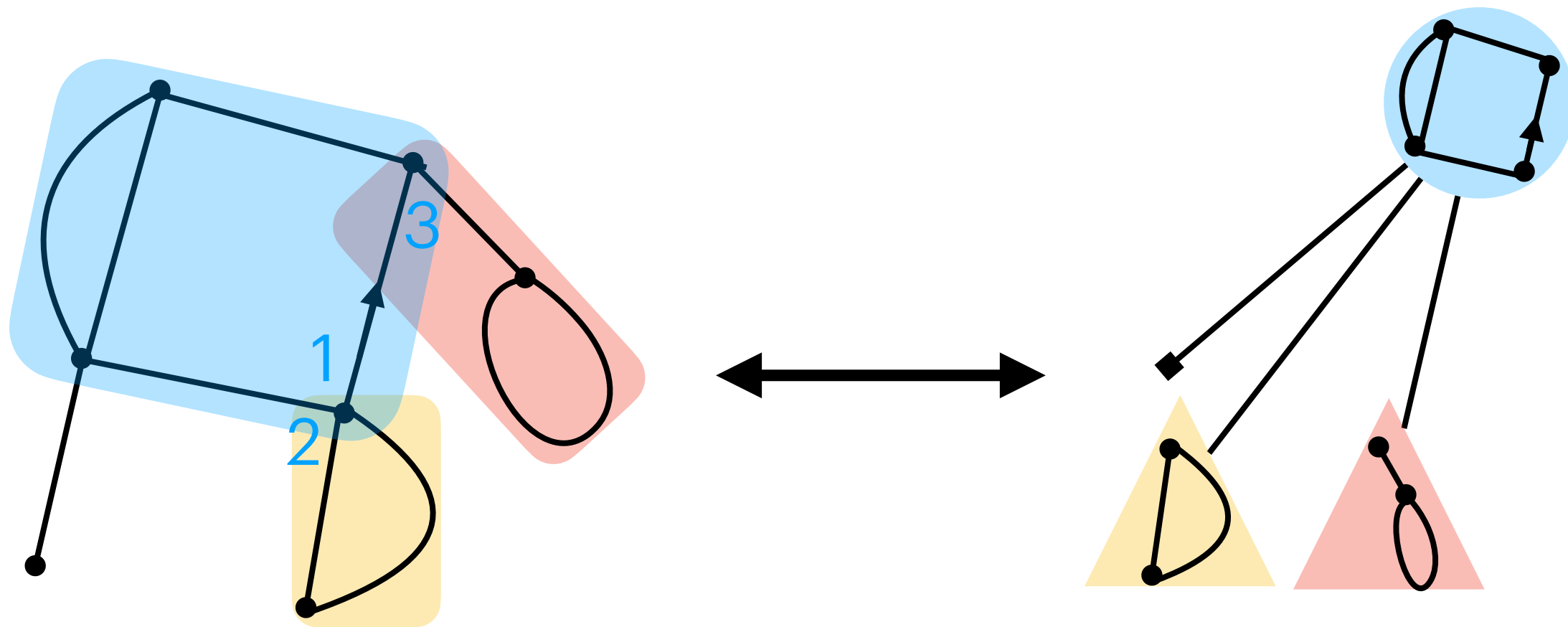
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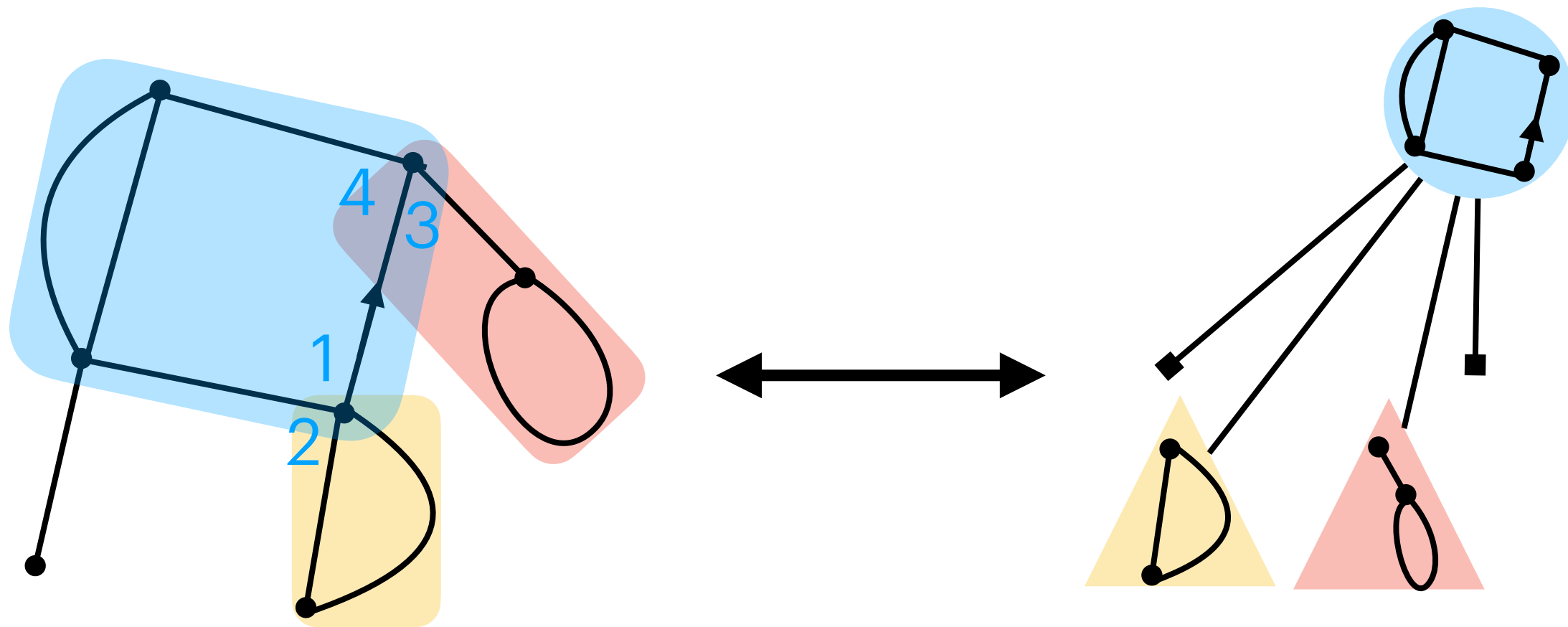
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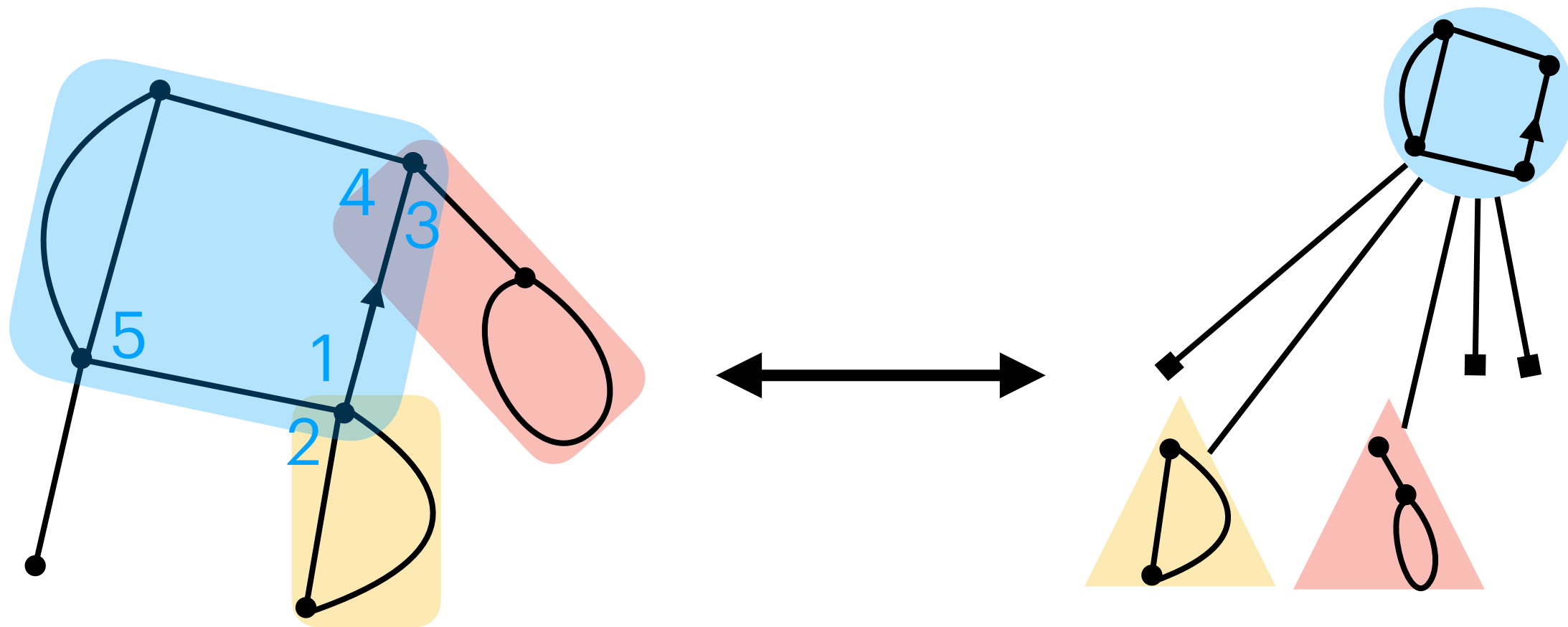
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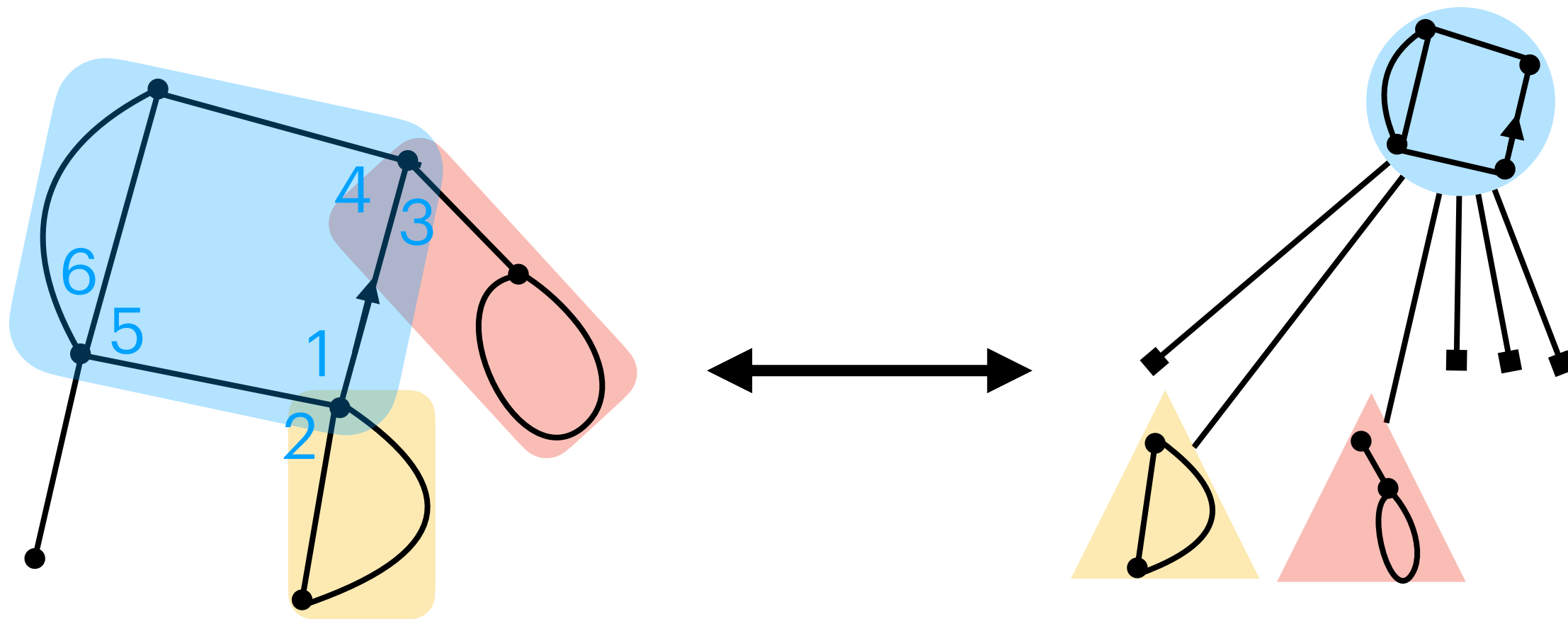
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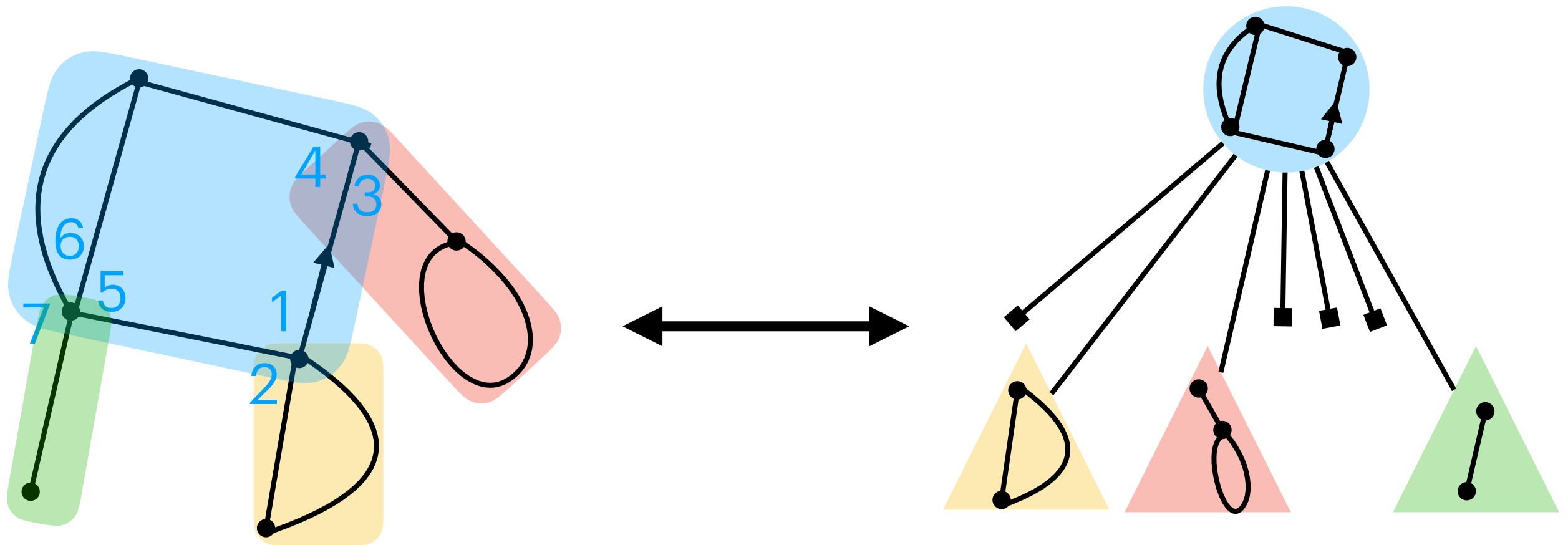
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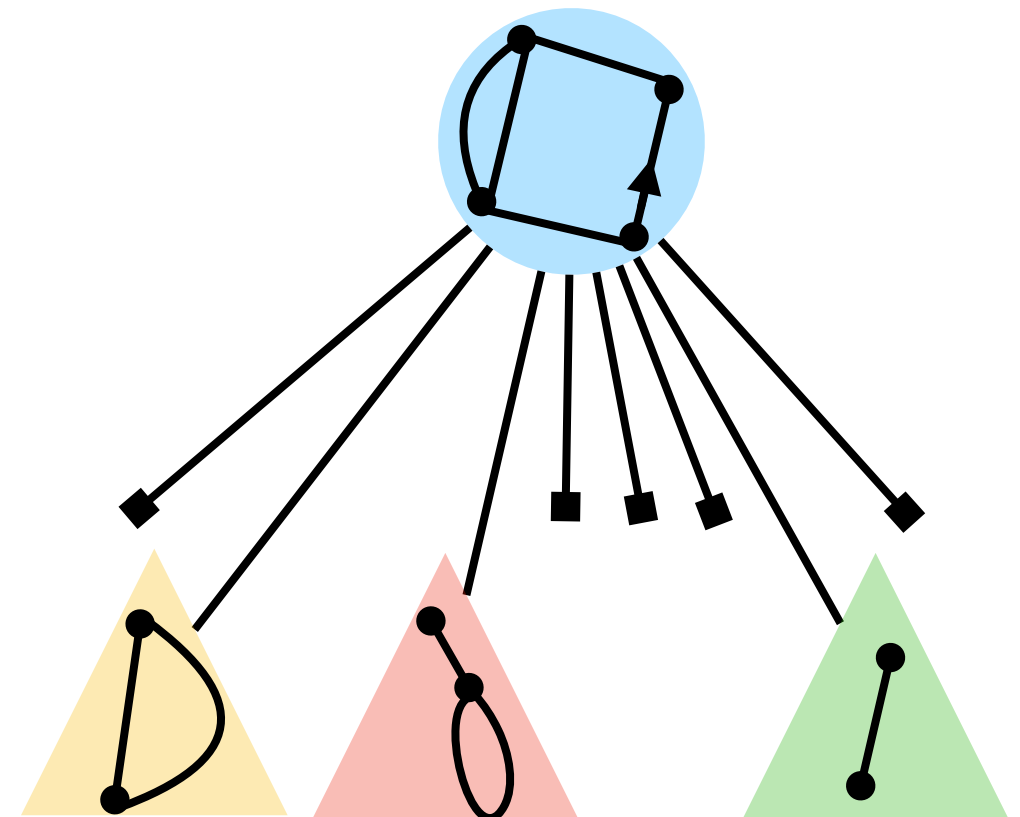
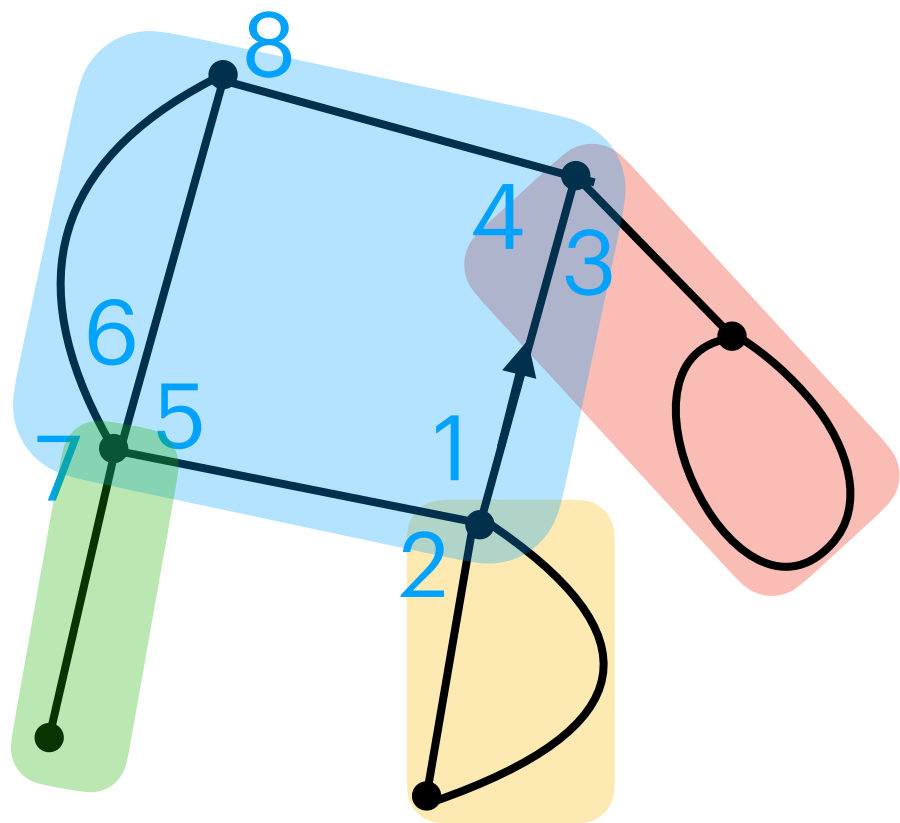
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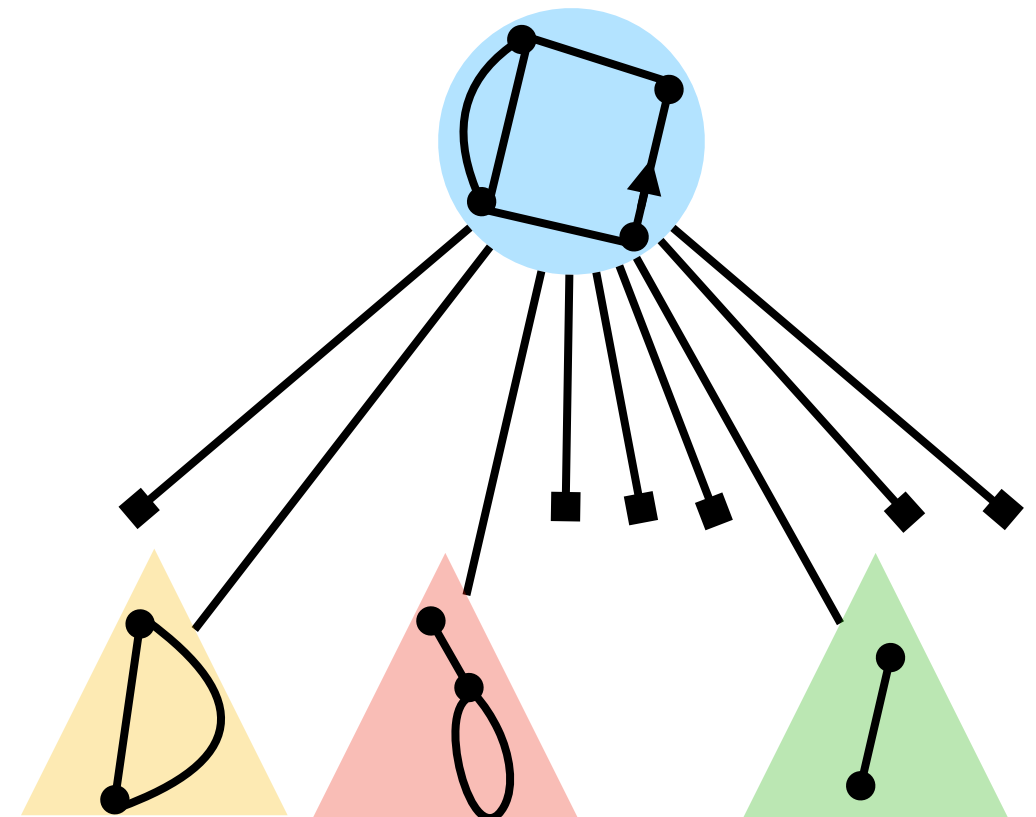
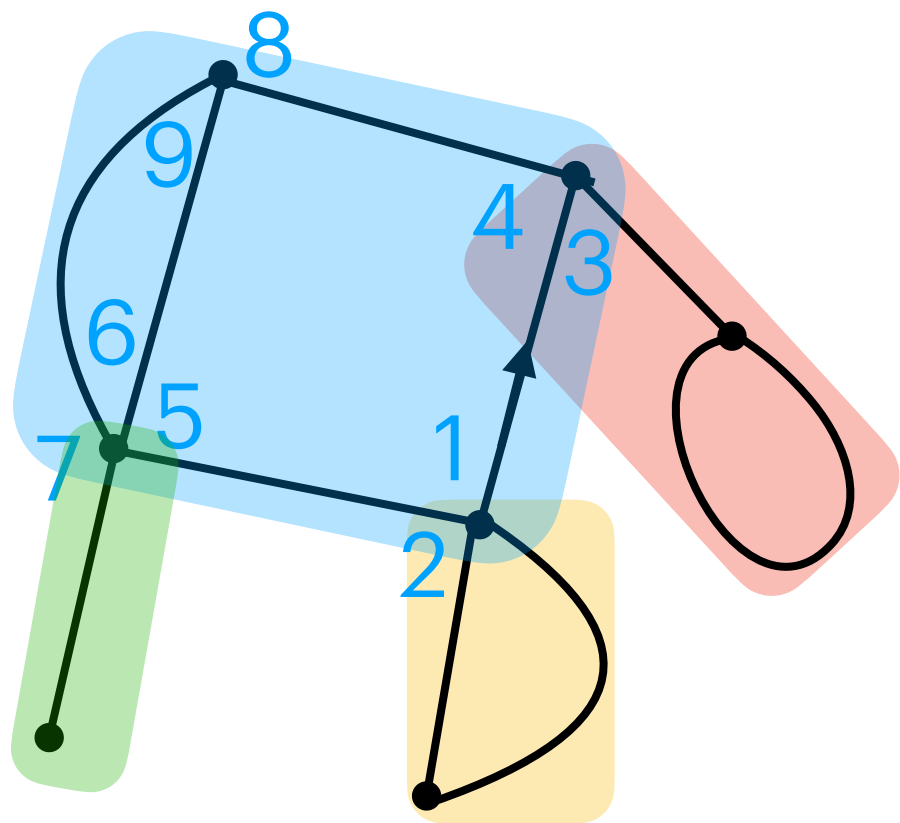
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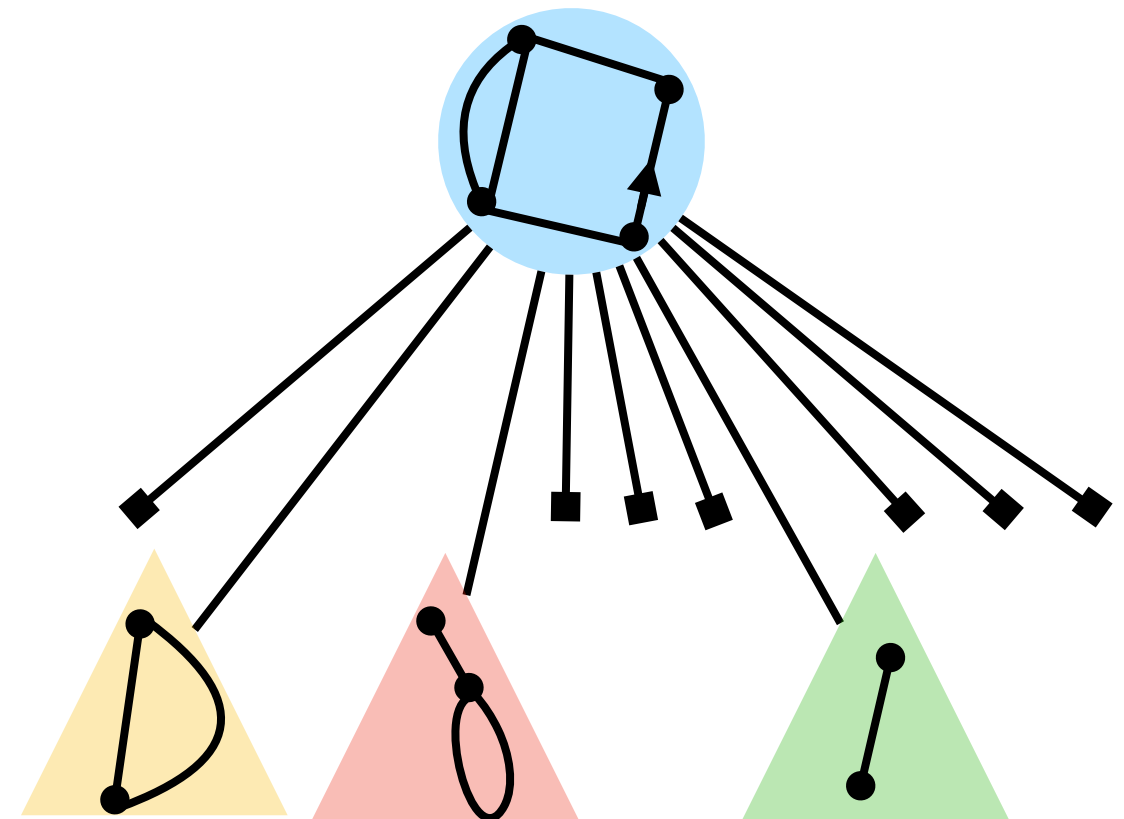
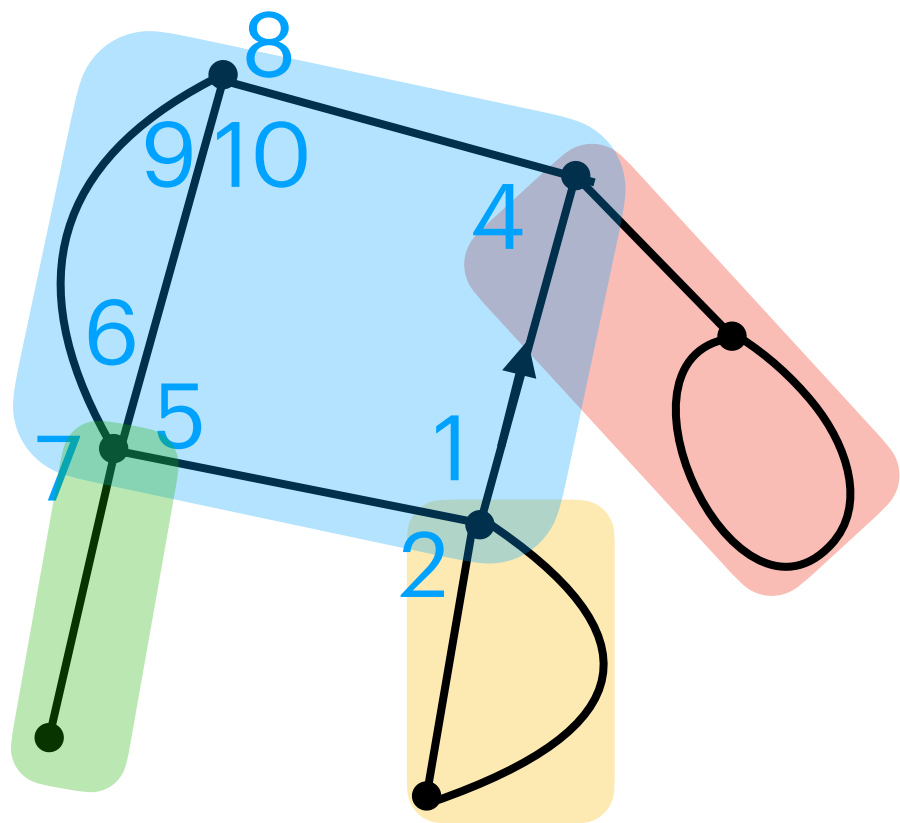
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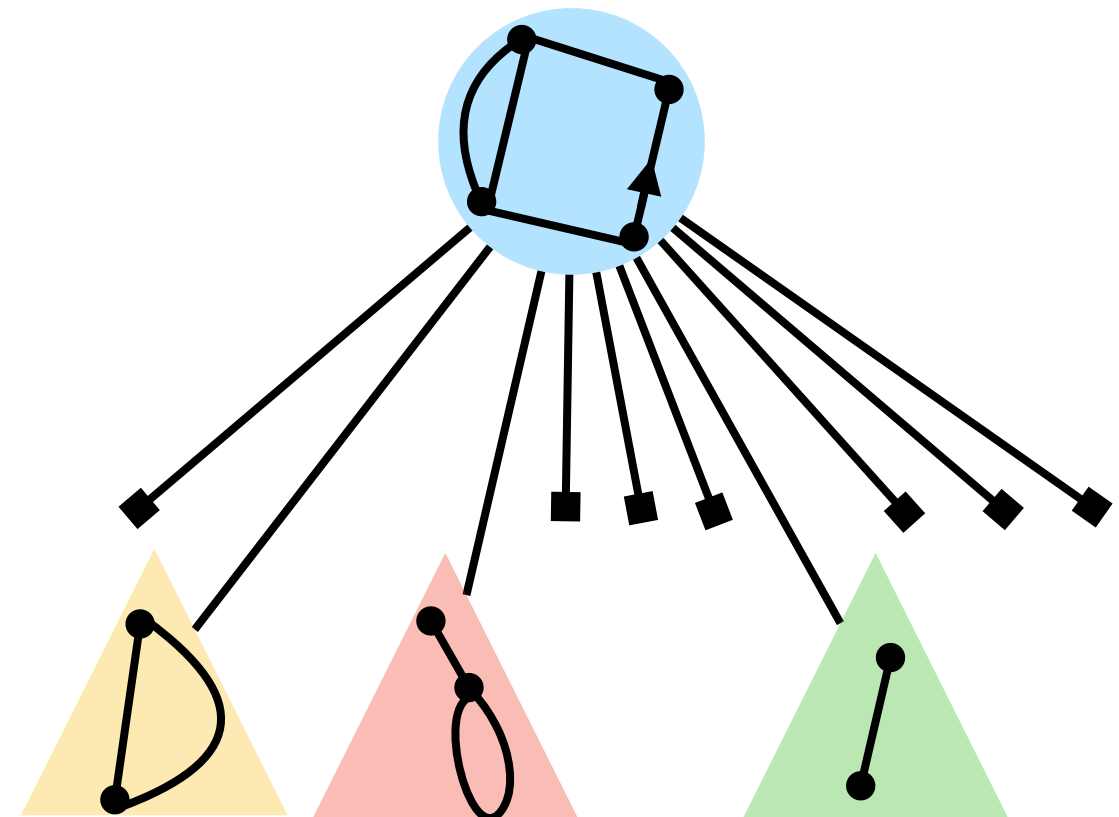
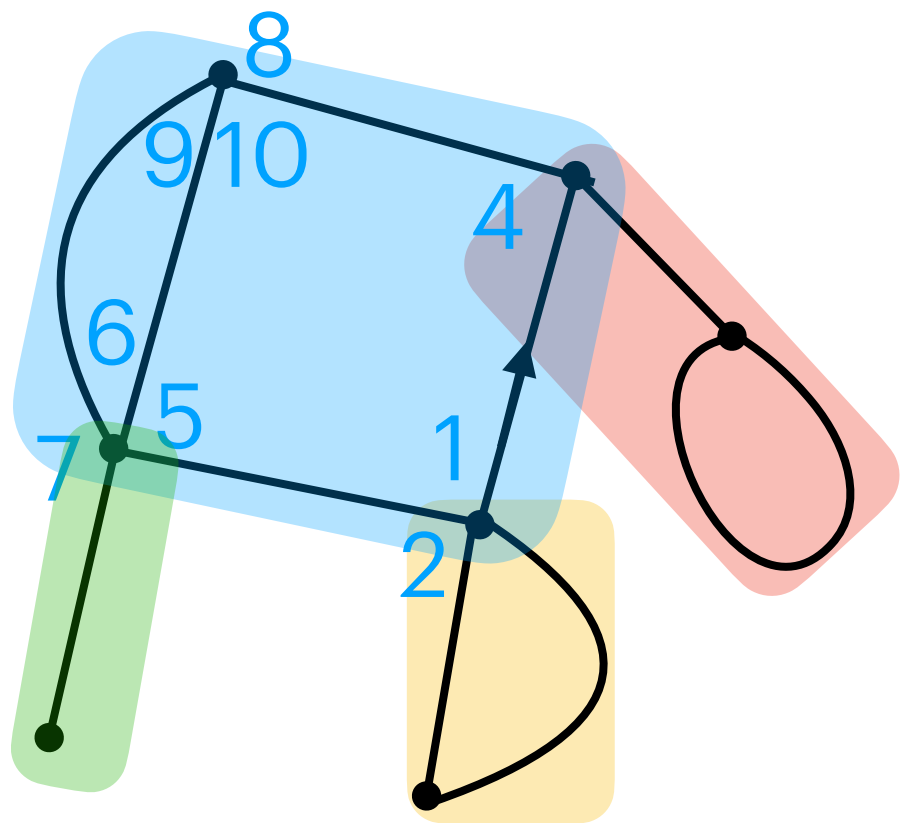
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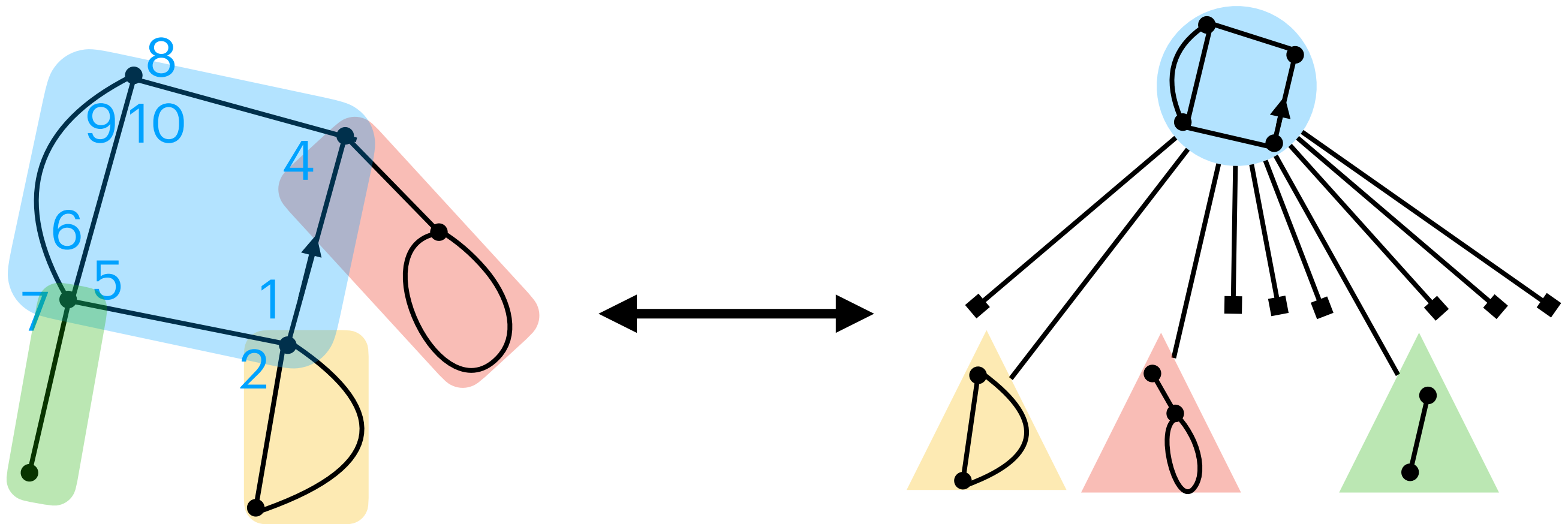
GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

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⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

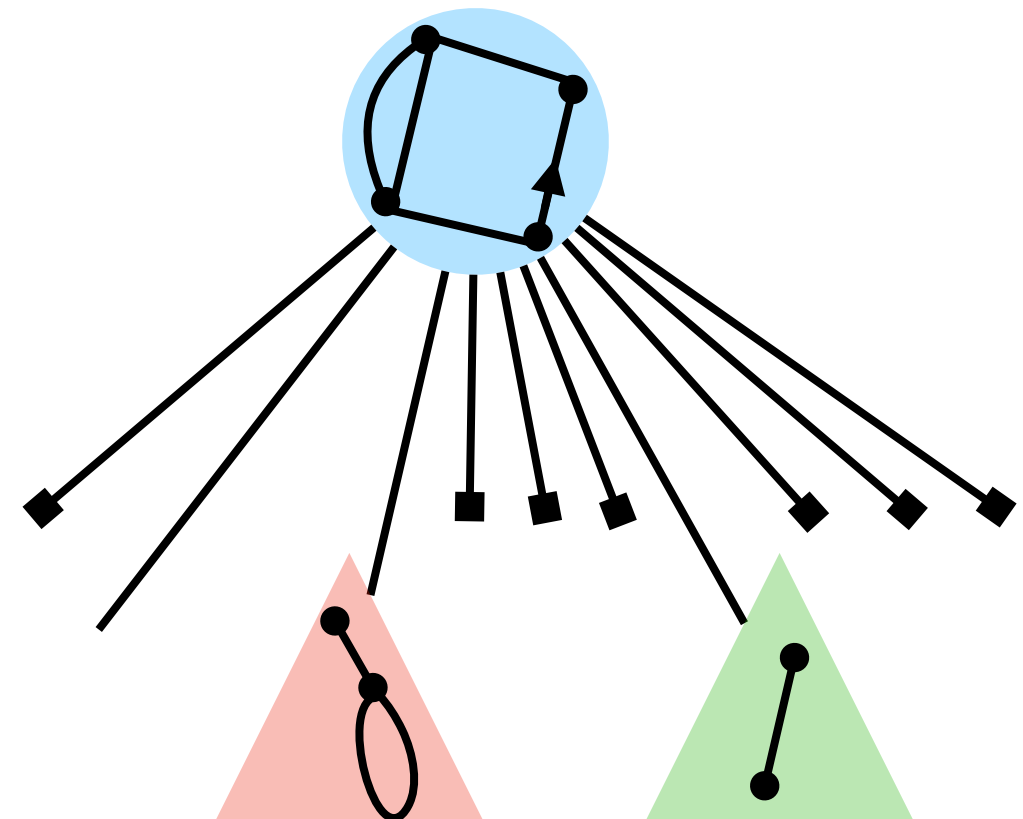
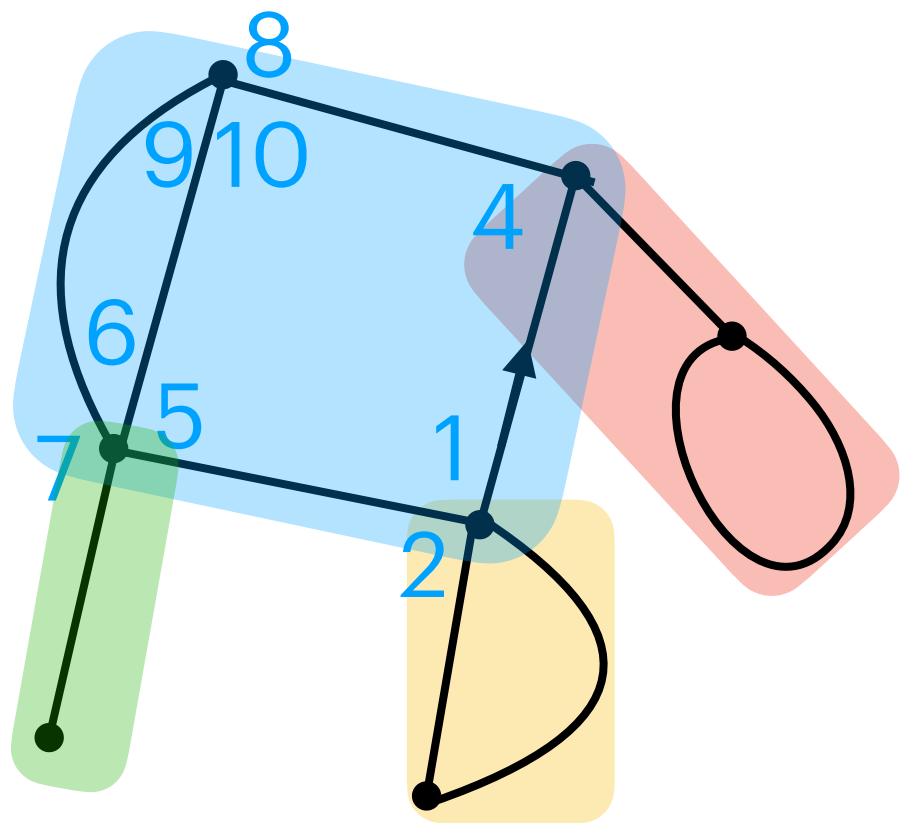
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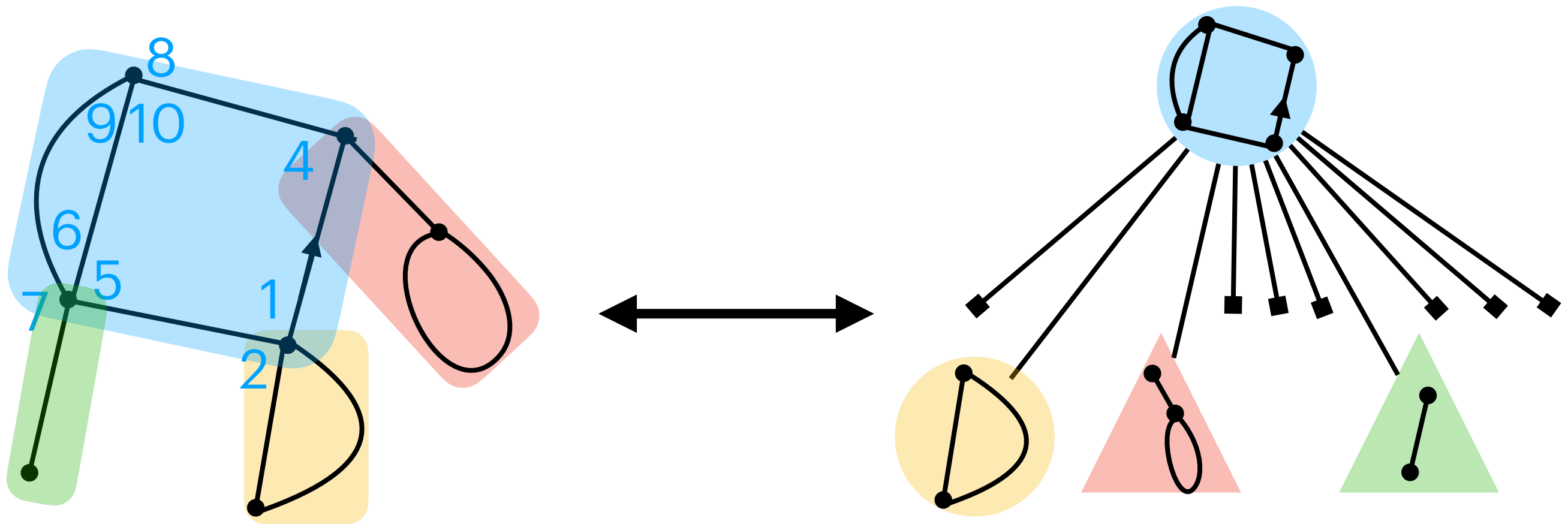
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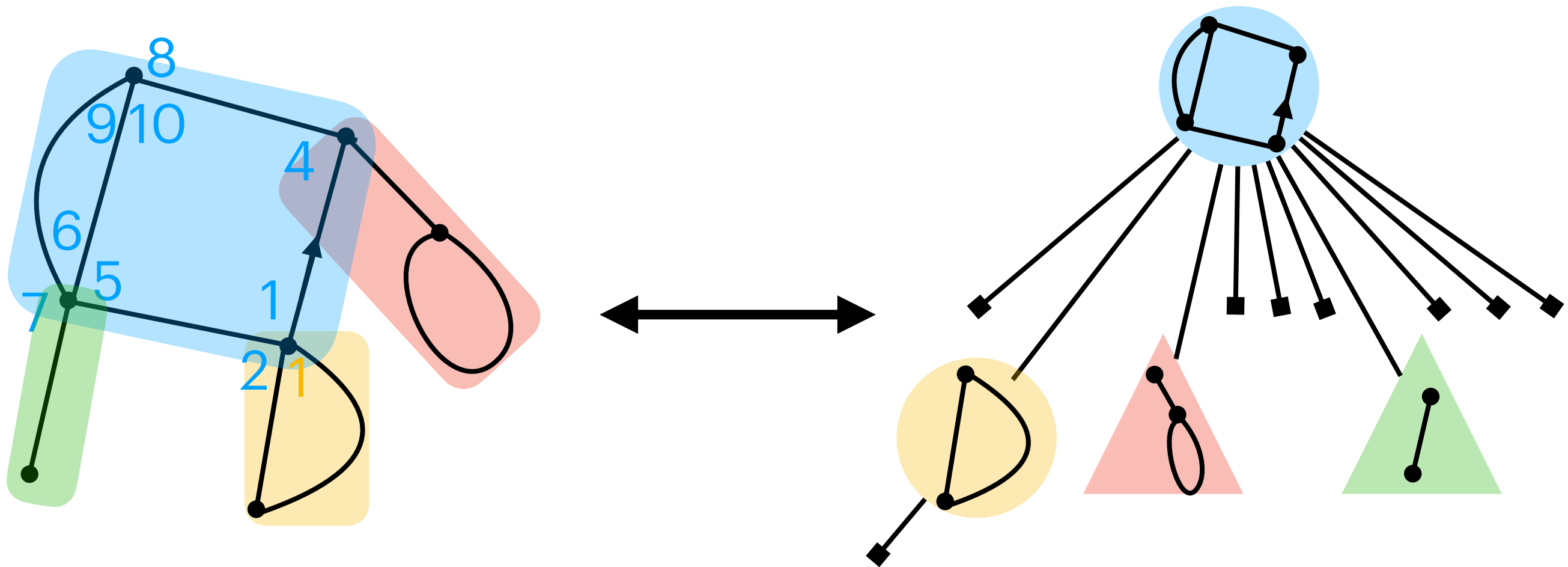
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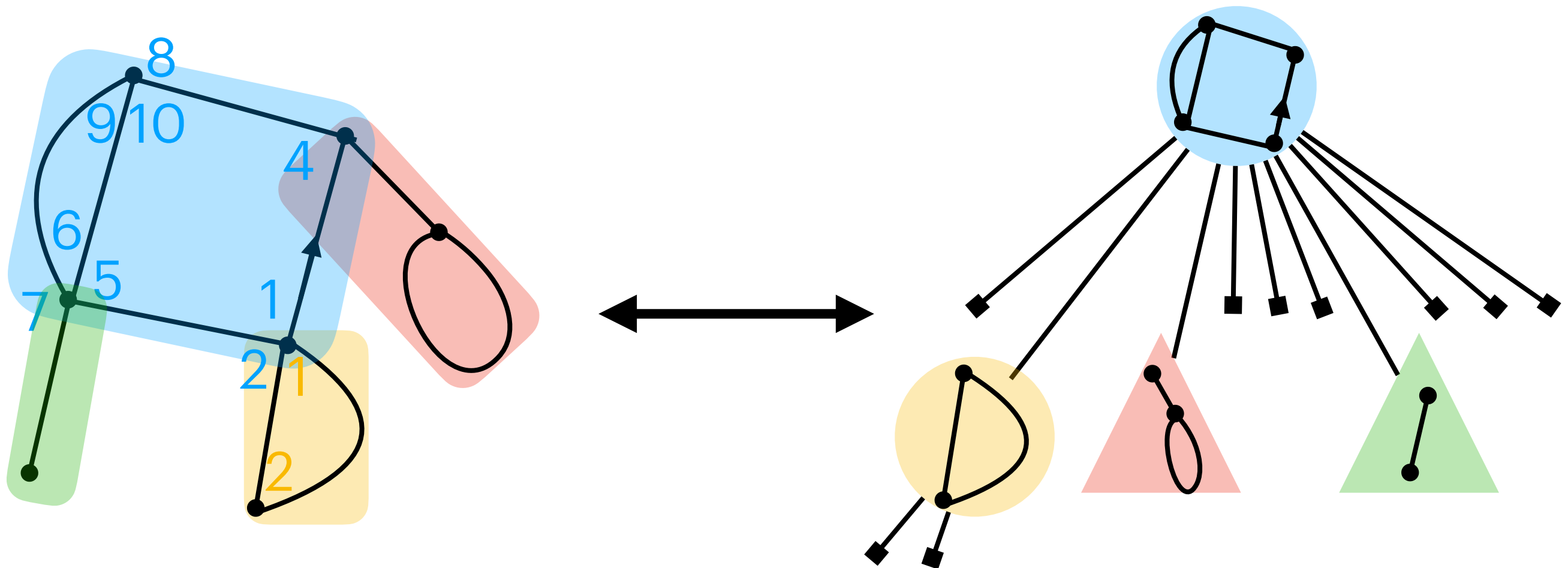
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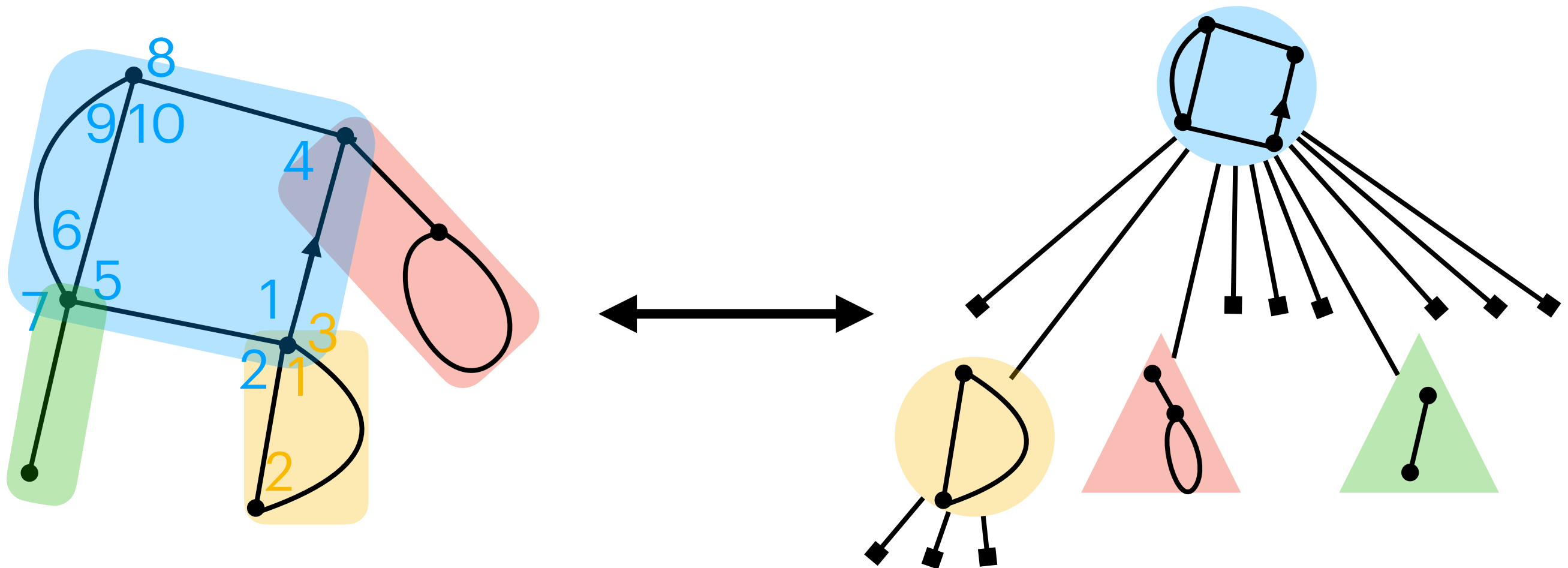
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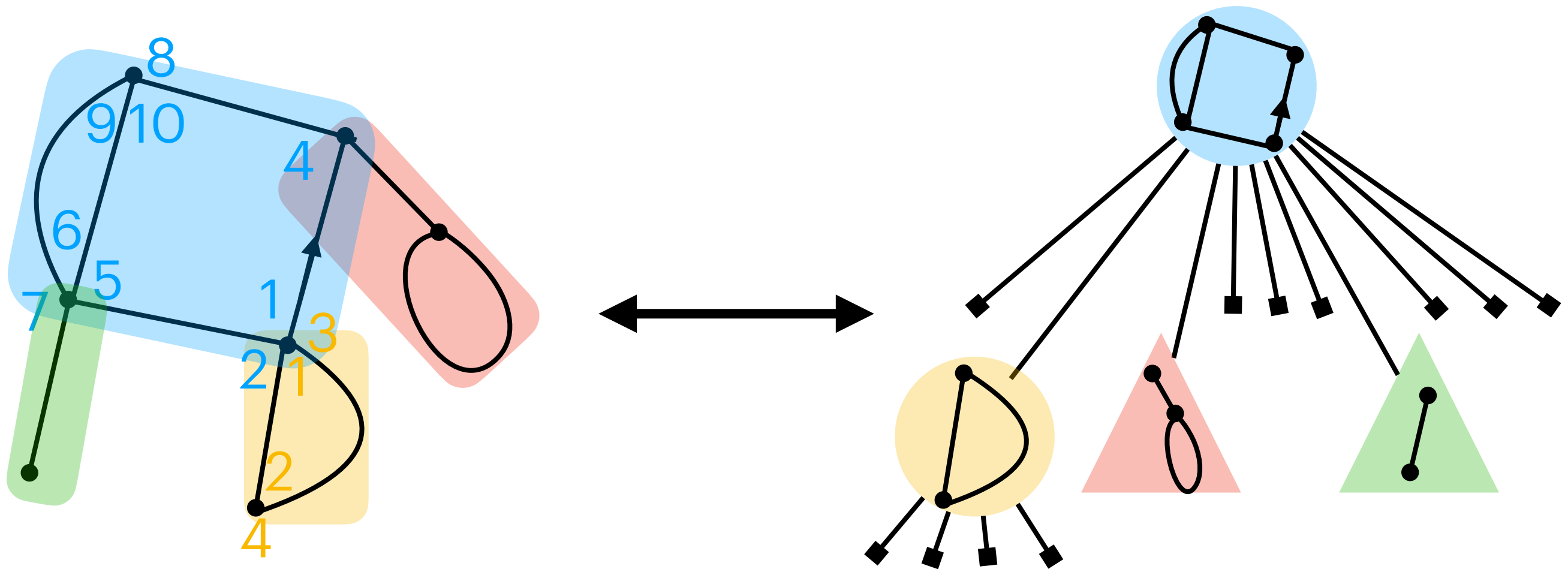
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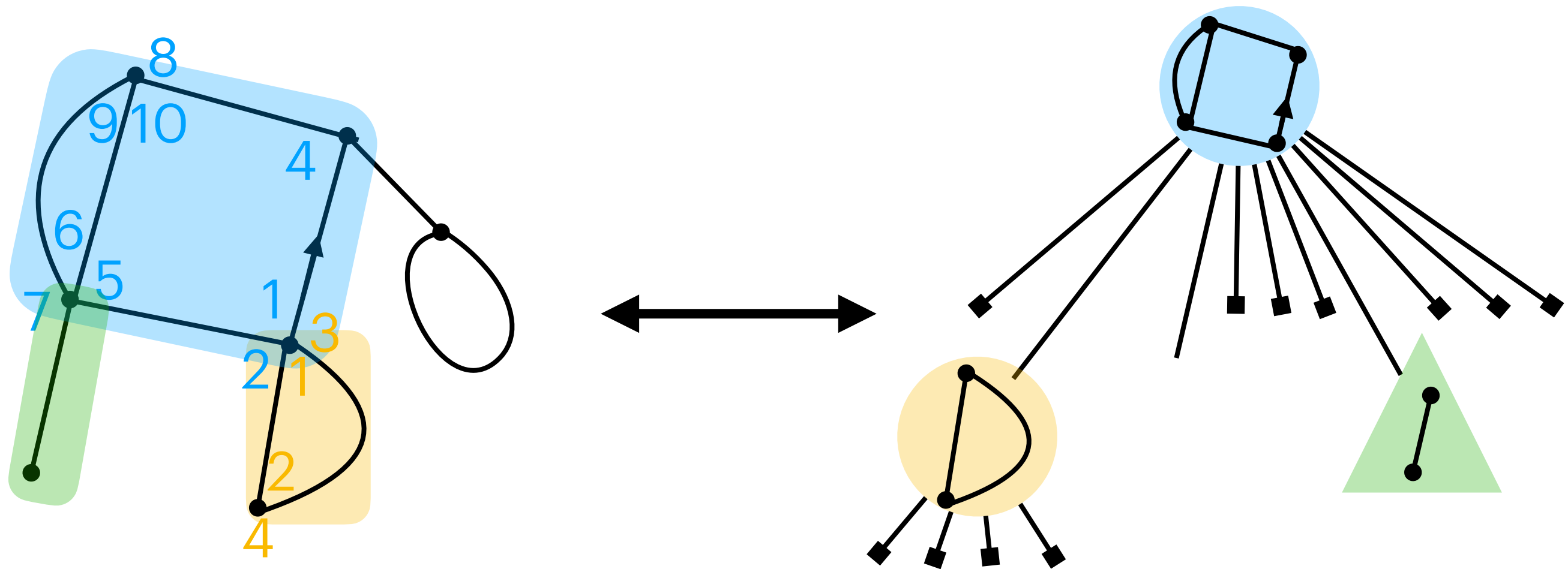
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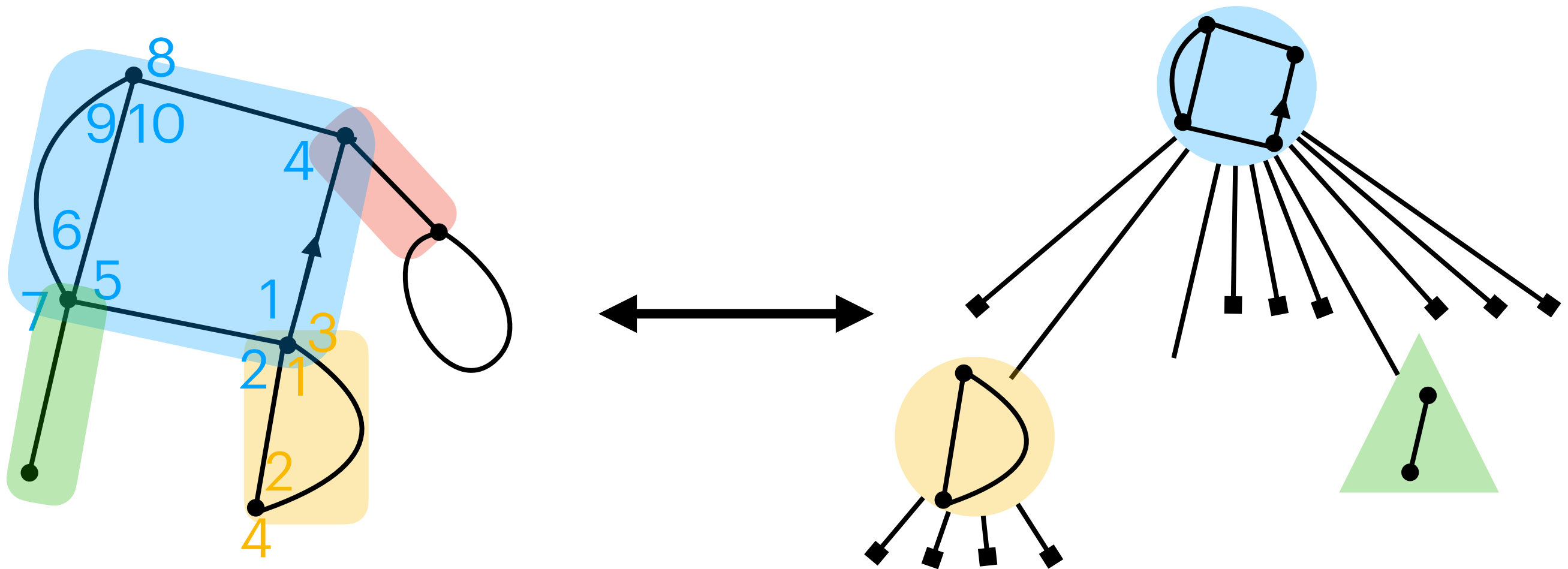
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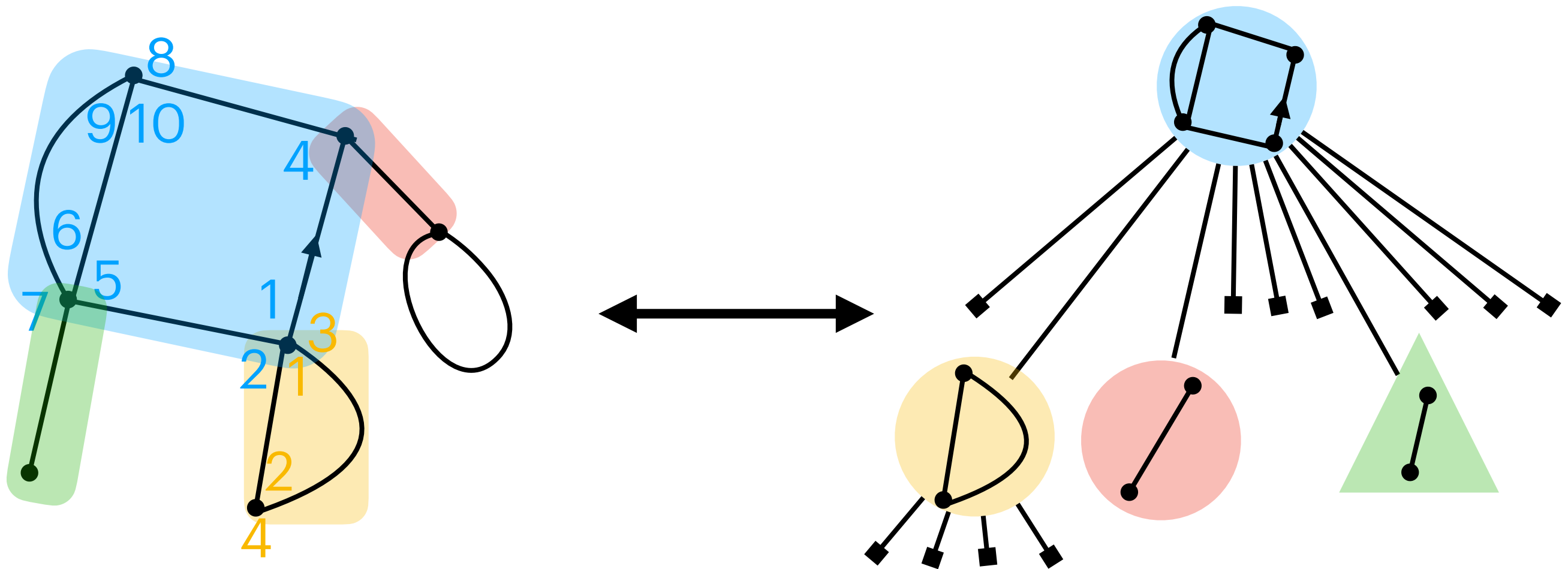
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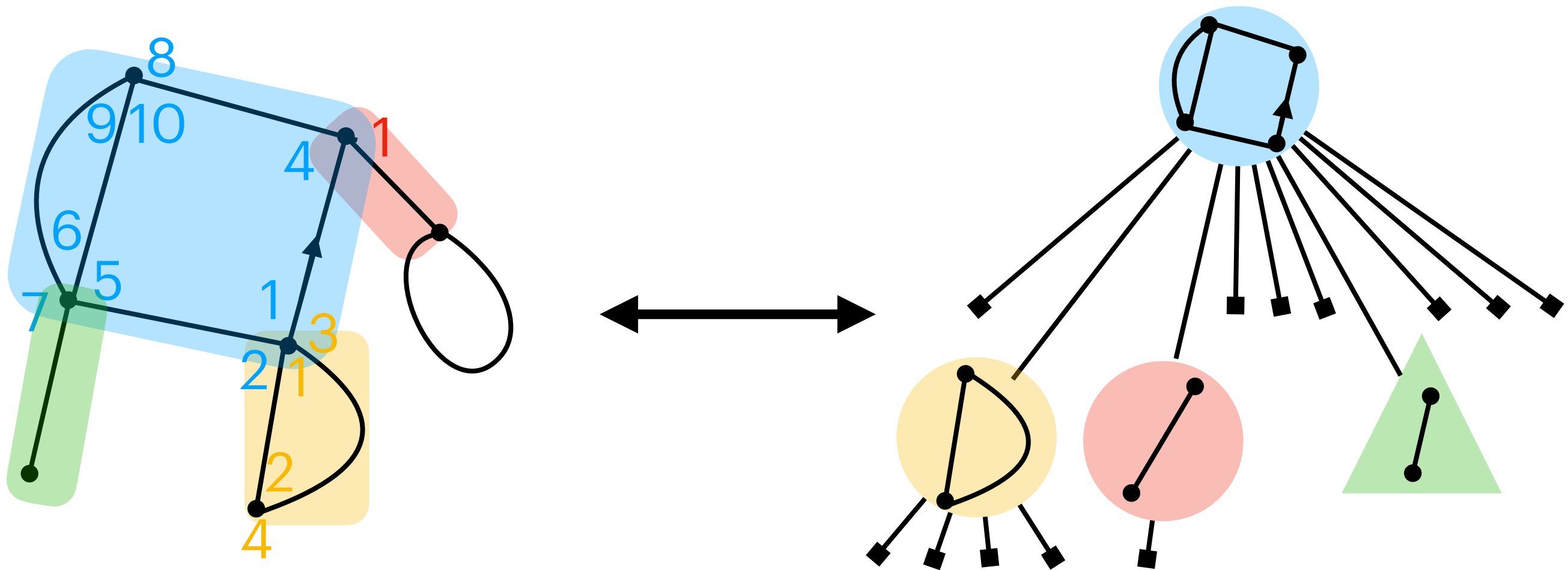
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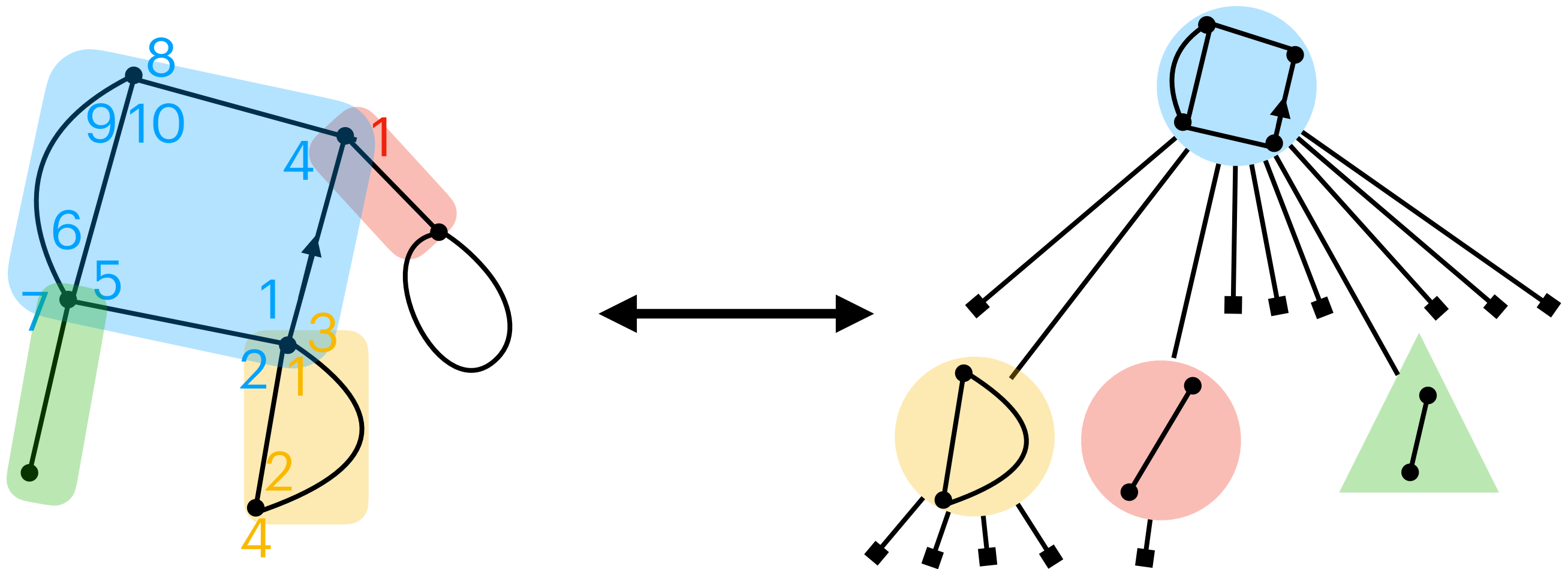
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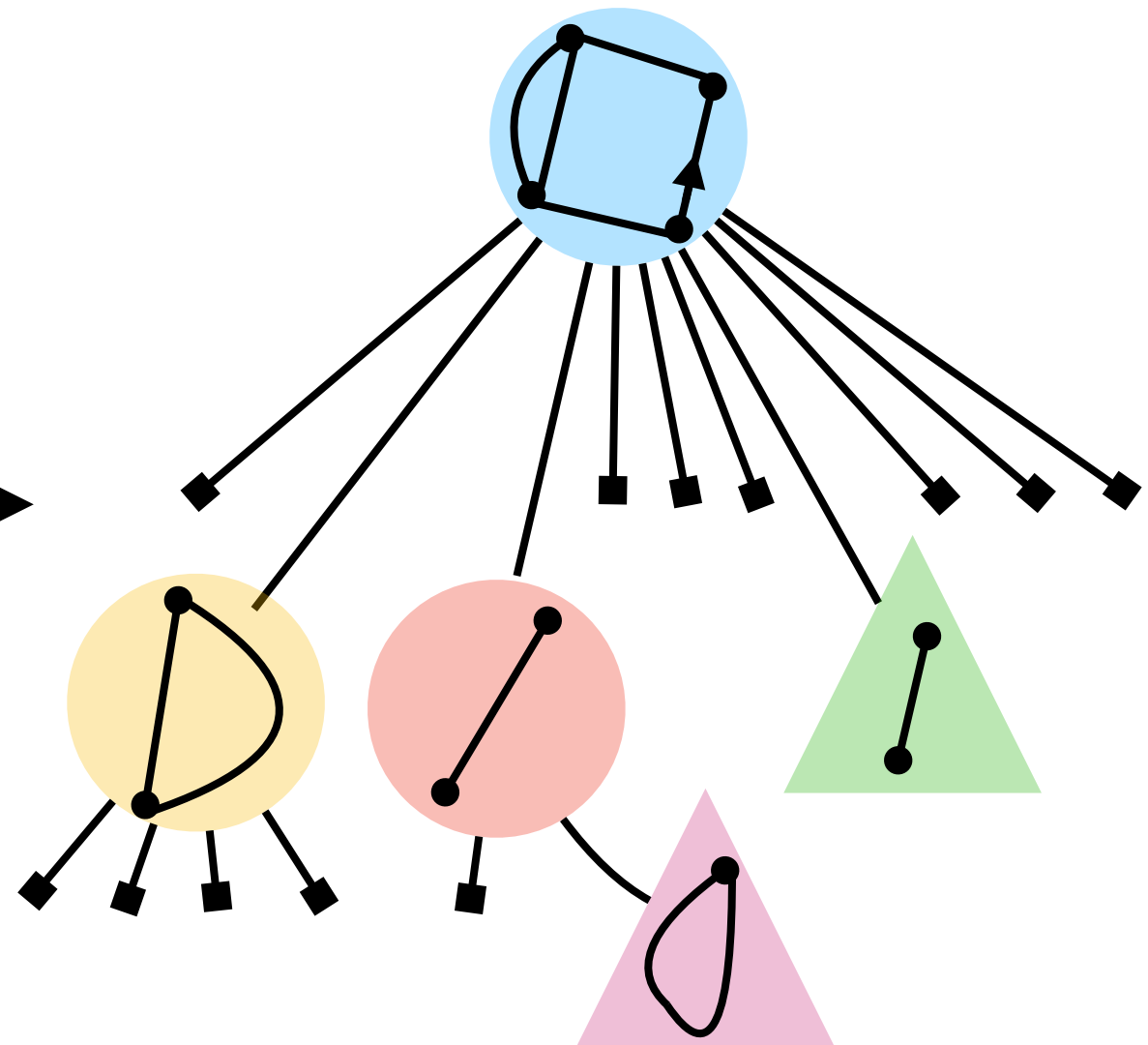
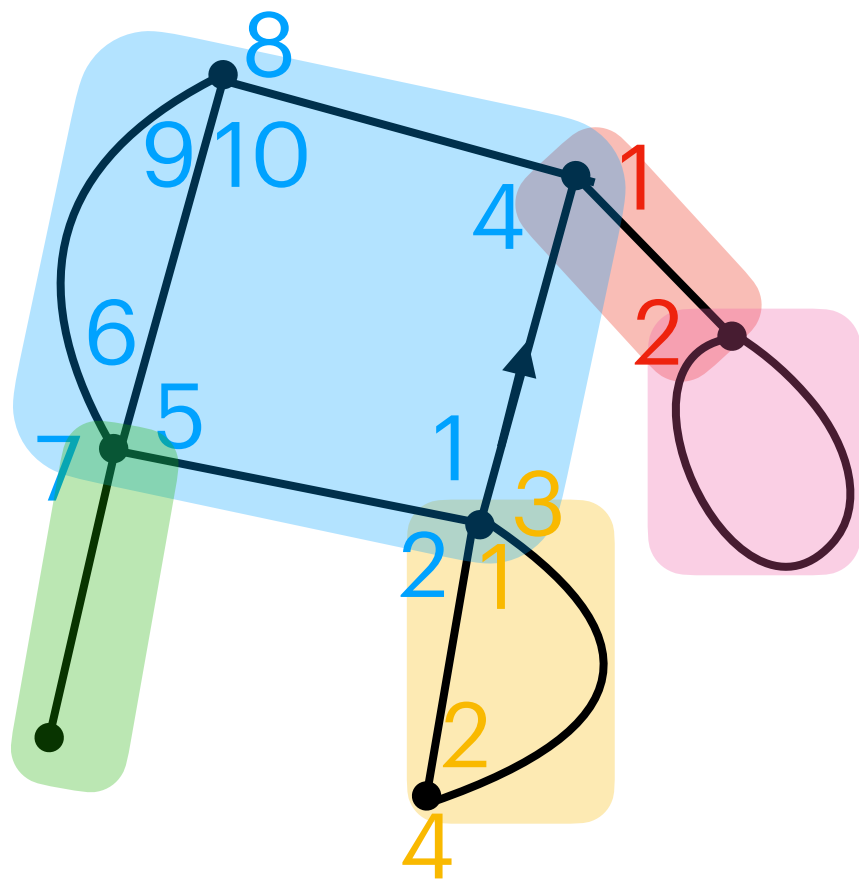
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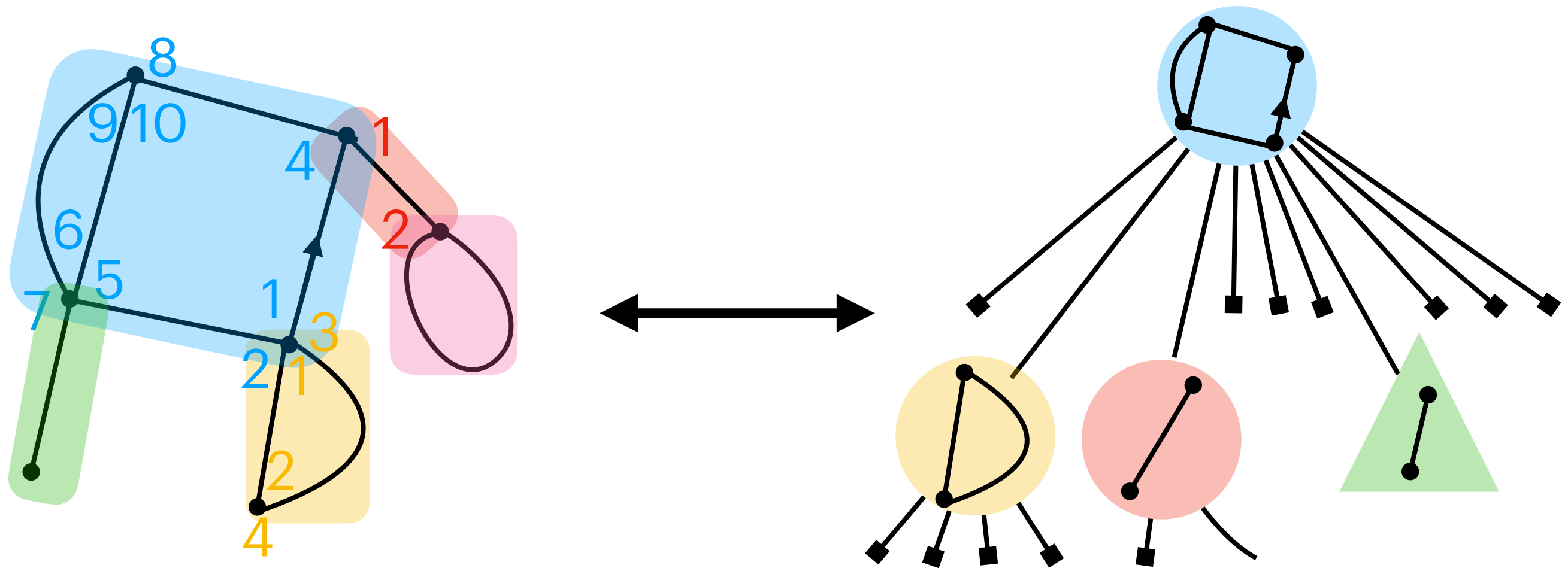
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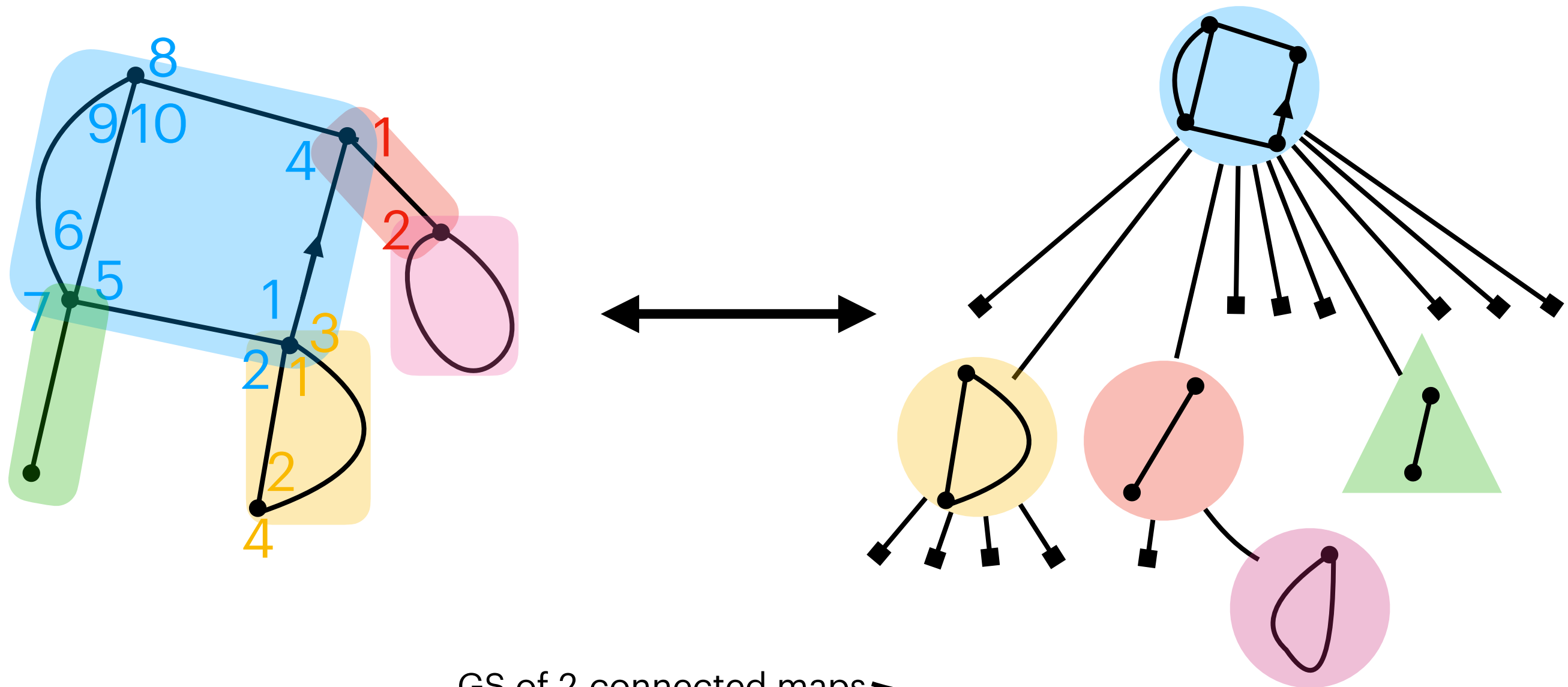
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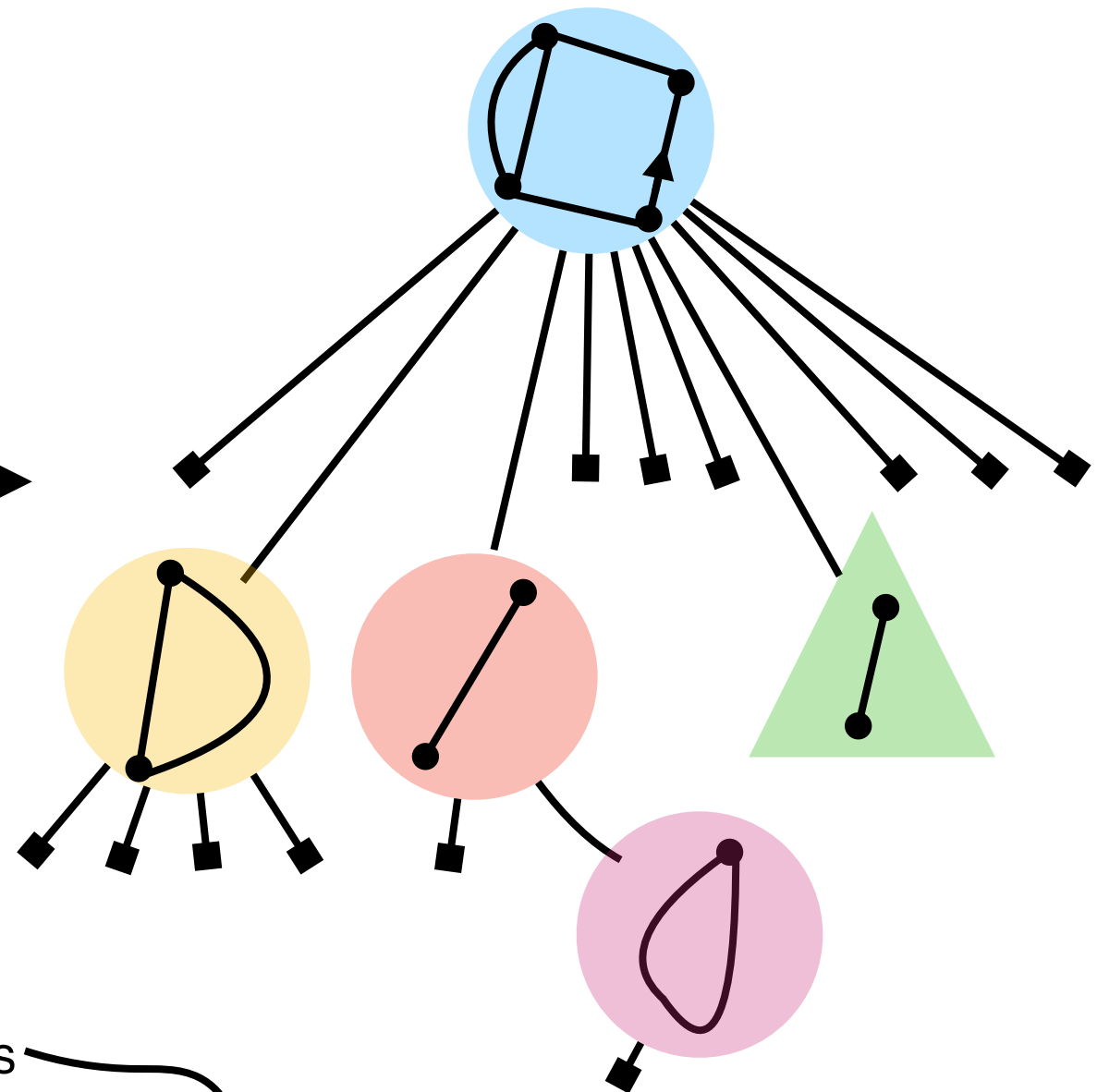
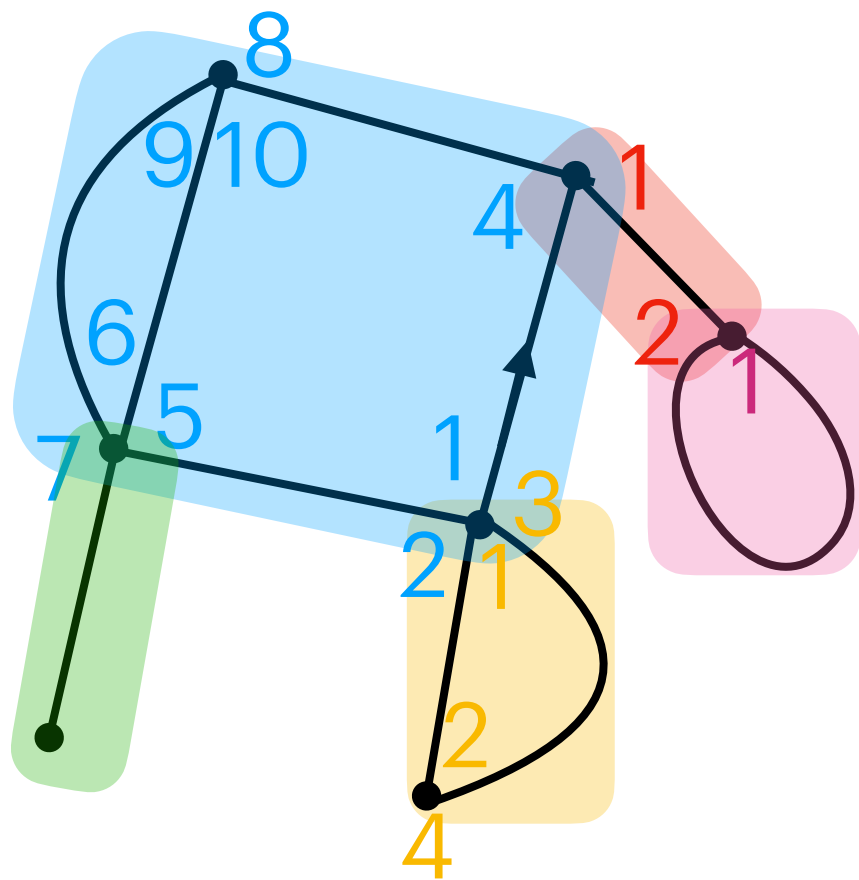
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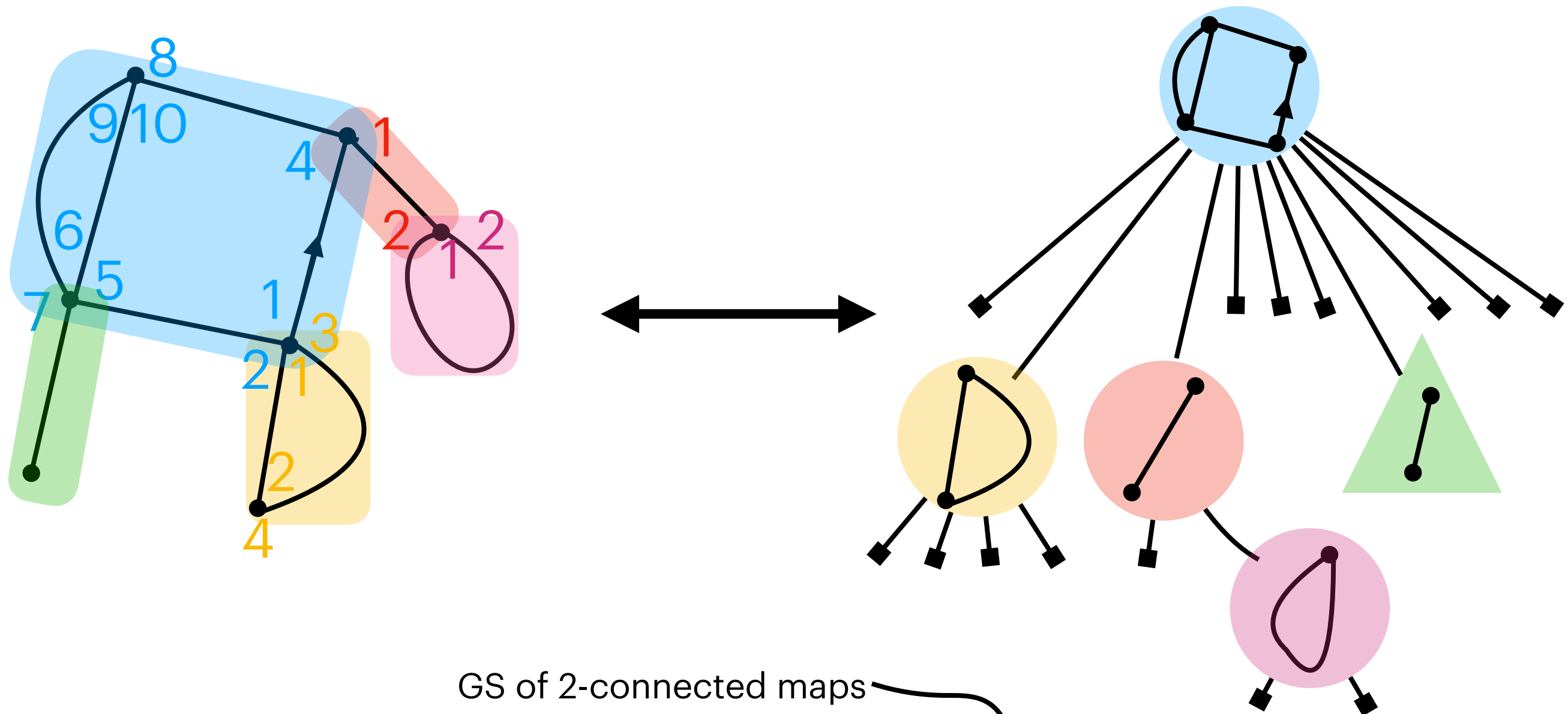
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Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z, u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



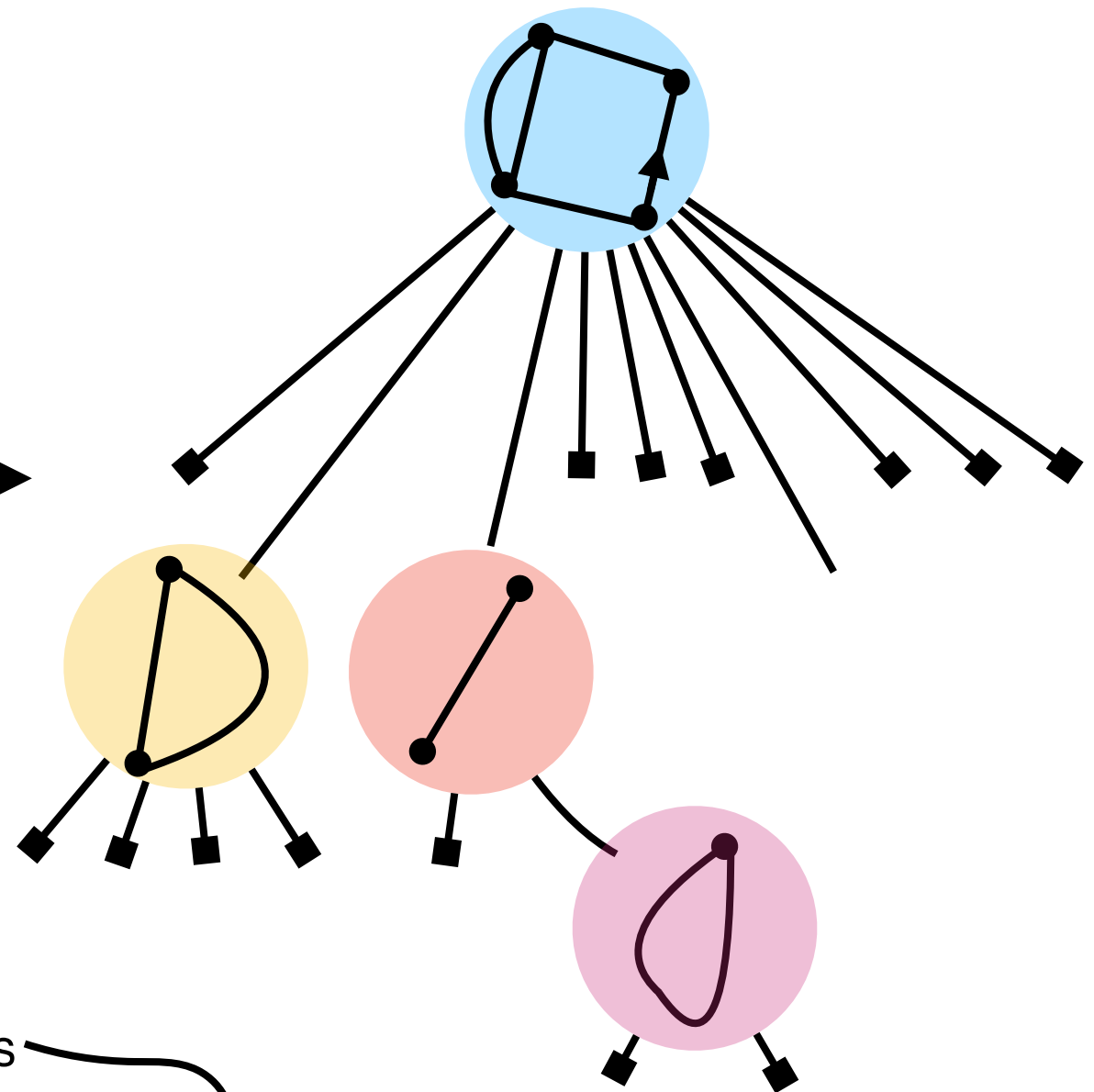
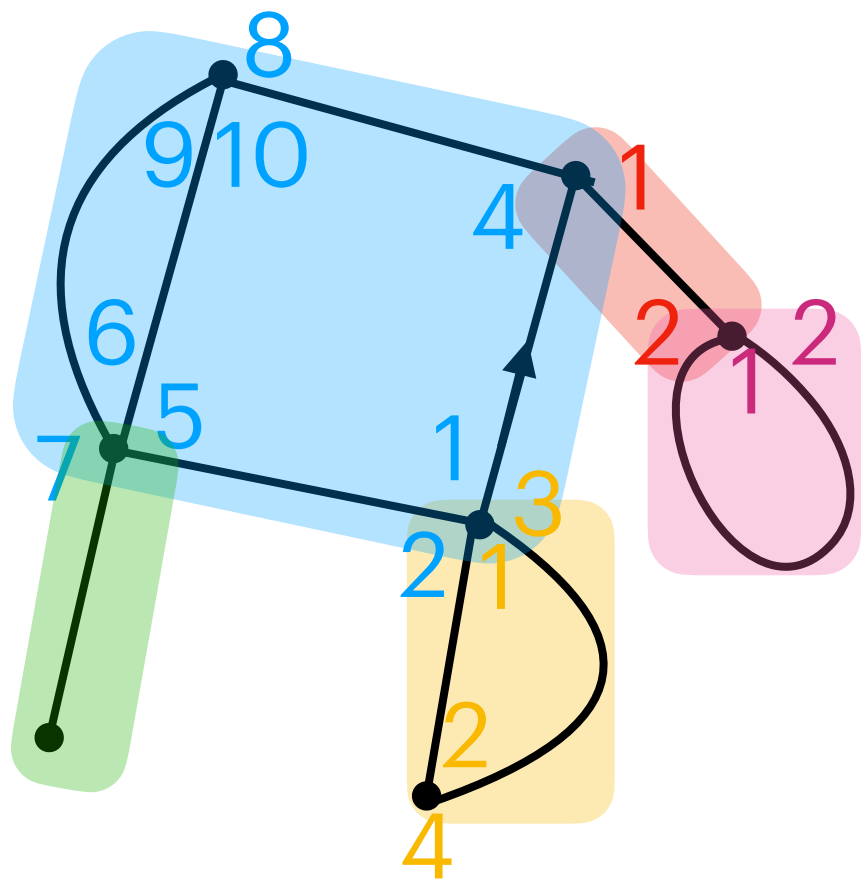
GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Decomposition of a map into blocks

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$$M(z, u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



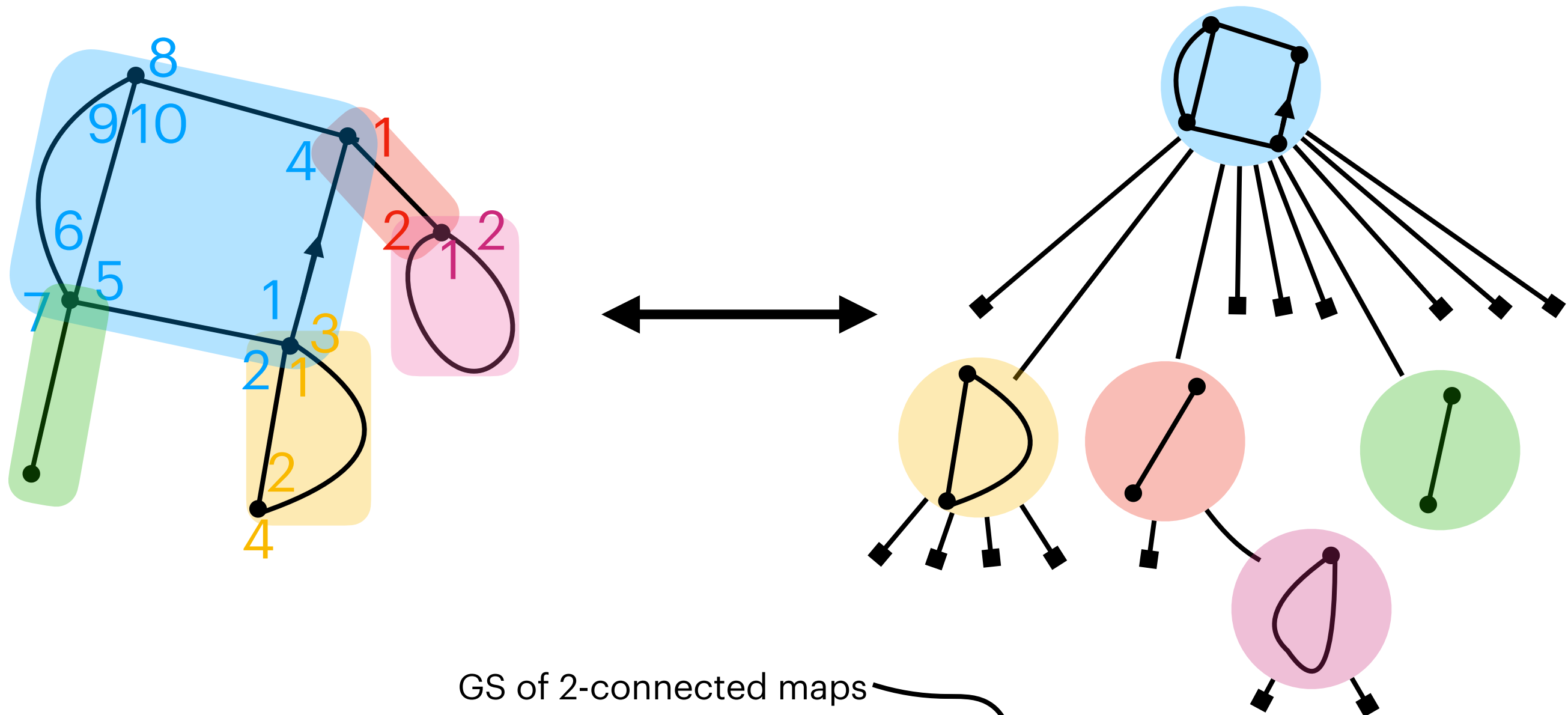
GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Decomposition of a map into blocks

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$$M(z, u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



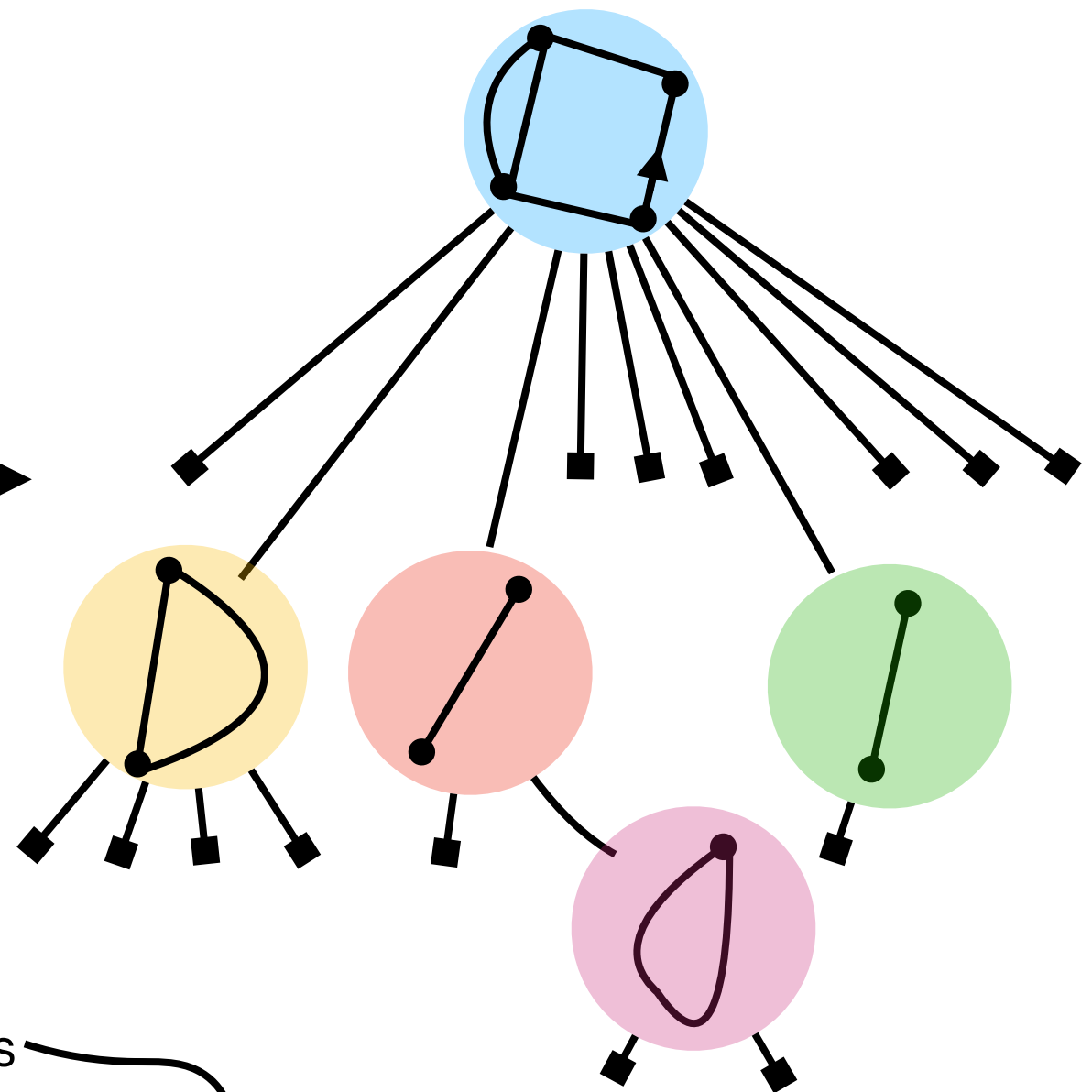
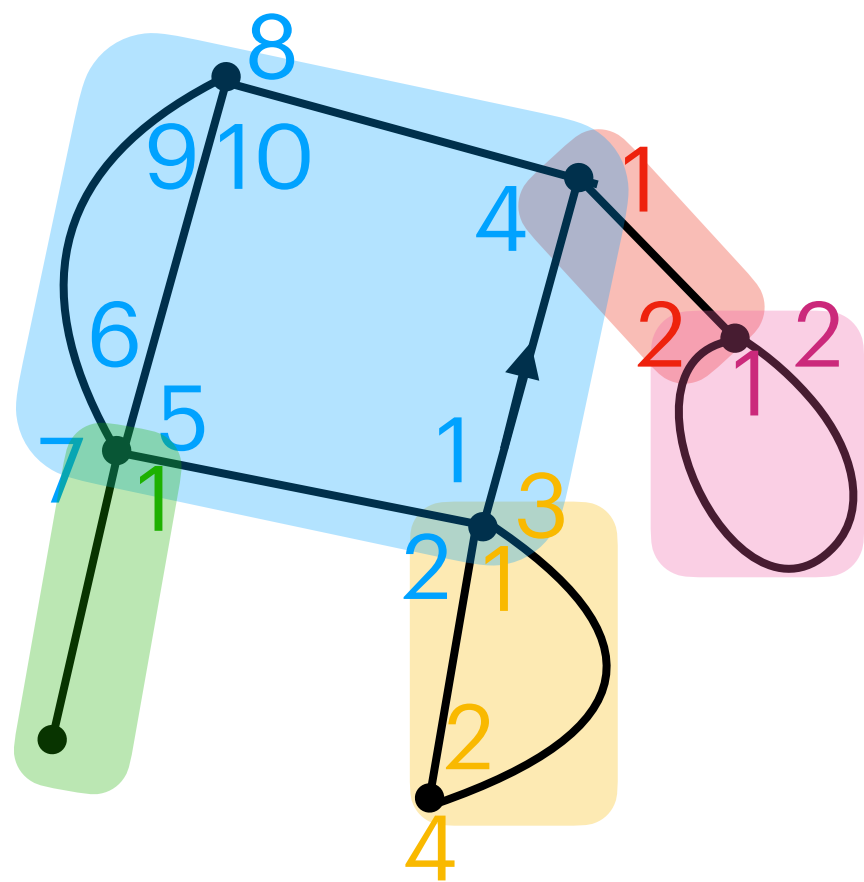
GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z, u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



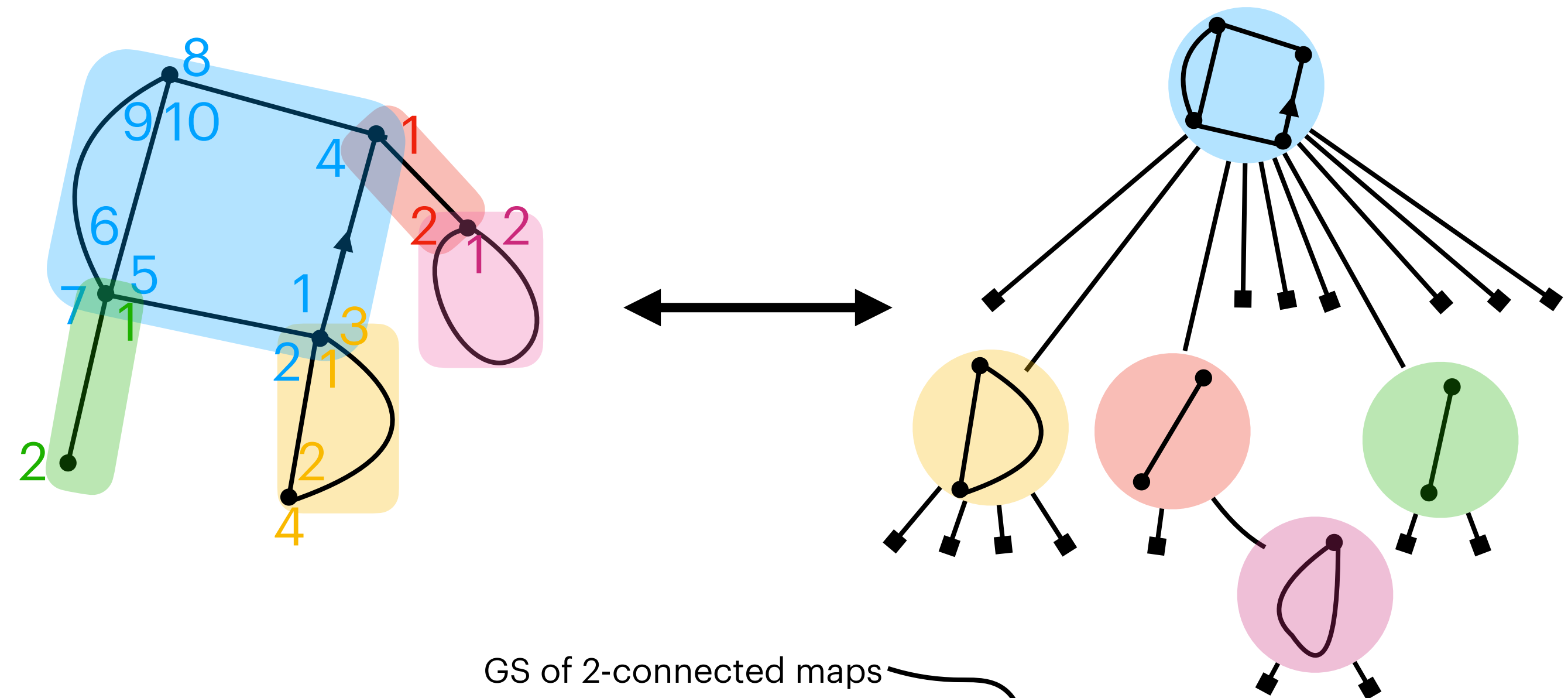
GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z, u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



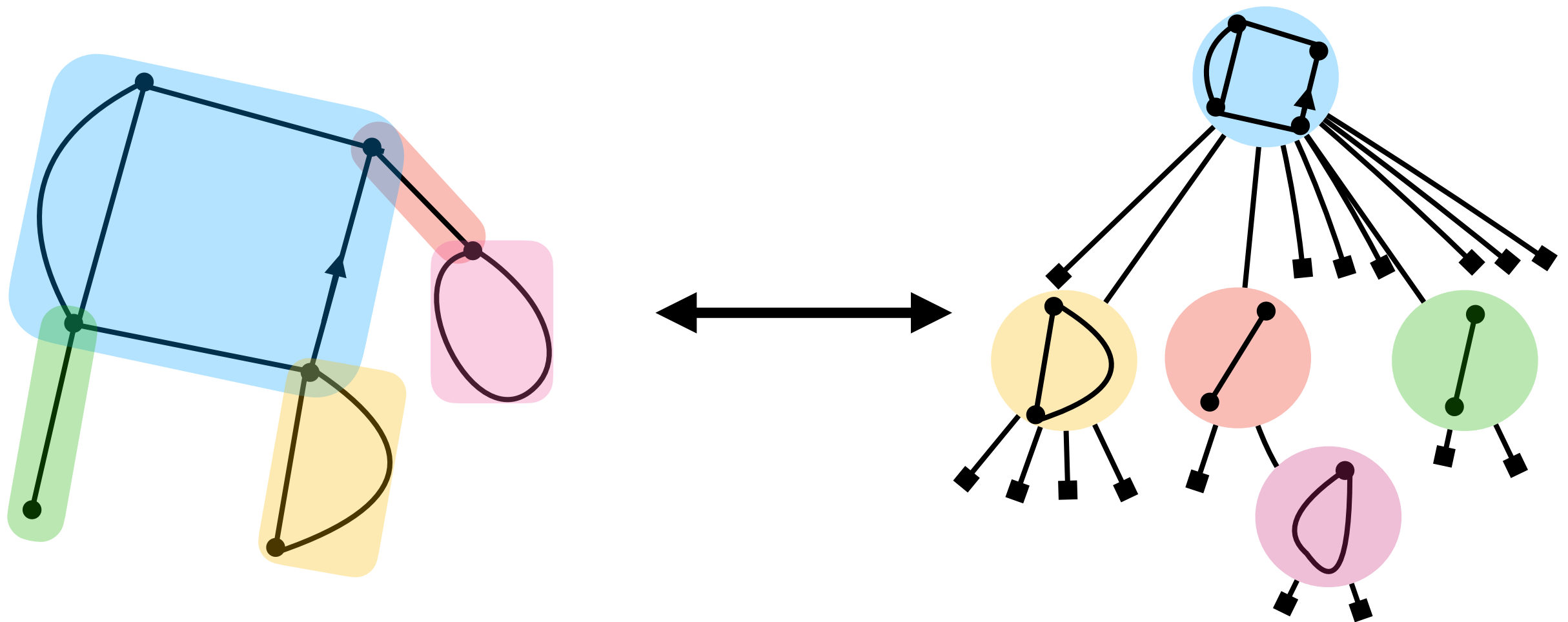
GS of 2-connected maps

With a weight u on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks: properties



- Internal node (with k children) of $T_{\mathfrak{m}}$ $\leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$;
- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$.

T_{M_n} gives the block sizes of a random map M_n .

Galton-Watson trees for map blocks

μ -Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with $\mu =$ probability law on $\mathbb{N}$.

=> Choice of y ?

$$u > 0$$

$$y \in (0, 4/27]$$

Galton-Watson trees for map blocks

μ -Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with $\mu =$ probability law on $\mathbb{N}$.

Theorem

If $M_n \hookrightarrow \mathbb{P}_{n,u}$ then T_{M_n} has the law of a Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{\mathbf{1}_{k \neq 0}}}{uB(y) + 1 - u}.$$

=> Choice of y ?

$$u > 0$$

$$y \in (0, 4/27]$$

Phase transition

When is $\mu^{y,u}$ critical? ($= \mathbb{E}(\mu) = 1$?)

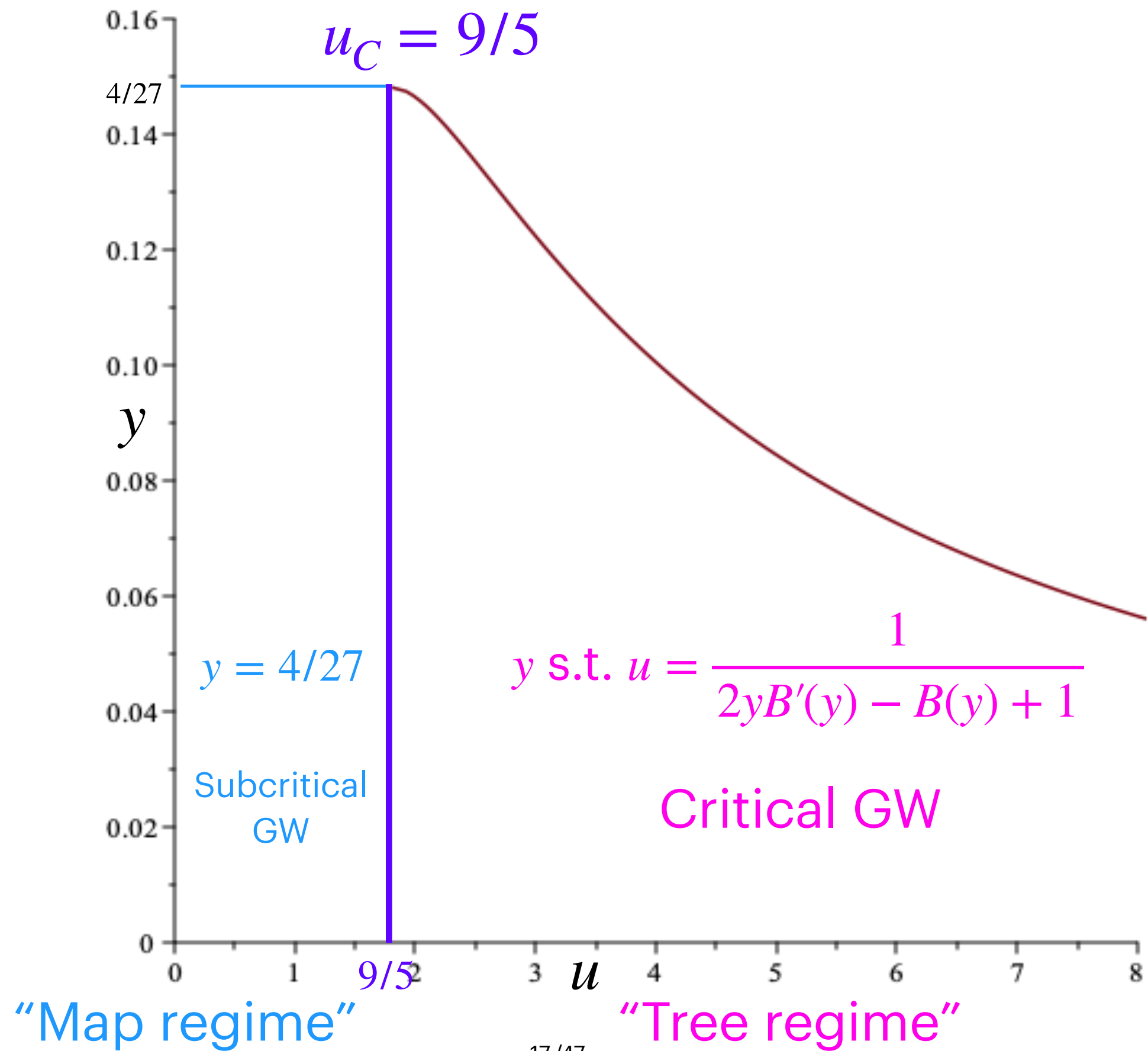
$$\mathbb{E}(\mu^{y,u}) = 1 \Leftrightarrow u = \frac{1}{2yB'(y) - B(y) + 1}$$

covers $[9/5, +\infty)$ when y covers $(0, \rho_B = 4/27]$.

Theorem

- If $u < 9/5$, then $\mathbb{E}(\mu^{y,u}) < 1$. The mean is maximal for $y = 4/27$ and then $\mu^{y,u}(2k) \sim c_u k^{-5/2}$;
- If $u = 9/5$ and $y = 4/27$, then $\mathbb{E}(\mu^{y,u}) = 1$ and $\mu^{y,u}(2k) \sim c_u k^{-5/2}$;
- If $u > 9/5$ and y is well chosen, then $\mathbb{E}(\mu^{y,u}) = 1$ and $\mu^{y,u}(2k) \sim c_u \pi_u^k k^{-5/2}$.

Phase transition



II. Largest blocks

Properties of T_{M_n}

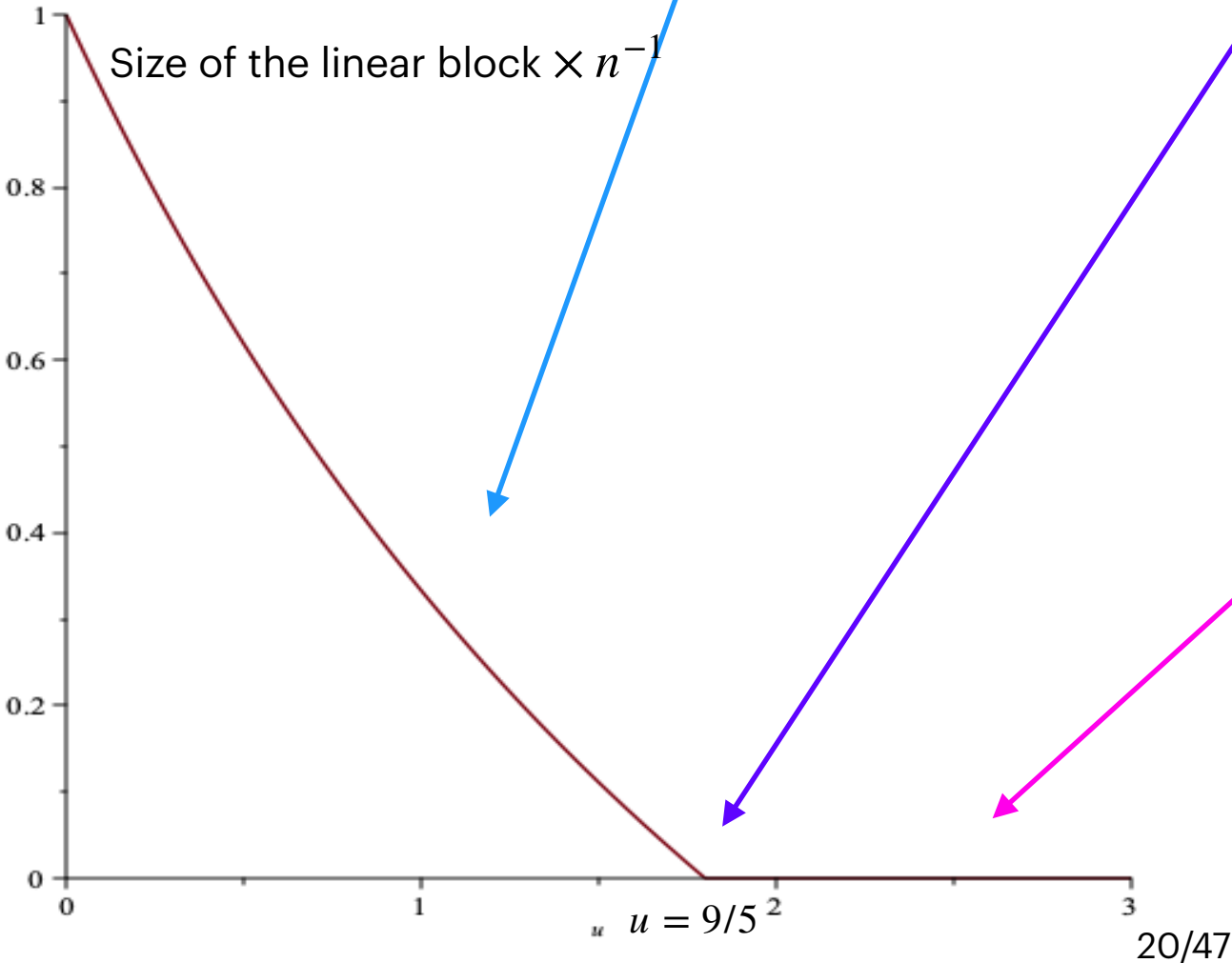
	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
Variance	∞		$< \infty$
Galton-Watson tree	subcritical	critical	

Tool: [Janson 2012] = extensive study of the degrees in Galton-Watson trees

Properties on trees give properties of maps.

Size $L_{n,k}$ of the k -th largest block

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$ [Stufler 2020]		



Rough intuition

	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
Galton-Watson tree	subcritical	critical	

Dichotomy between situations:

- Subcritical: condensation, cf [Jonsson Stefánsson 2011];
- Supercritical: behaves as maximum of independent variables.

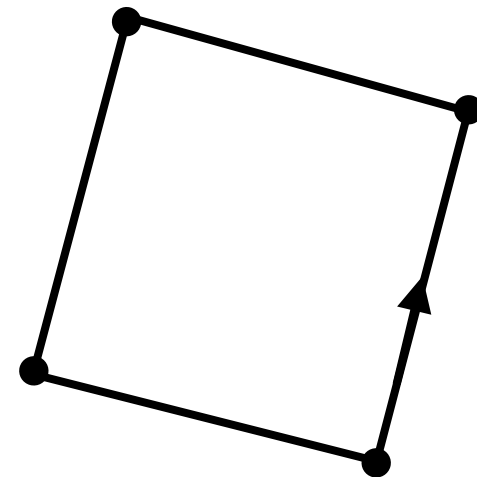
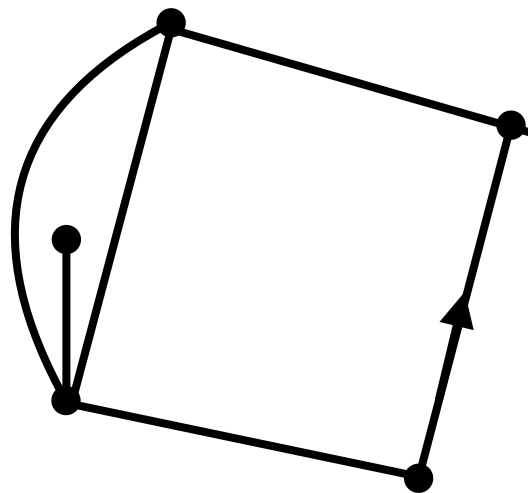
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

III. Similar model: quadrangulations

Quadrangulations

Def: map with all faces of degree 4.

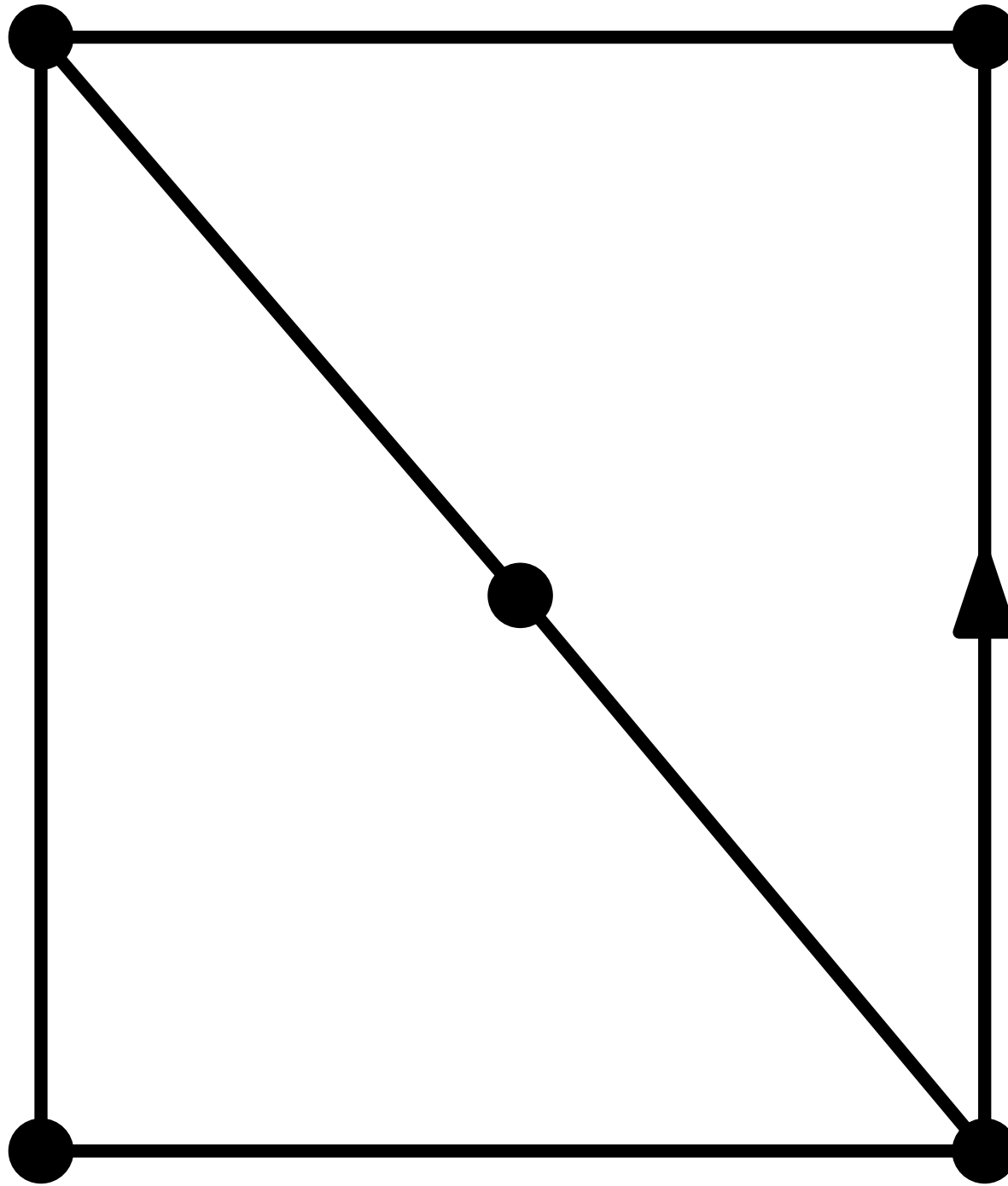


Simple quadrangulation = no multiple edges.

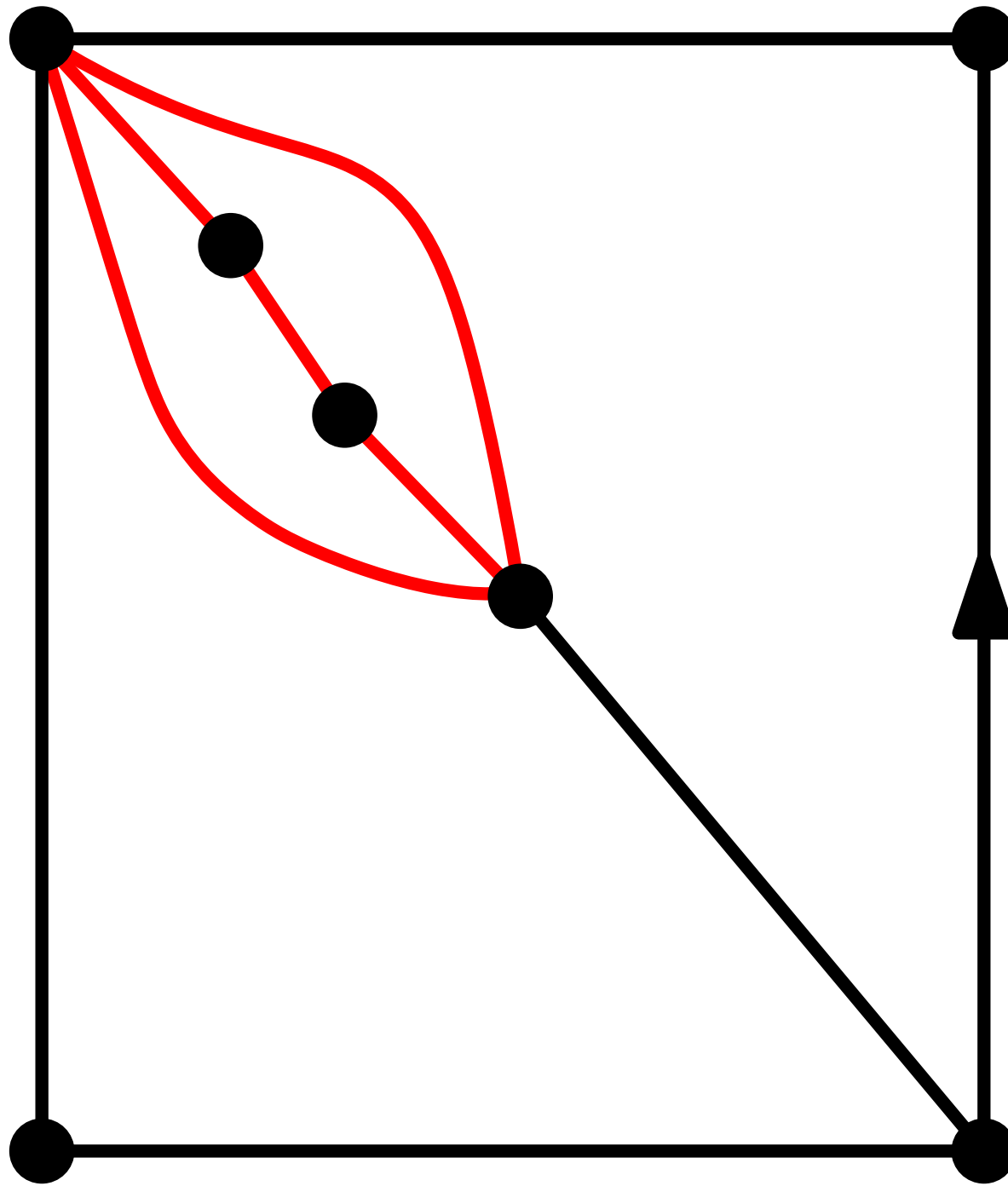
Size $|\mathfrak{q}|$ = number of *faces*.

$$|V(\mathfrak{q})| = |\mathfrak{q}| + 2, |E(\mathfrak{q})| = 2|\mathfrak{q}|.$$

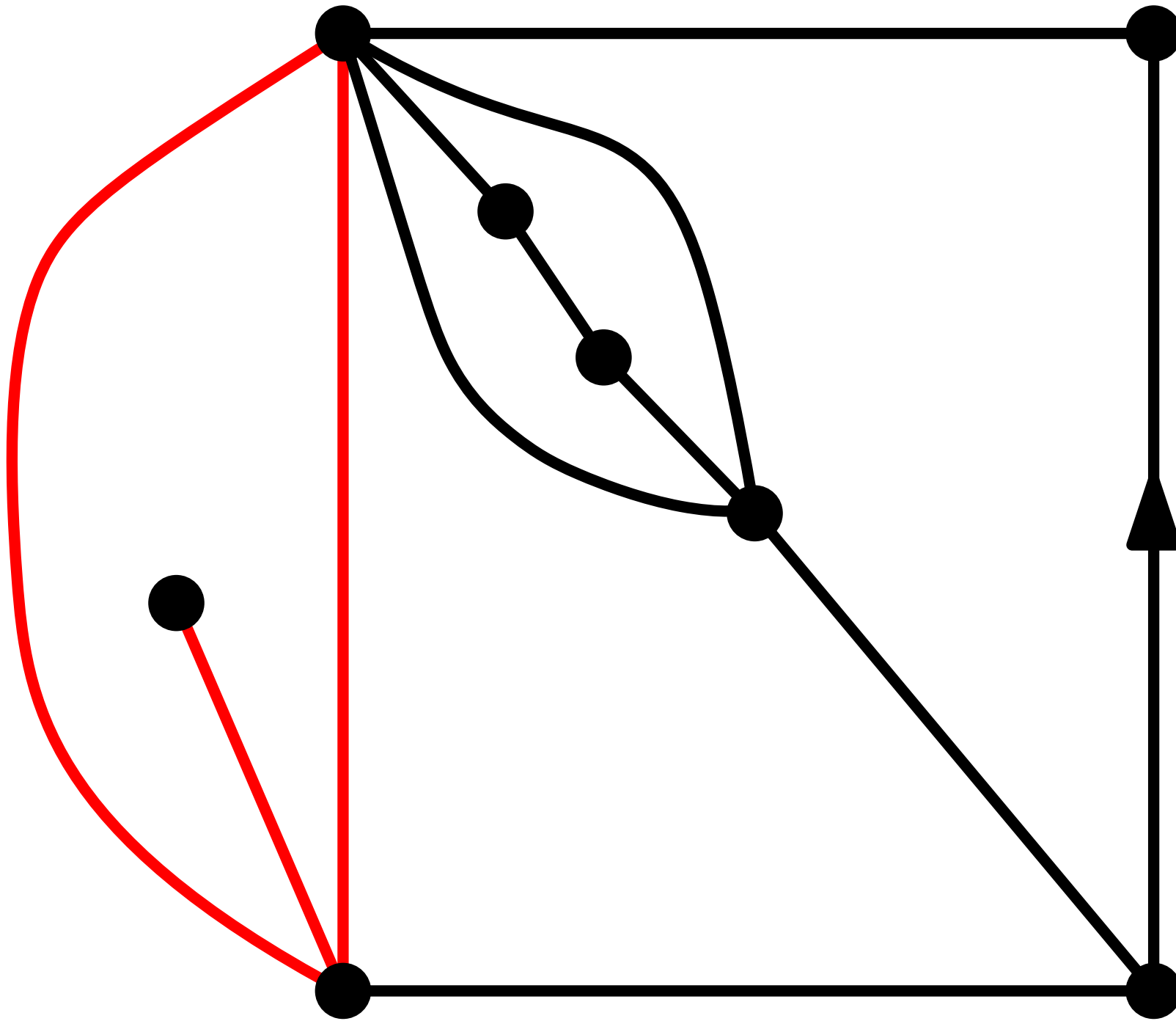
Construction of a quadrangulation from a simple core



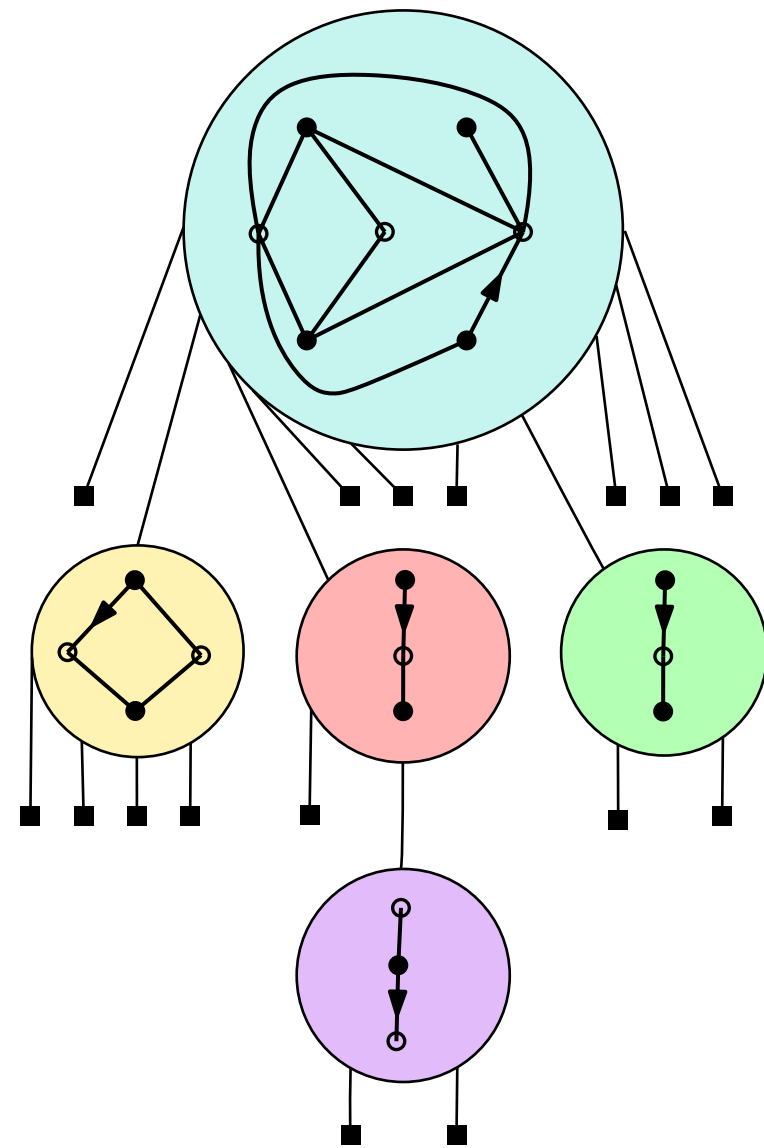
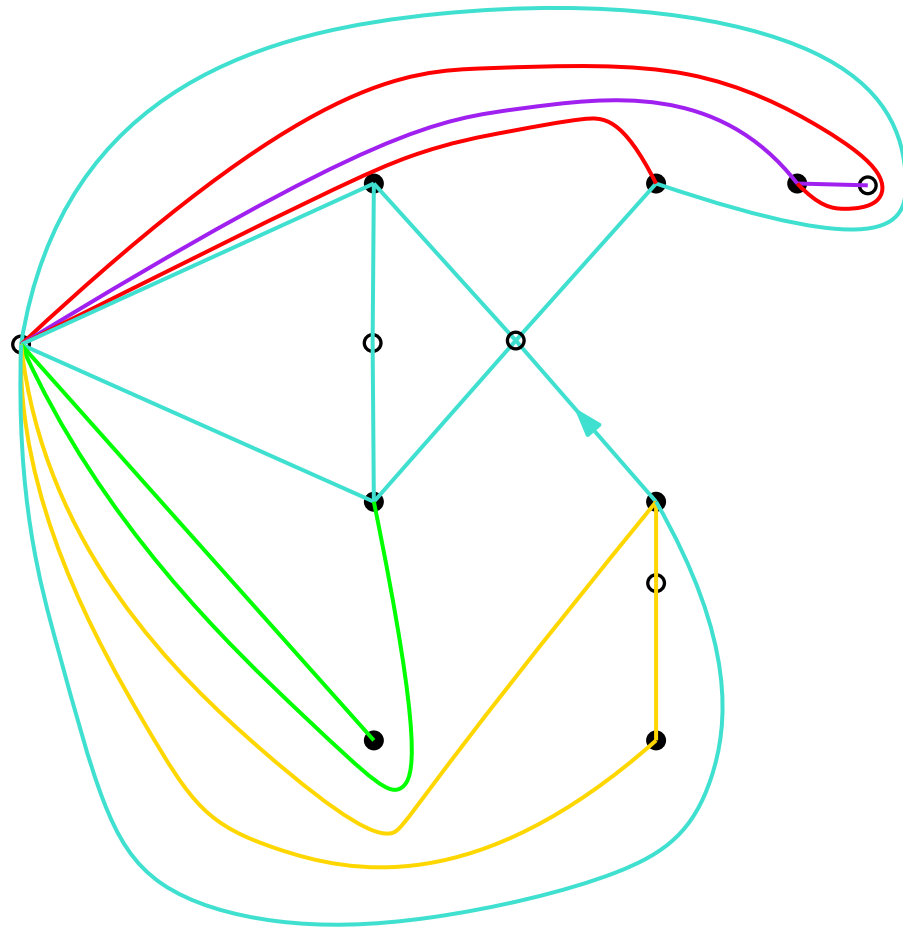
Construction of a quadrangulation from a simple core



Construction of a quadrangulation from a simple core



Block tree for a quadrangulation



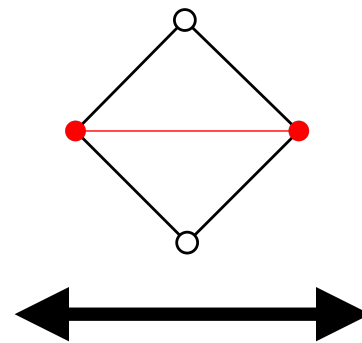
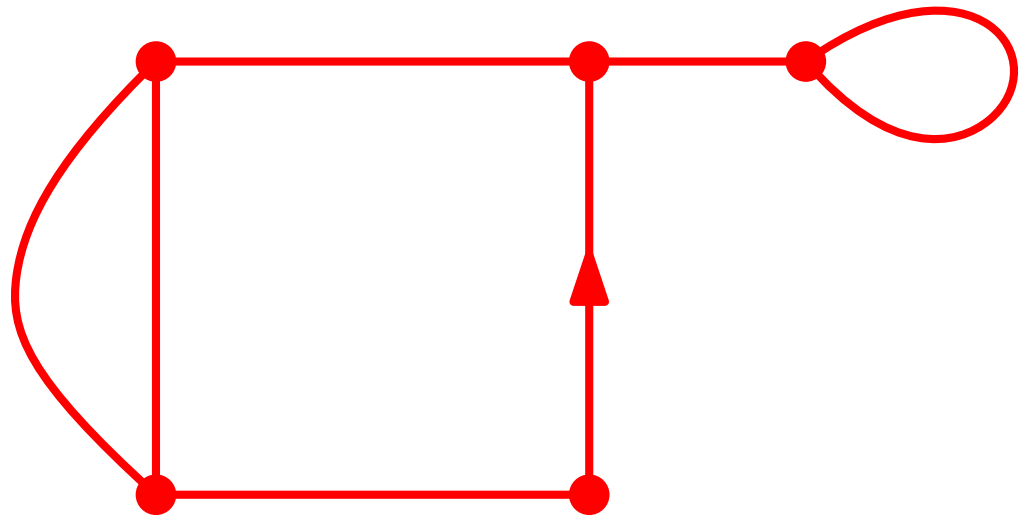
With a weight u on blocks: $Q(z, u) = uS(zQ^2(z, u)) + 1 - u$

Remember: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

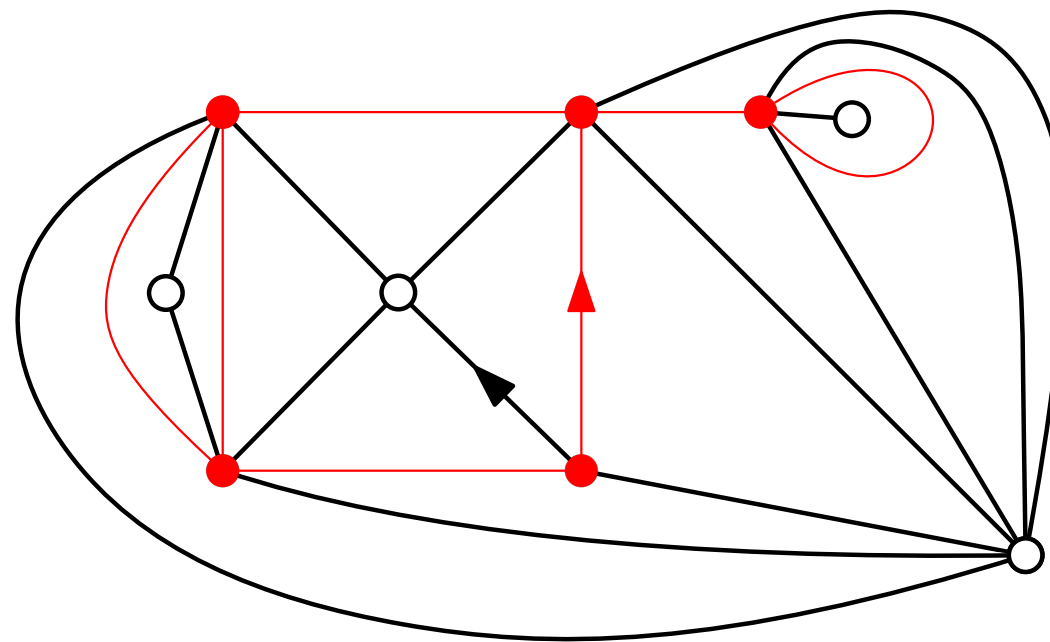
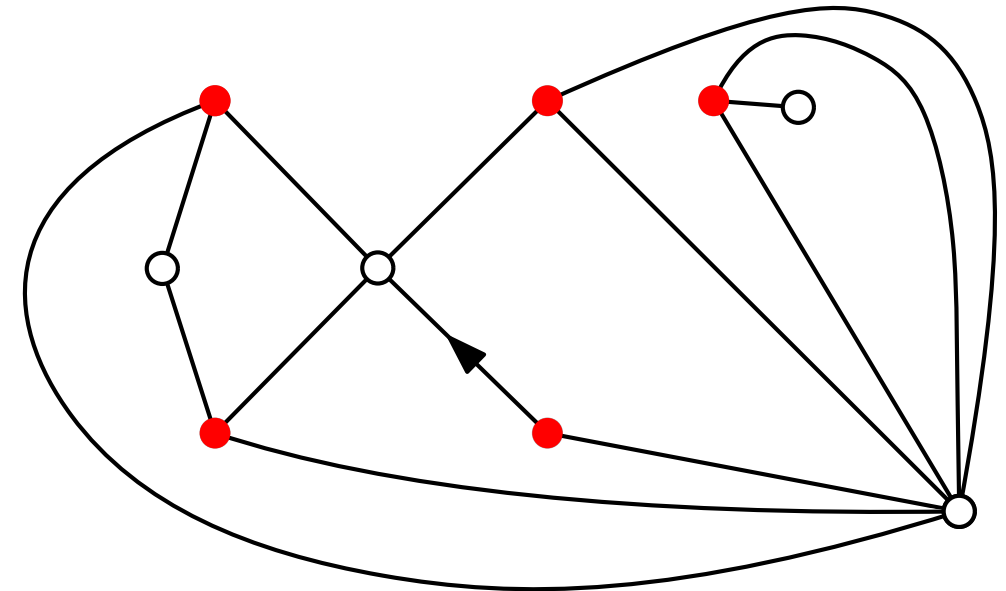
Tutte's bijection



Map

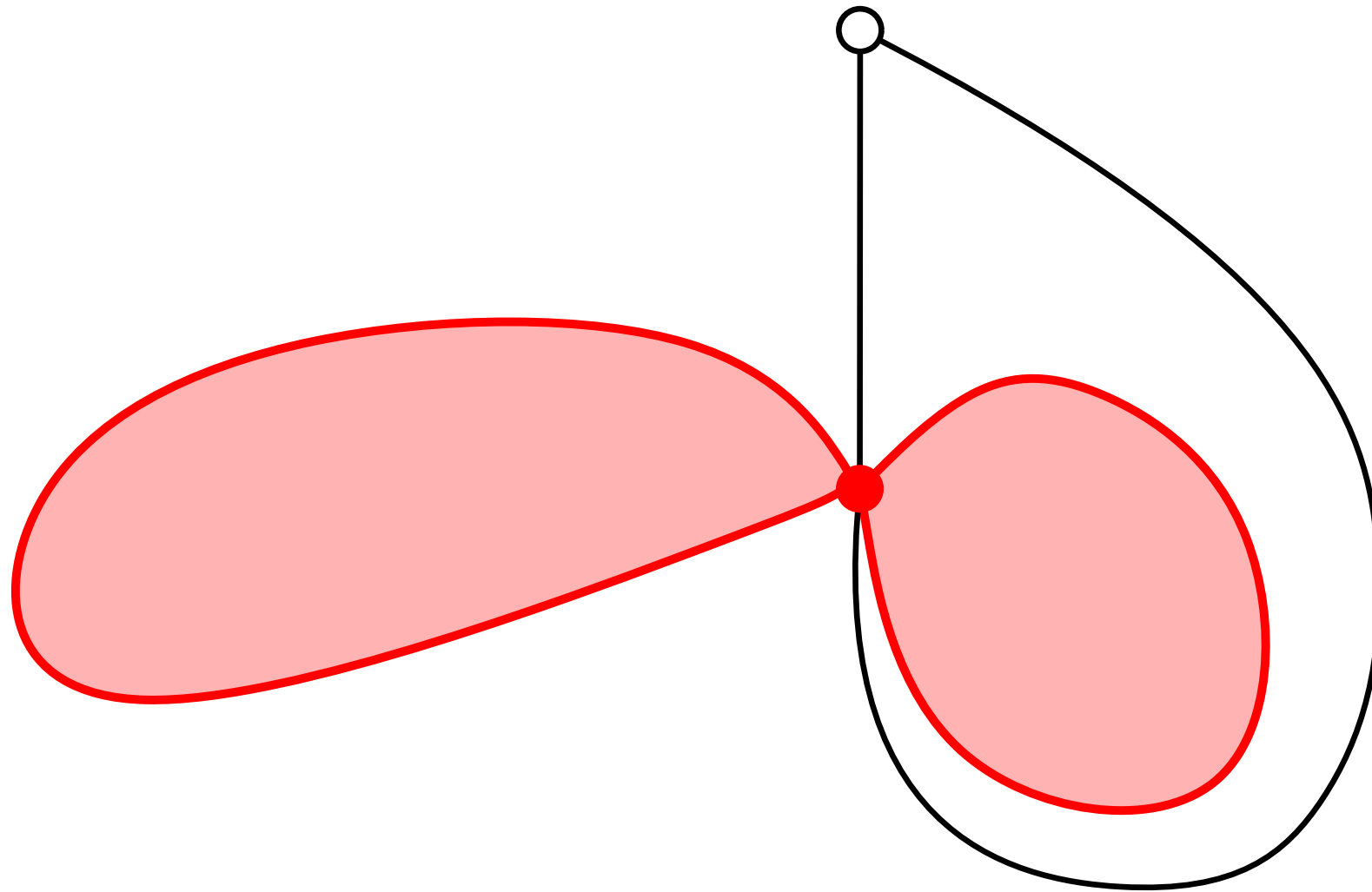


Quadrangulation



[Tutte 1963]

Tutte's bijection for 2-connected maps

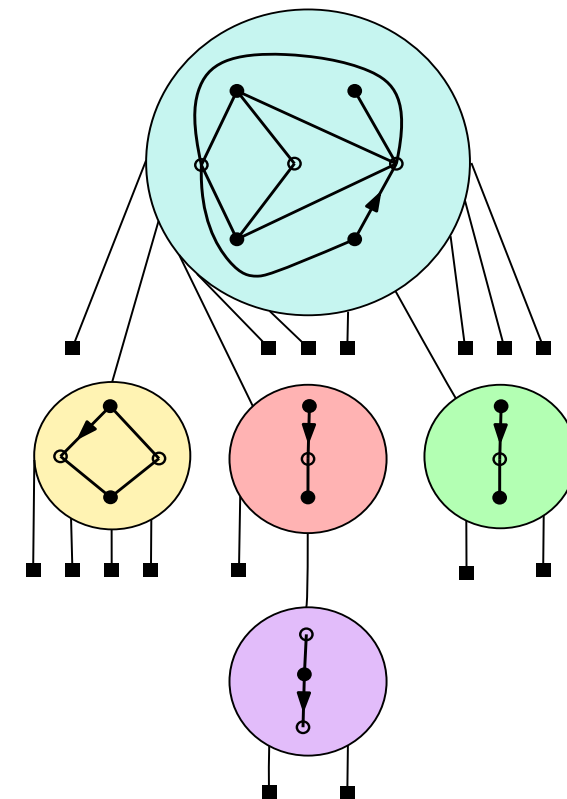
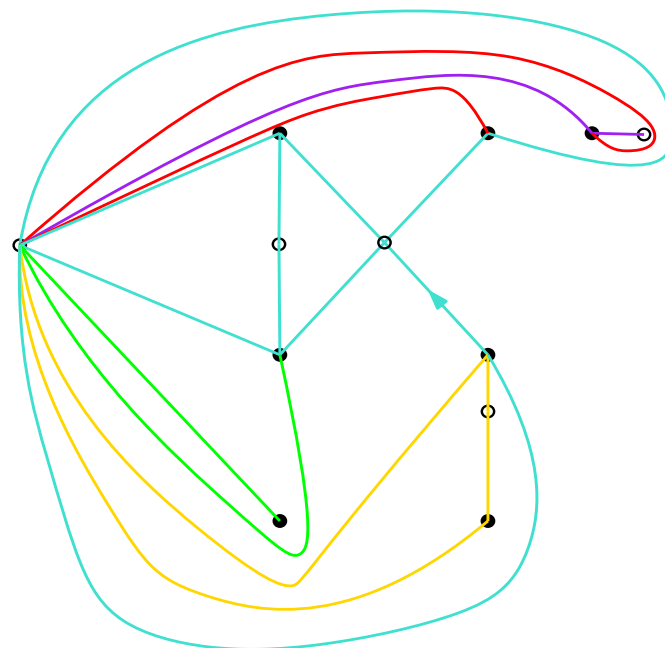
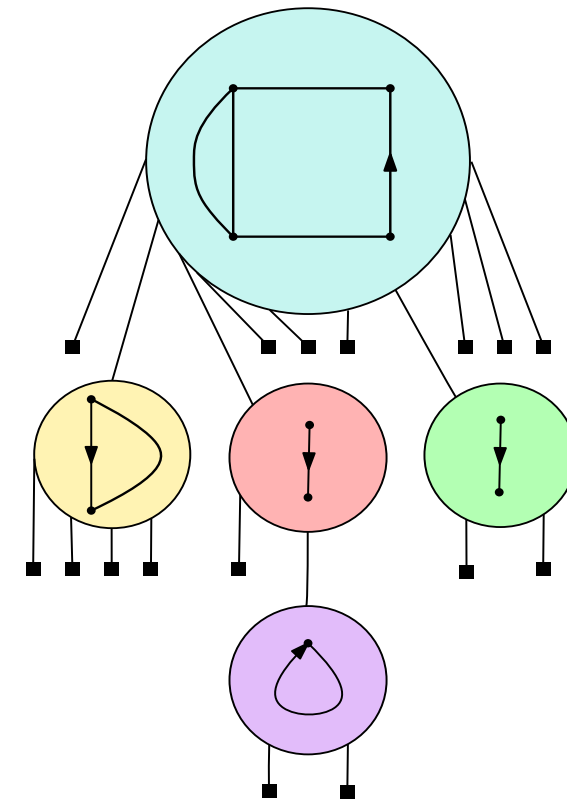
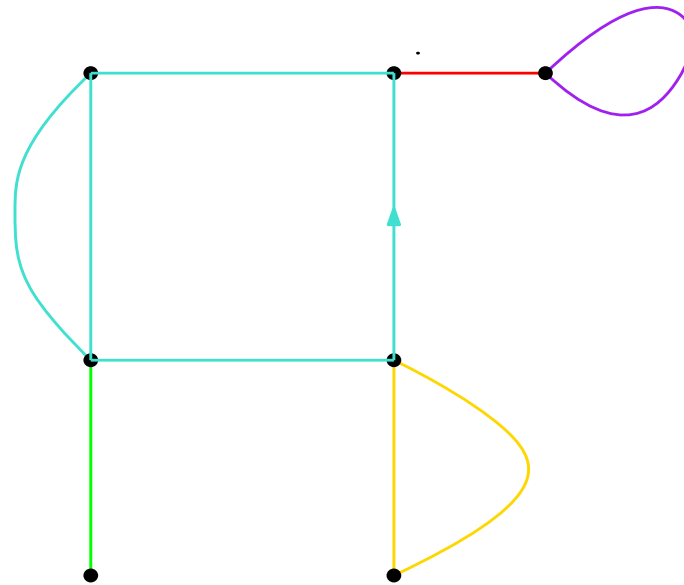


Cut vertex $\Rightarrow$ multiple edge

2-connected maps $\Leftrightarrow$ simple quadrangulations

[Brown 1965]

Block trees under Tutte's bijection



Implications on results

We choose: $\mathbb{P}_{n,u}(\mathbf{q}) = \frac{u^{\#blocks(\mathbf{q})}}{Z_{n,u}}$ where

$u > 0$,

$\mathcal{Q}_n = \{\text{quadrangulations of size } n\}$,

$\mathbf{q} \in \mathcal{Q}_n$,

$Z_{n,u} = \text{normalisation.}$

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.

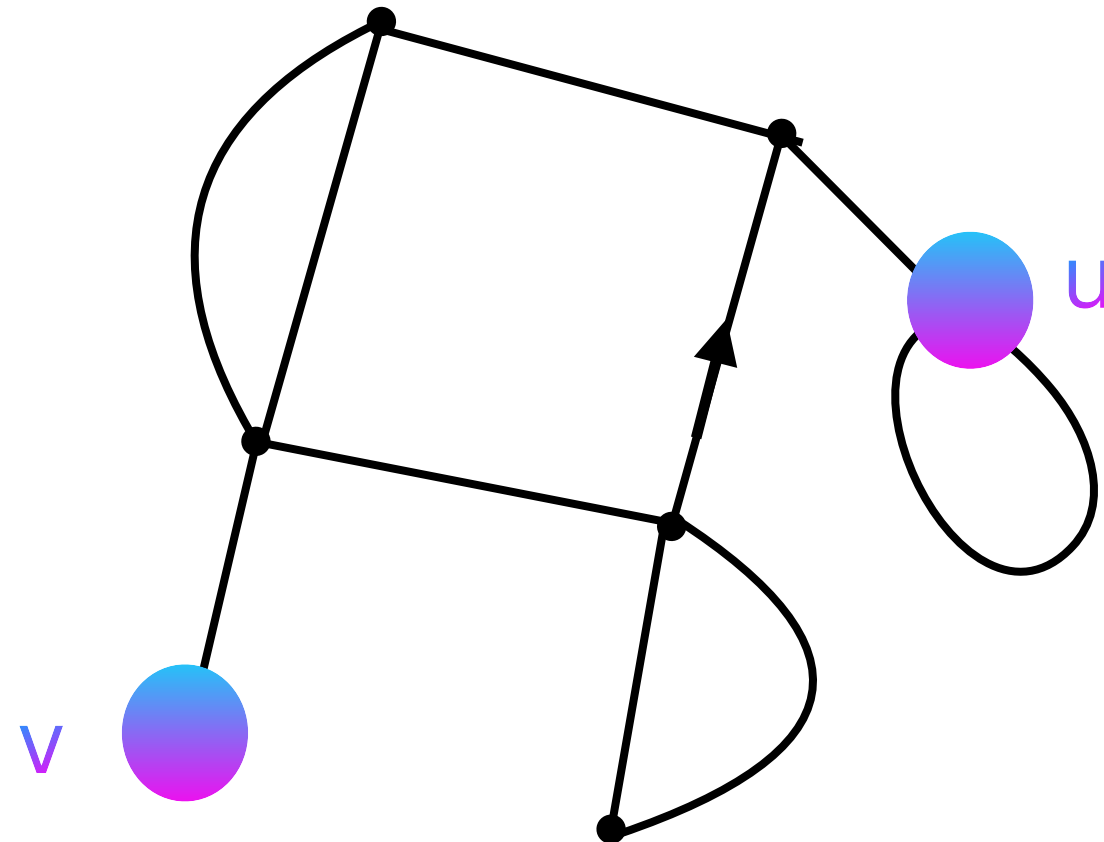
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016] for 2-c case	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

IV. Scaling limits

Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.



$$d(u, v) = 4$$

$$M_n \hookrightarrow \mathbb{P}_{n,u}$$

What is the limit of the sequence of metric spaces $((M_n, d/n^?))_{n \in \mathbb{N}}$?

(Convergence for Gromov-Hausdorff metric)

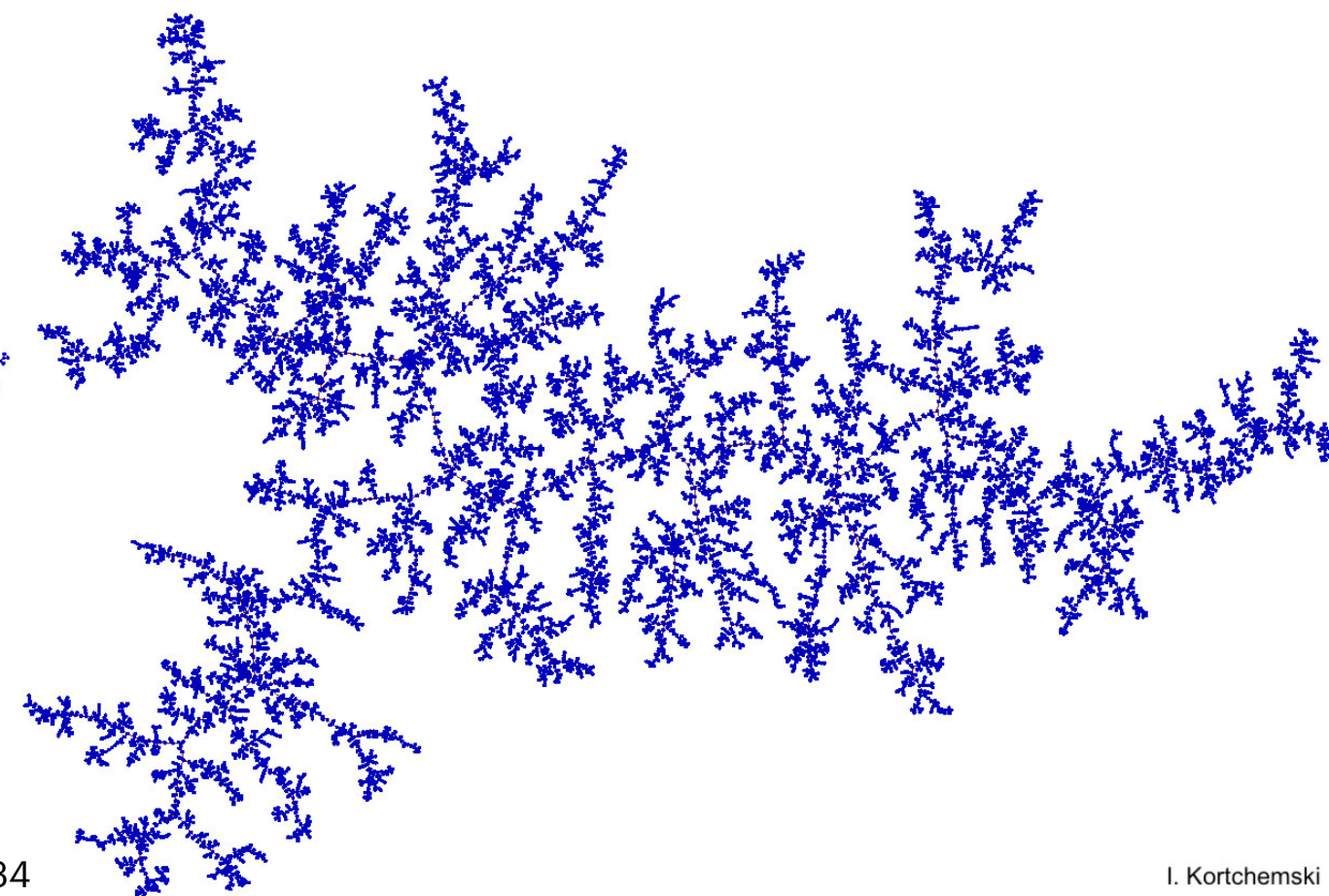
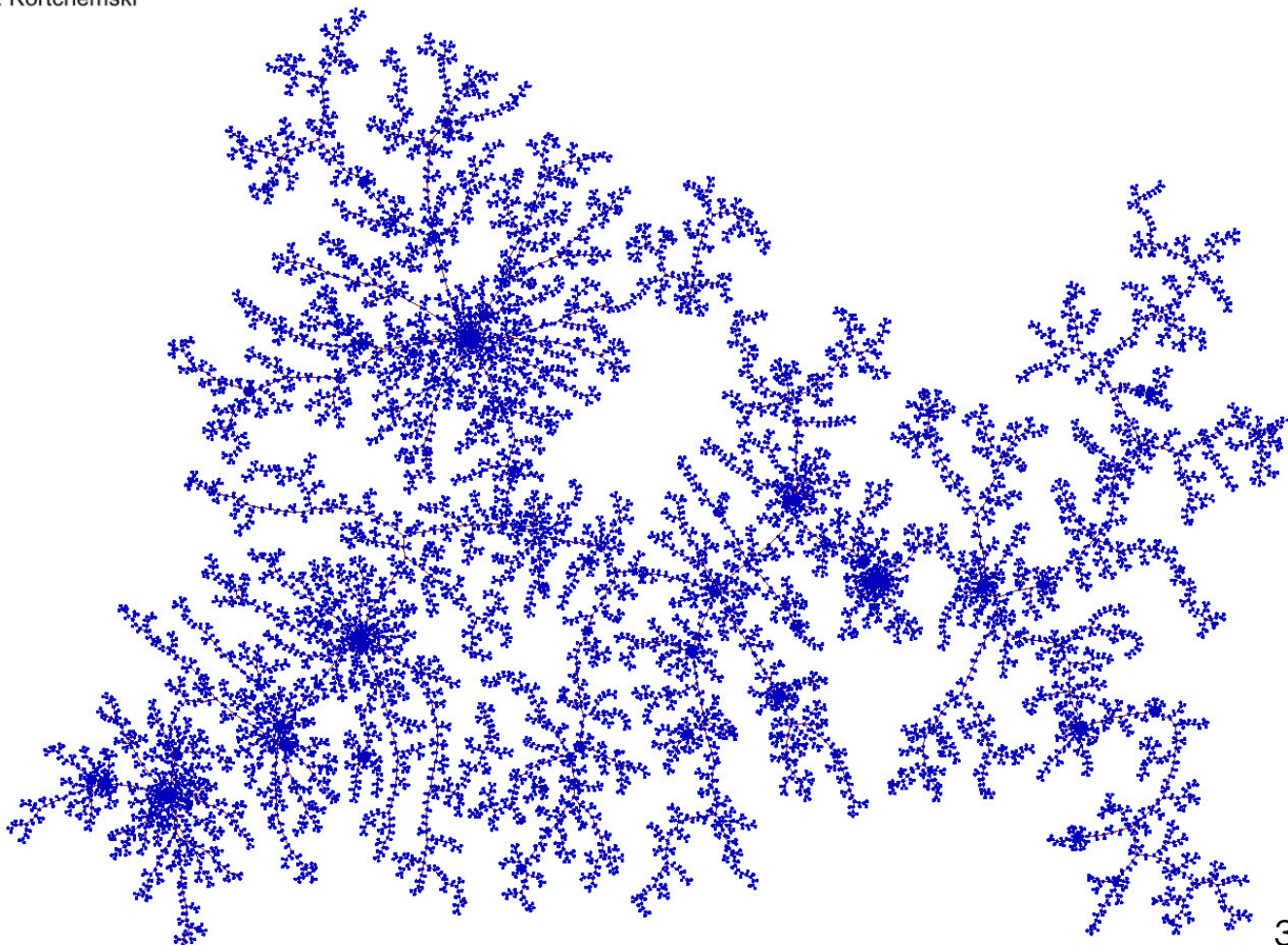
Scaling limits of Galton-Watson trees

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$, $\frac{c_3(u)}{n^{1/2}} T_{M_n} \rightarrow \mathcal{T}_e$.
- If $u = 9/5$, $\frac{c_2}{n^{2/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2}$.

Stable tree of index $3/2$ $\mathcal{T}_{3/2}$

Brownian tree $\mathcal{T}_e$ (Aldous's CRT)



Scaling limits of Galton-Watson trees

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$, $\frac{c_3(u)}{n^{1/2}} T_{M_n} \rightarrow \mathcal{T}_e$.
- If $u = 9/5$, $\frac{c_2}{n^{2/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2}$.

Proof

- Scaling limit of **critical** Galton-Watson trees with finite variance [Aldous 1993, Le Gall 2006];
- Scaling limit of **critical** Galton-Watson with infinite variance and nice tails [Duquesne 2003].

Scaling limit of supercritical and critical maps

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$,

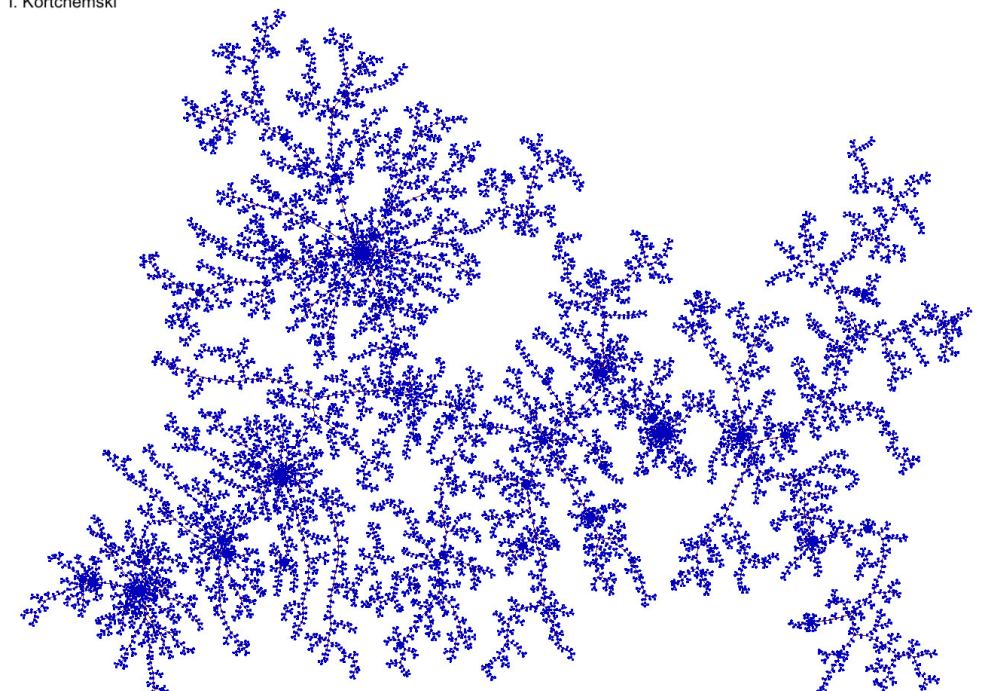
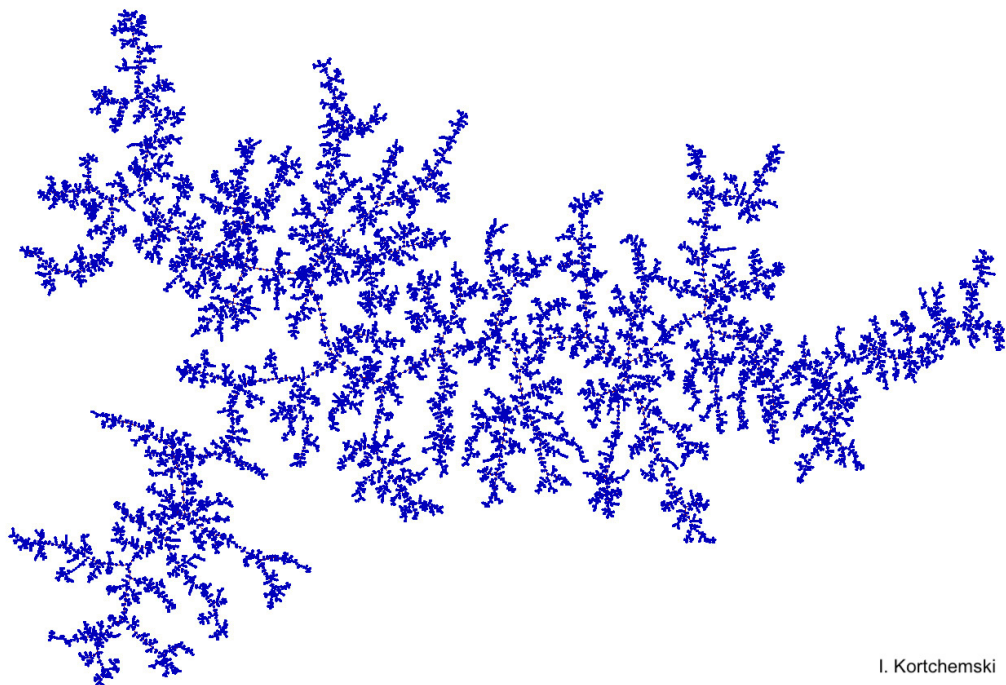
$$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{C_2}{n^{2/3}} M_n \rightarrow \mathcal{T}_{3/2}.$$

[Stufler 2020]

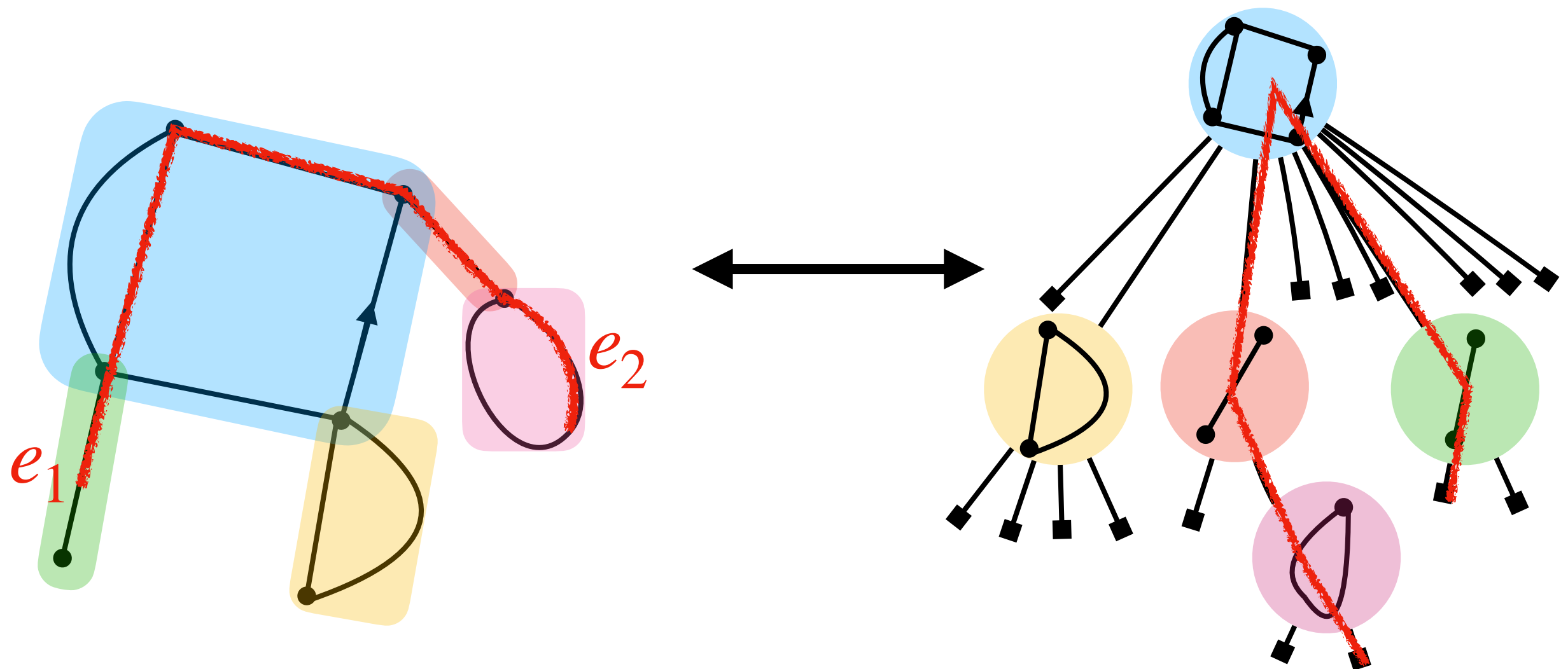
I. Kortchemski



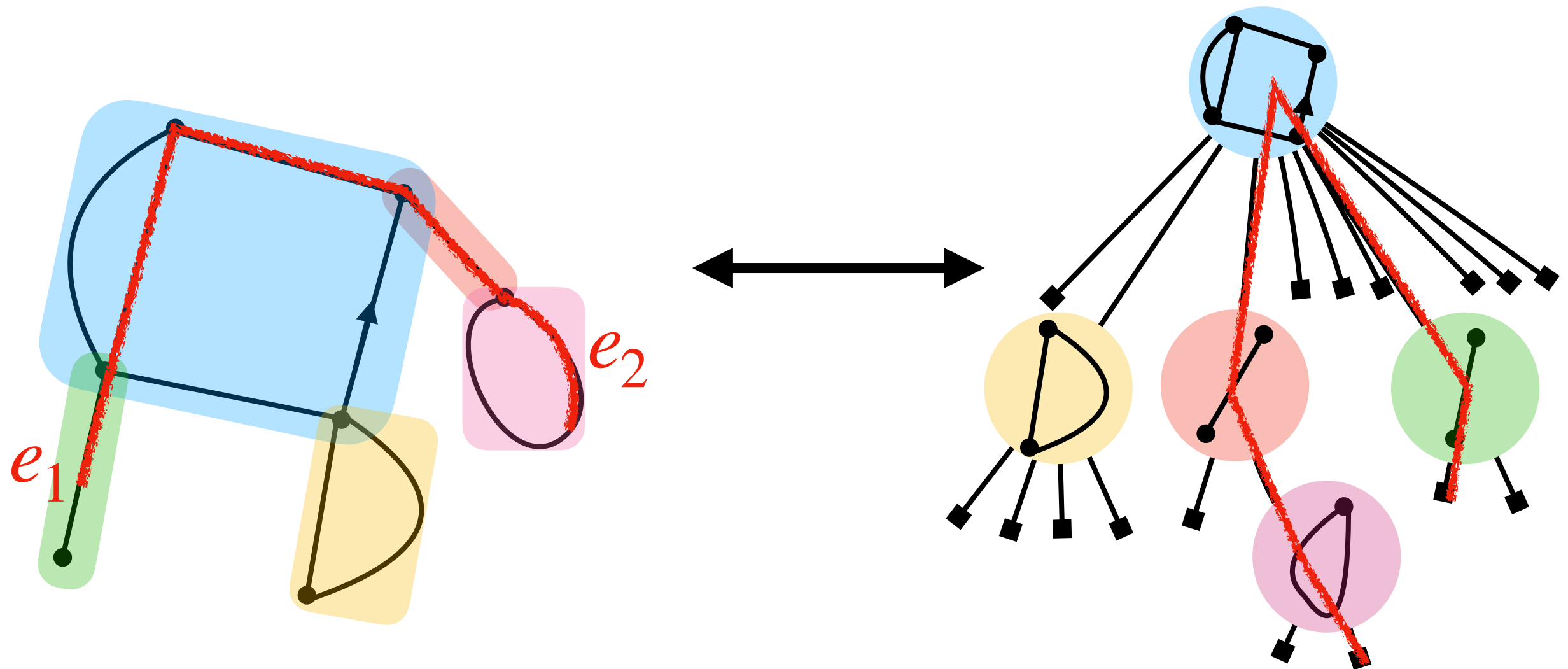
Supercritical and critical cases (1)

Difficult part = show that distances in $\mathfrak{M}$ behave like distances in $T_{\mathfrak{M}}$. We show

$$\forall e_1, e_2 \in \vec{E}(M_n), d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$



Supercritical and critical cases (2)



Let $\kappa = \mathbb{E}(\text{"diameter" bipointed block})$. By a "law of large numbers"-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$

Difficult for the critical case => use diameter estimates

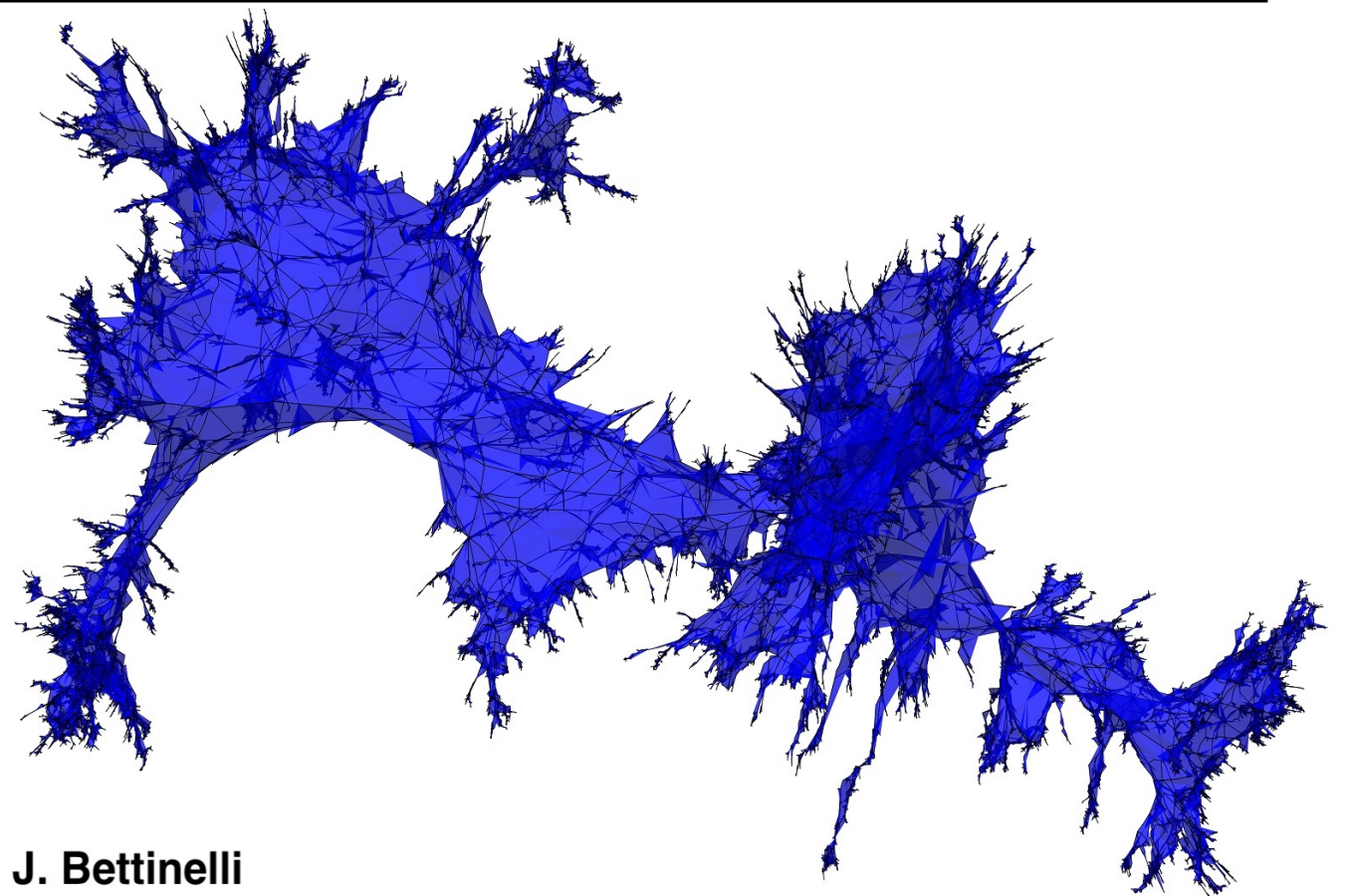
Scaling limits of subcritical maps

Theorem If $u < 9/5$, for $M_n \hookrightarrow \mathbb{P}_{n,u}$ a quadrangulation,

$$\frac{C_1(u)}{n^{1/4}} M_n \rightarrow \mathcal{S}_e.$$

Moreover, M_n and its simple core converge jointly to the same Brownian sphere.

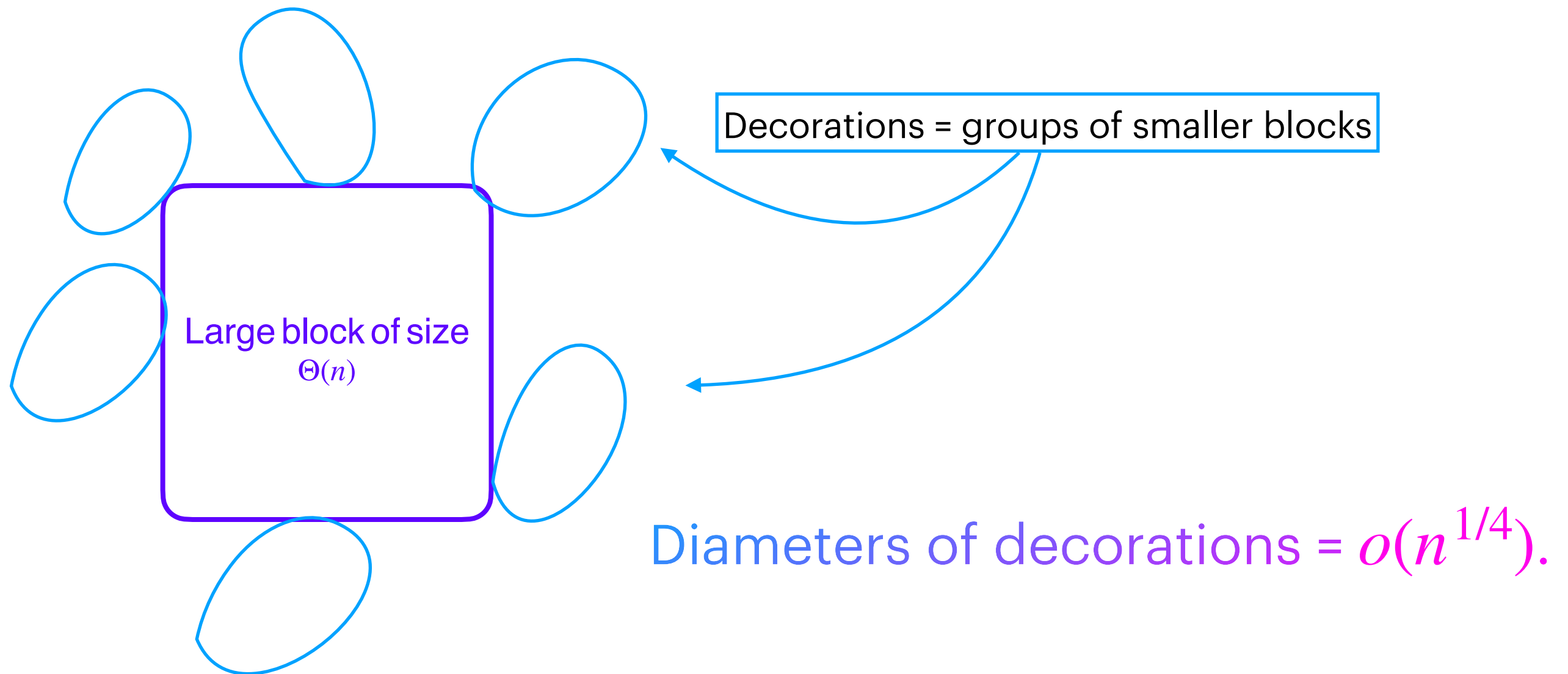
We expect the same scaling limits for maps but the scaling limit of 2-connected maps is not yet proved.



J. Bettinelli

See [Addario-Berry, Wen 2019] for a similar result and method

Subcritical case (1)



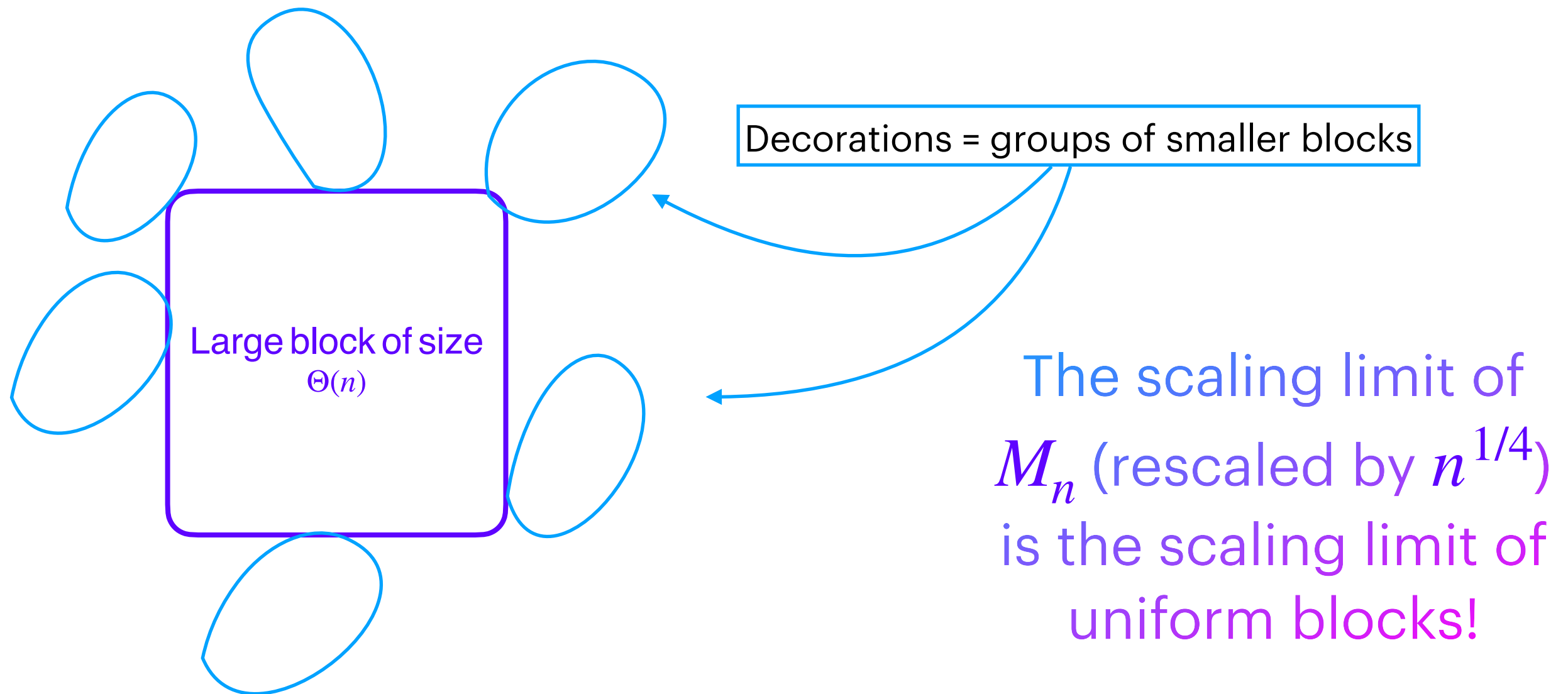
Diameter of a decoration $\leq$ number of blocks $\times$ max diameter of blocks

$$\leq \text{diam}(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = \text{diam}(T_{M_n}) \times O(n^{1/6+\delta})$$

T_{M_n} is a subcritical Galton-Watson tree $= O(n^{1/6+2\delta}) = o(n^{1/4})$.

[Chapuy Fusy Giménez Noy 2015]

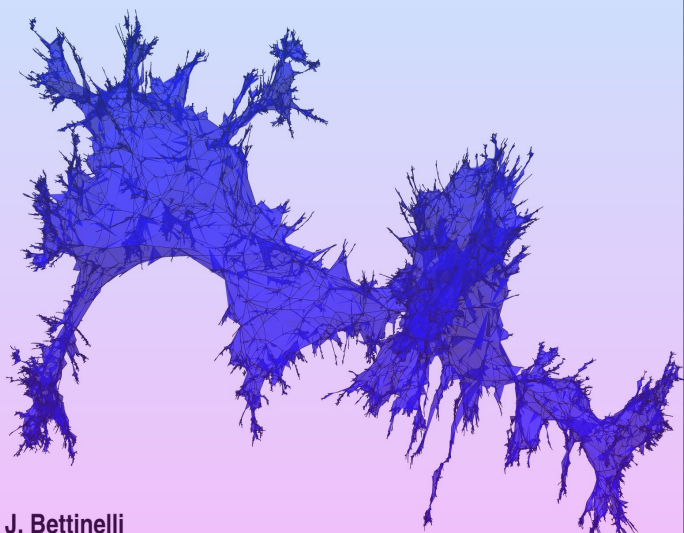
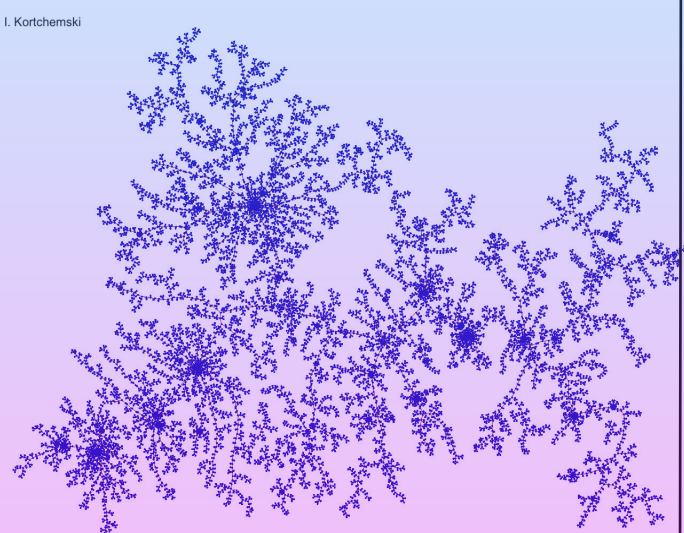
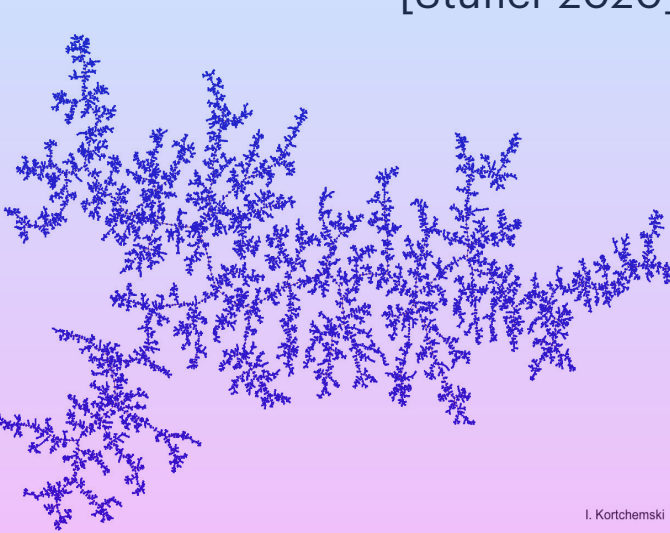
Subcritical case (2)



Scaling limit of uniform $\sim$ (rescaled by $n^{1/4}$)

- 2-connected maps = brownian sphere (assumed);
- Simple quadrangulations = Brownian sphere [Addario-Berry Albenque 2017].

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration Bonzom 2016 for 2-c case	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \rightarrow \mathcal{S}_e$  J. Bettinelli Assuming the convergence of 2-connected maps towards the brownian sphere	$\frac{C_2}{n^{1/3}} M_n \rightarrow \mathcal{T}_{3/2}$  I. Kortchemski	$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$  I. Kortchemski [Stufler 2020]

V. Perspectives

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, $M_2(z)$ or loopless	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1 + M)$	—
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1 + M)^2$	—
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1 + M)$	—
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1 + M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1 + M)$	—
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1 + M))^2$	$z(1 + M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1 + M)^2$	—
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1 + M)$	—
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1 + M)^3$	—
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1 + M)^2$	—

Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n)$, $\varepsilon^3 n \rightarrow \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, conjecture $L_{n,1} \sim 2.7648 \varepsilon^{-2} \ln(\varepsilon^3 n)$ when $\varepsilon^3 n \rightarrow \infty$ (analogous to [Bollobás 1984]’s result for Erdős-Rényi graphs!);
- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?

Perspectives

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u_n = 9/5 - \varepsilon(n)$ $\varepsilon^3 n \rightarrow \infty$	$u = 9/5$	$u_n = 9/5 + \varepsilon(n)$ $\varepsilon^3 n \rightarrow \infty$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$		$\Theta(n^{2/3})$	$\sim 2.7648 \, \varepsilon^{-2} \ln(\varepsilon^3 n)$	$\frac{\ln(n)}{2 \ln\left(\frac{4}{27y}\right)} - \frac{5 \ln(\ln(n))}{4 \ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$				
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \rightarrow \mathcal{S}_e$	$\varepsilon(n) = n^{-\alpha}$ $\frac{C_4}{n^{(1-\alpha)/4}} M_n \rightarrow \mathcal{S}_e$	$\frac{C_2}{n^{1/3}} M_n \rightarrow \mathcal{T}_{3/2}$	stable tree ?	$\frac{C_3(u)}{n^{1/2}} M_n \rightarrow \mathcal{T}_e$
	Admitting the convergence of 2-connected maps towards the brownian map				

Pink = work in progress

Thank you!