Der Wissenschaftsfonds.

The maximal degree of random unlabelled series-parallel graphs

Journées Aléa

11 mars 2026

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Joint work with Michael Drmota, Veronika Kraus, Konstantinos Panagiotou and Benedikt Stufler

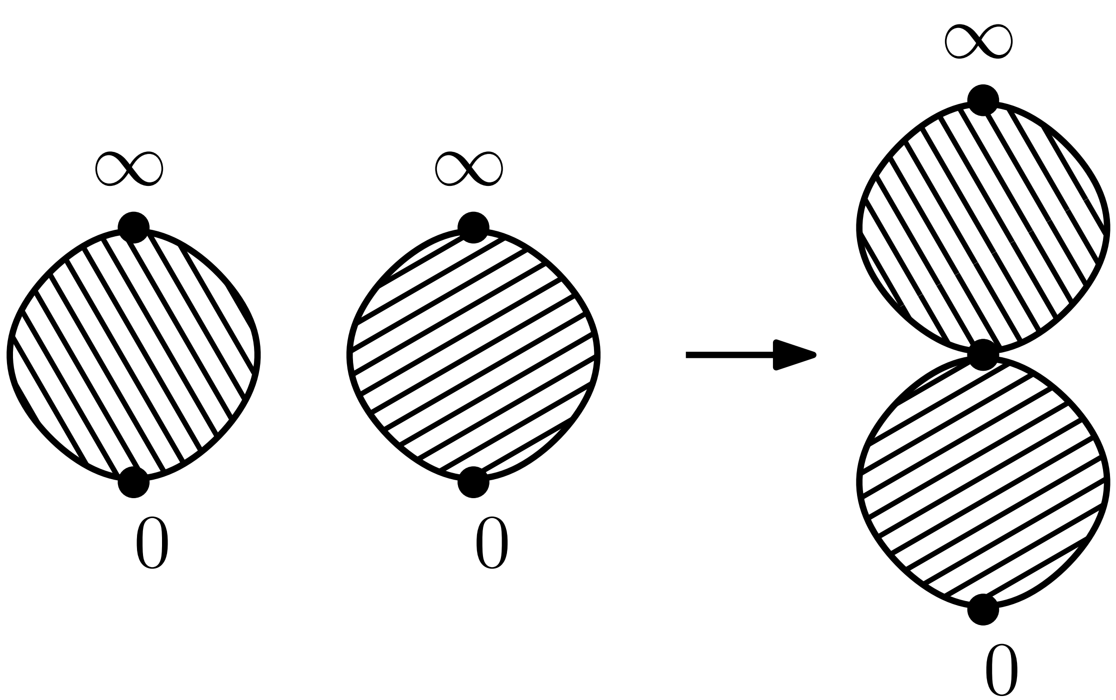
TU Wien, Austria

Series-parallel networks

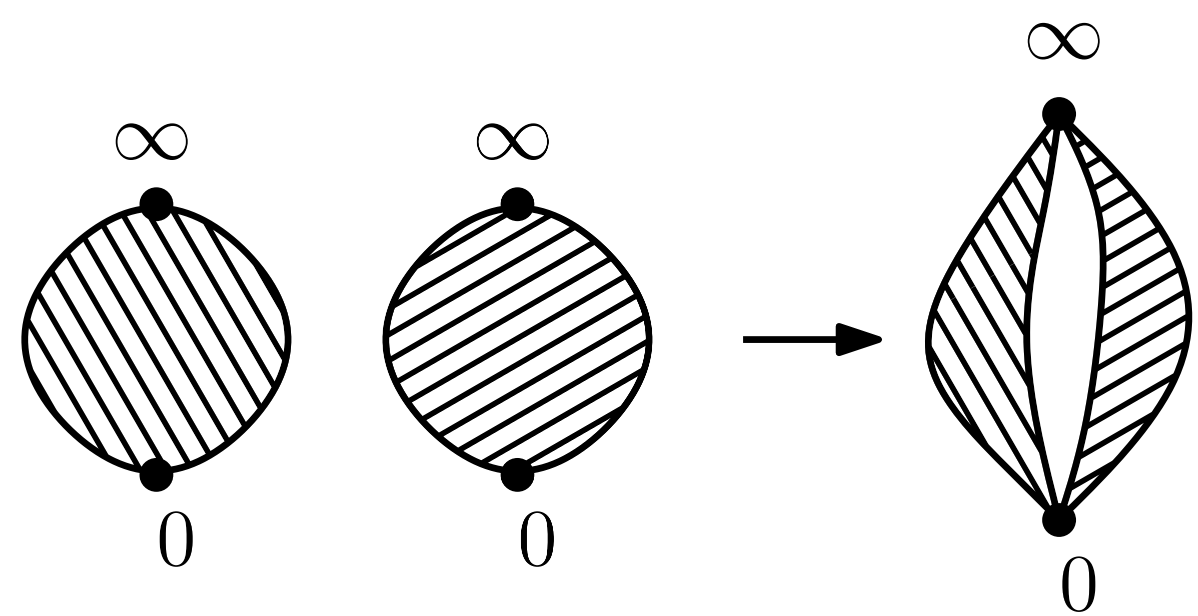
A bipointed (simple) graph is an SP network if it can be obtained from a sequence of these two operations:

- Series composition;
- Parallel composition;

starting from (several copies of) a bipointed edge.



Series composition



Parallel composition

Series-parallel networks

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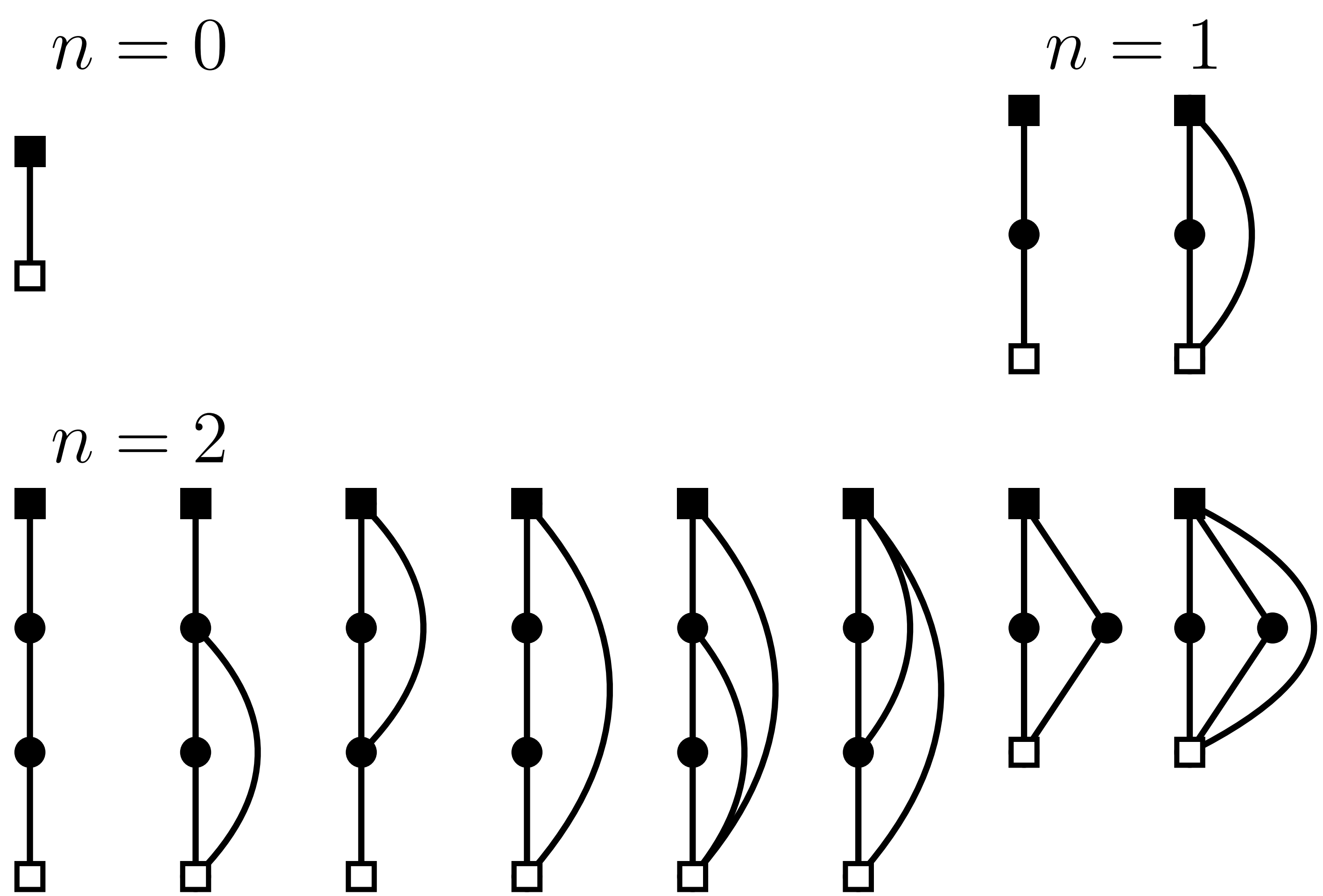

Series-parallel graphs

A (simple) graph C is a connected SP graph if it can be obtained from a sequence of these two operations:

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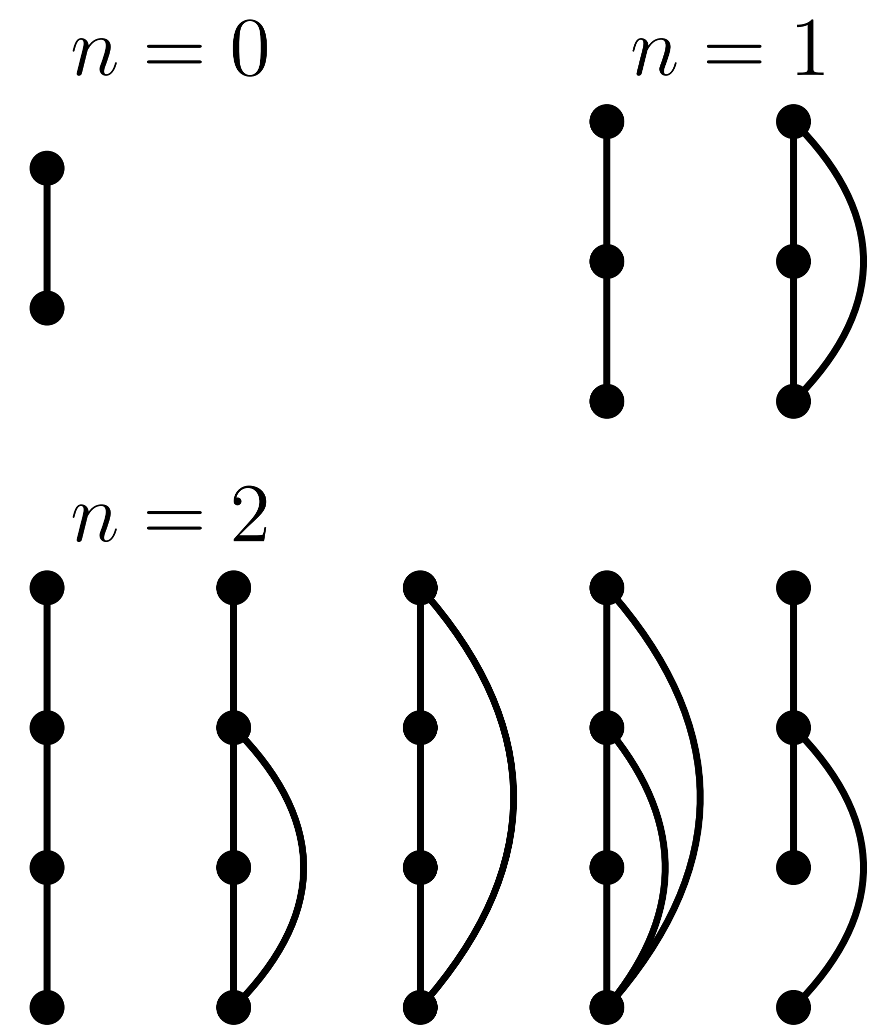
starting from (several copies of) bipointed trees; then forgetting the $0/\infty$ labelling.



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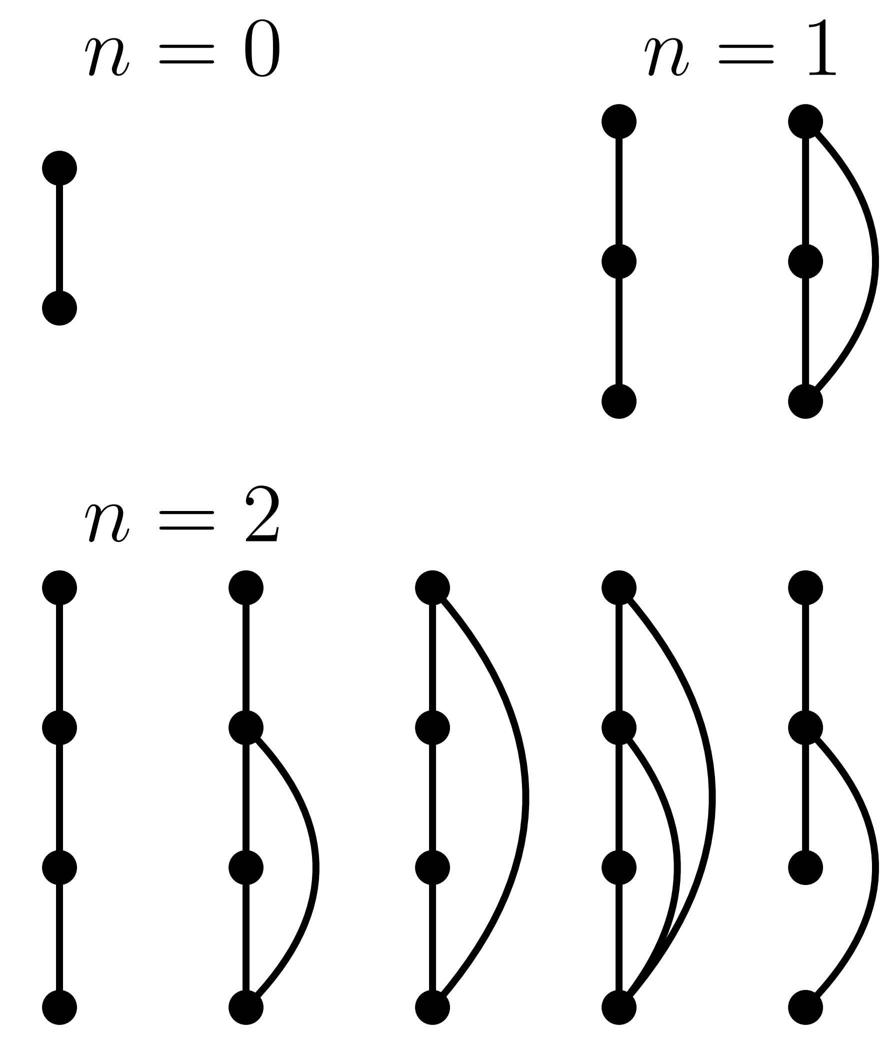


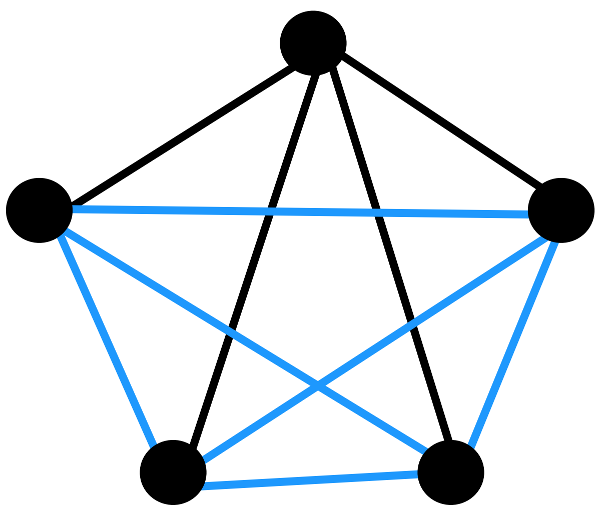
SP graphs = all connected components are connected SP graphs

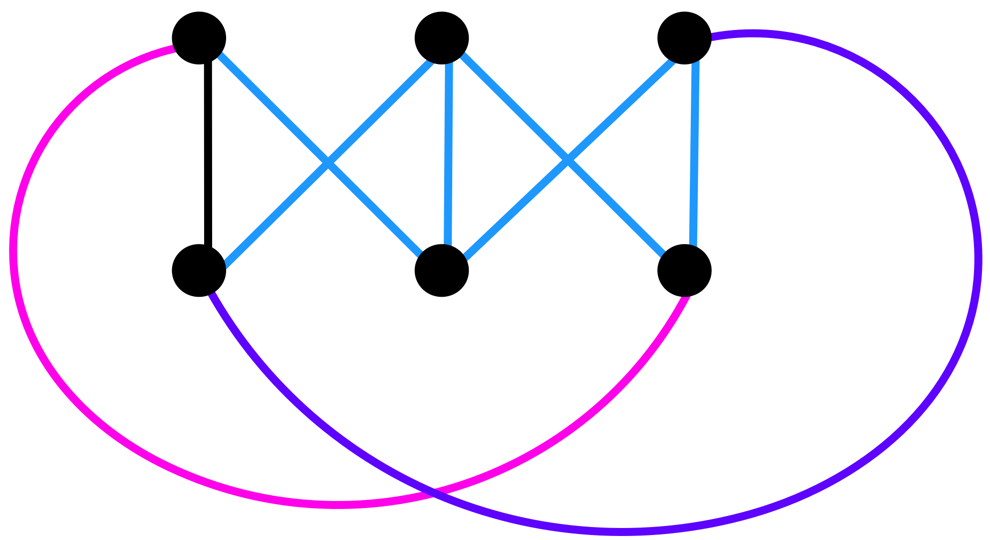
Motivation

SP graphs = no K_4 minor.

K_5 and $K_{3,3}$ have K_4 minor.

So SP graphs are planar!





=> Good first step before studying planar graphs

Results on series-parallel graphs

Labelled case | Unlabelled case

Subcritical class [Drmota Fusy Kang Kraus Rué 2011]

Results on series-parallel graphs

<u>Labelled case</u>	<u>Unlabelled case</u>

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Enumeration (by vertices)

Labelled case	Unlabelled case
$g_n \sim c_l \rho_l^{-n} n^{-5/2} n!$	$g_n \sim c_u \rho_u^{-n} n^{-5/2}$
[Bodirsky Giménez Kang Noy 2007]	[Drmota Fusy Kang Kraus Rué 2011]

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Scaling limit is the Brownian tree

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[Panagiotou Stufler Weller 2016]	rooted [Stufler 2018] & unrooted [Stufler 2019]

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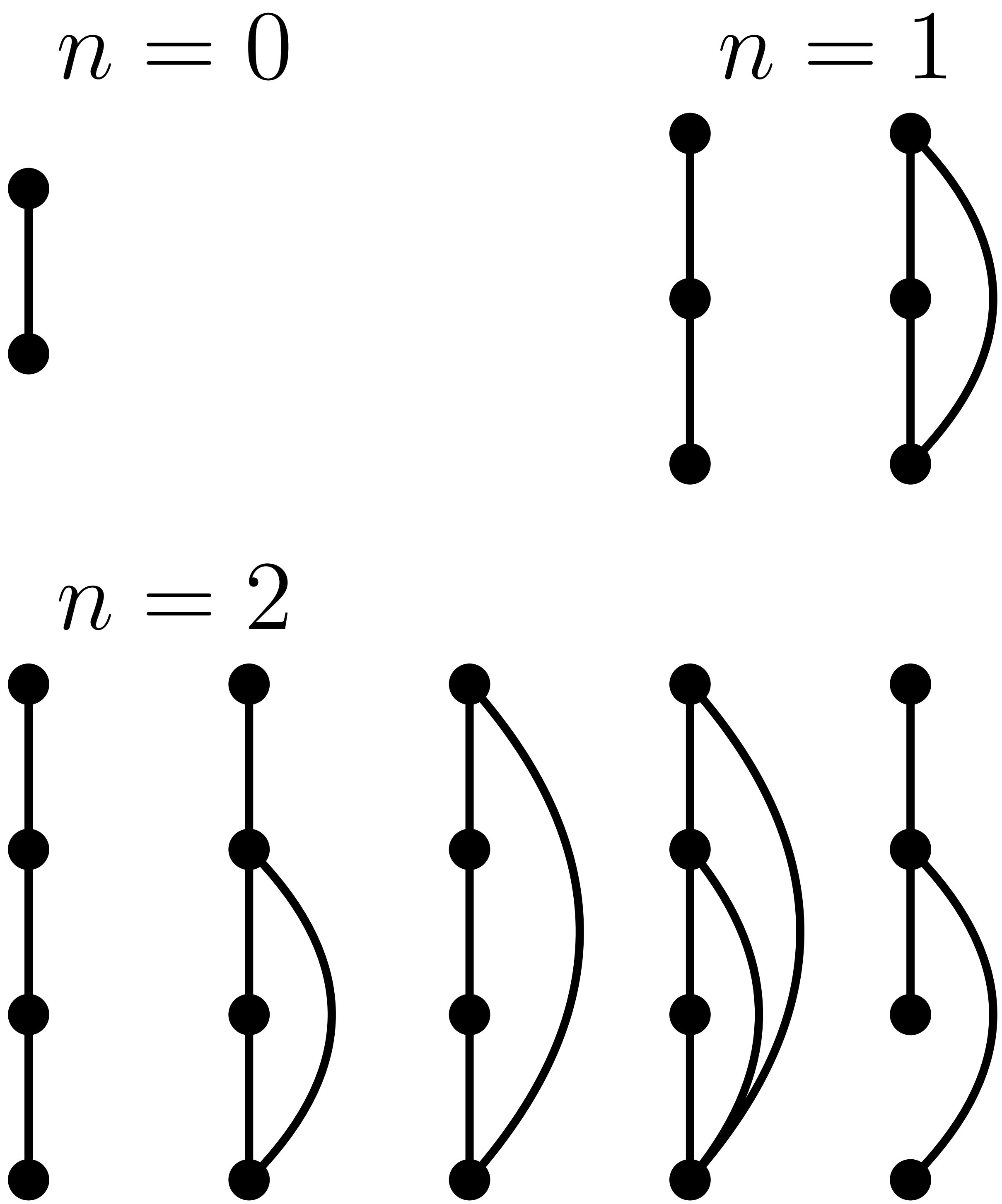
Local limit known [Georgakopoulos Wagner 2015, Stufler 2018]	
Maximum degree	
$\frac{\Delta_n}{\log(n)} \xrightarrow[n\to\infty]{\mathbb{P}} c > 0, \qquad \mathbb{E}(\Delta_n) \sim c \log(n)$ [Drmota Giménez Noy 2011]	this work

Labelled *vs* unlabelled

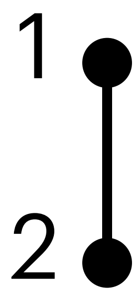
Unlabelled enum: $1, 2, 5\ldots$



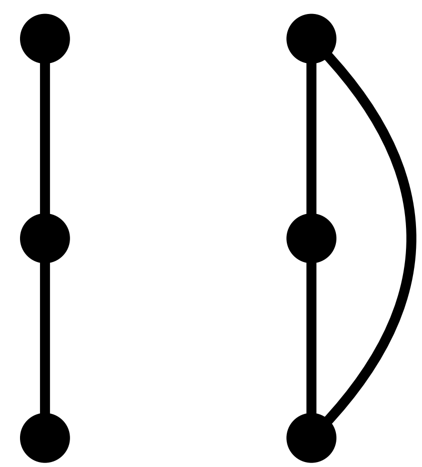
Labelled *vs* unlabelled

Unlabelled enum: $1, 2, 5\dots$

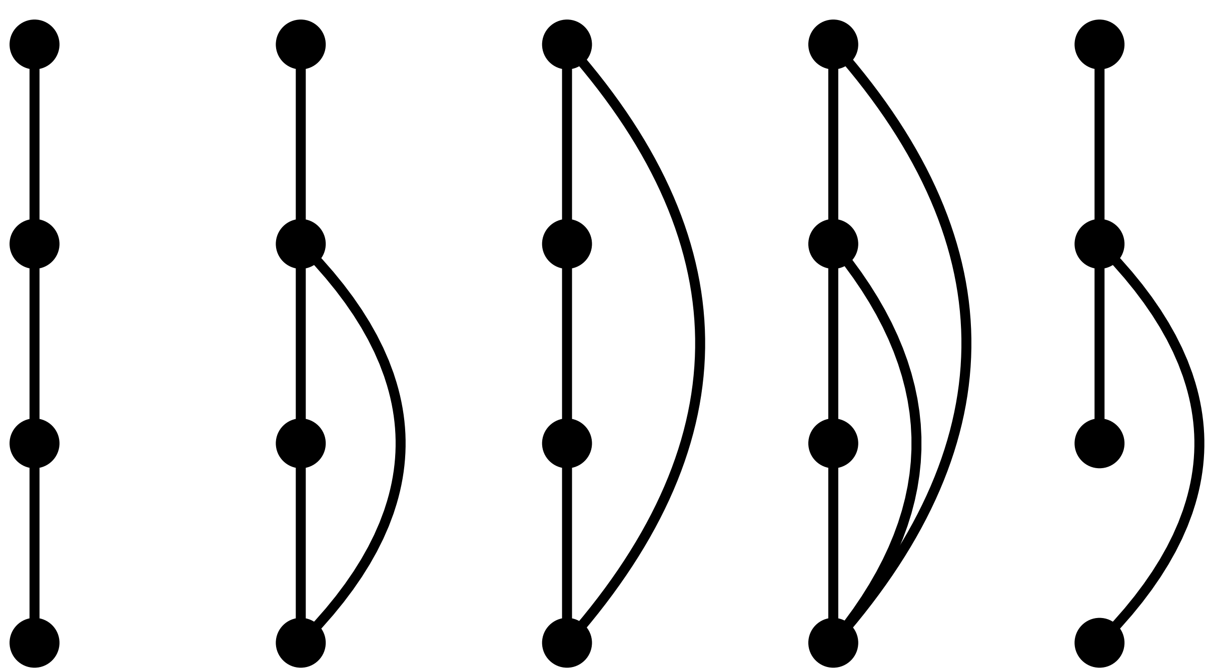
$n = 0$



$n = 1$



$n = 2$



Labelled *vs* unlabelled

Unlabelled enum: $1, 2, 5\ldots$

Labelled enum: 1

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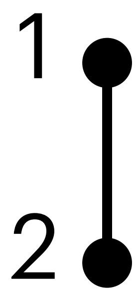

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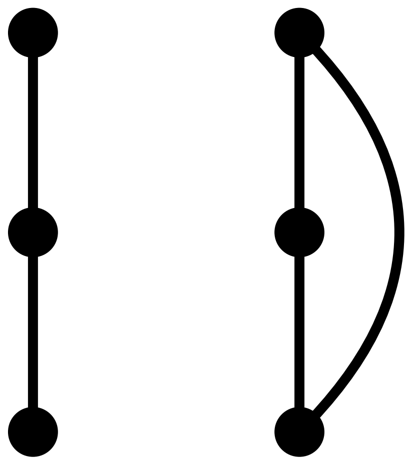

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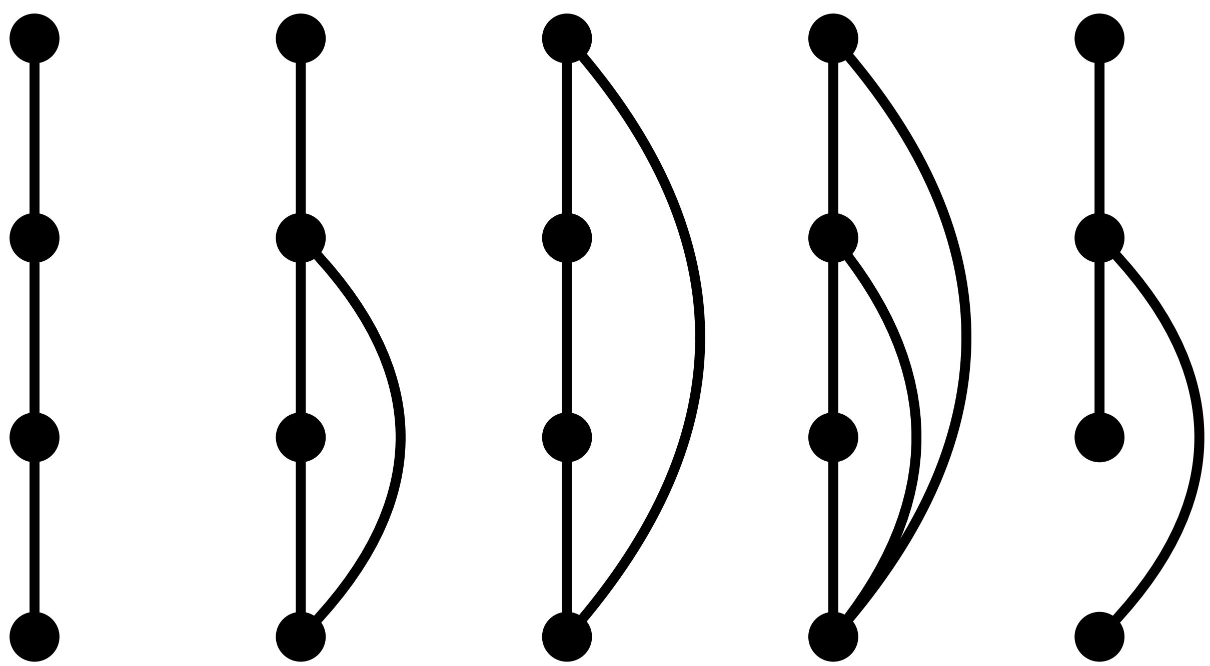

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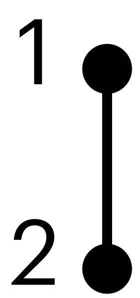

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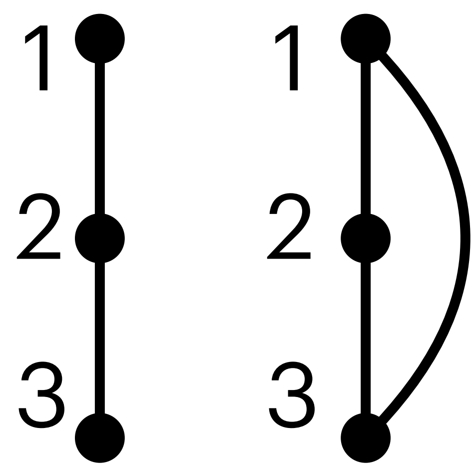

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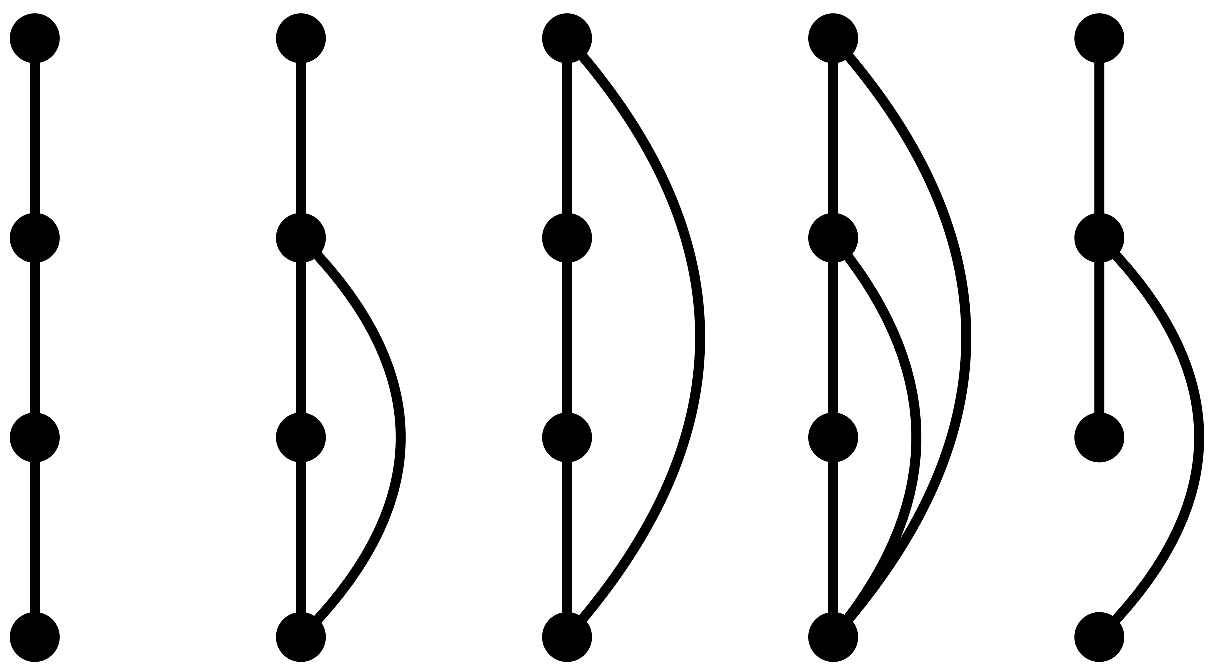

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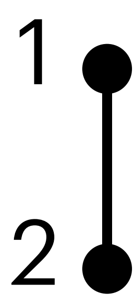

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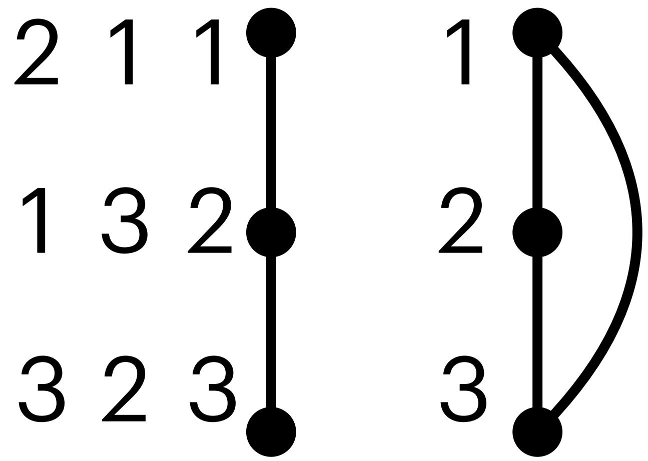
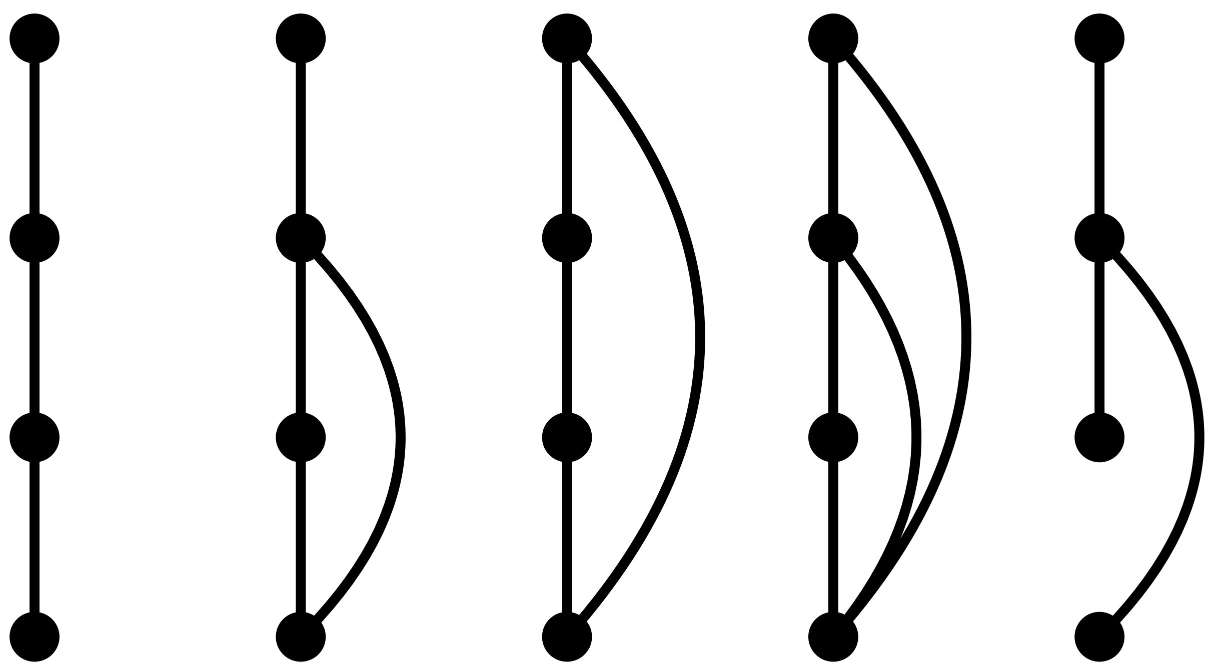


$n = 2$



Labelled *vs* unlabelled

Unlabelled enum: $1,\ 2,\ 5\ldots$

Labelled enum: $1,\ 4$

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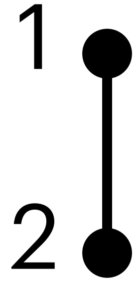

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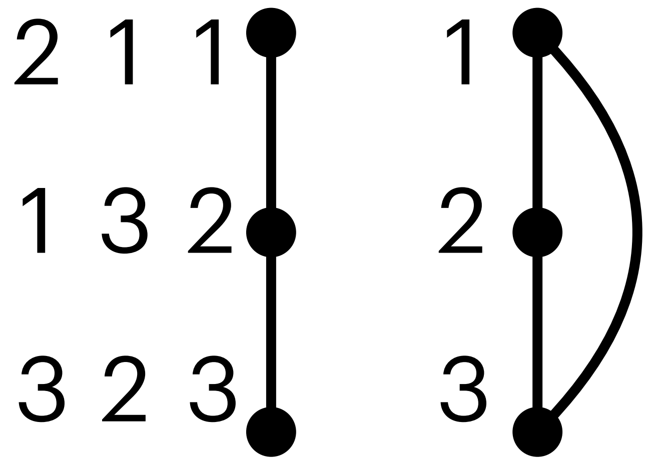

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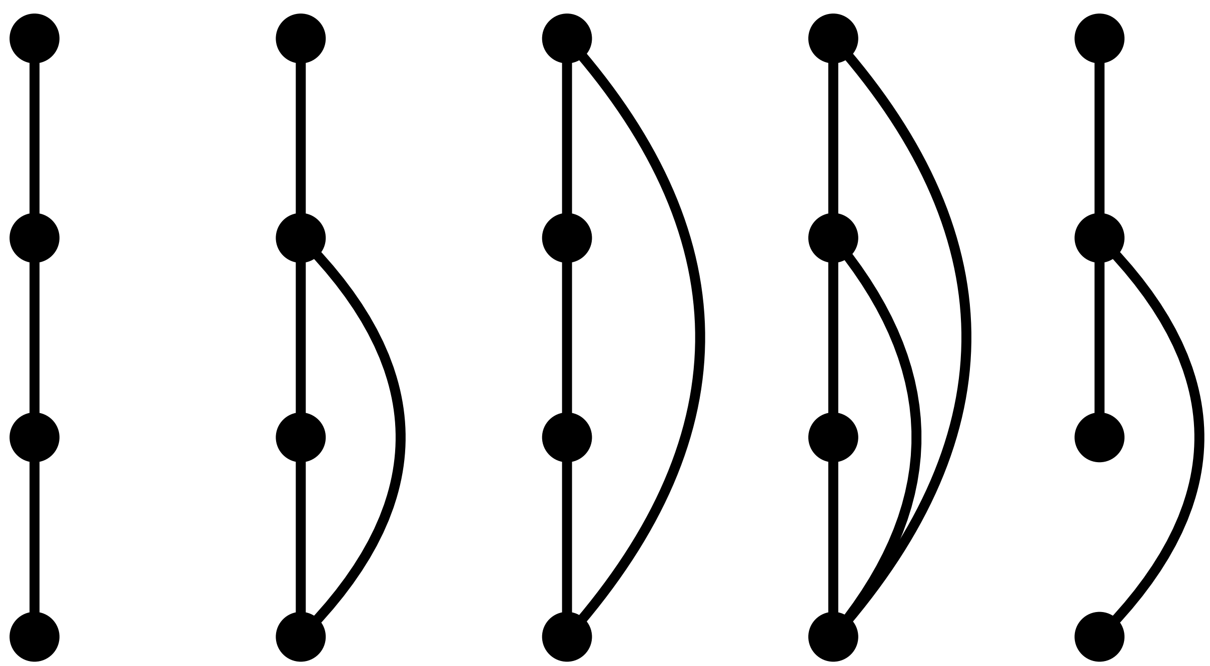

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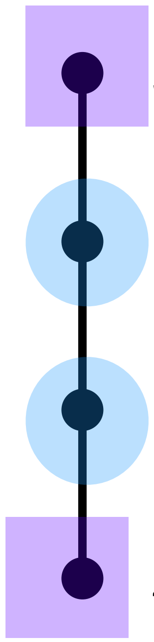

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2 1 4 3 1 1 4 4 3 3 2 2

1 2 1 1 4 3 3 2 4 2 4 3

4 4 3 4 3 4 2 3 2 4 3 4

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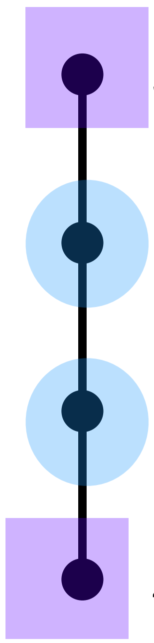

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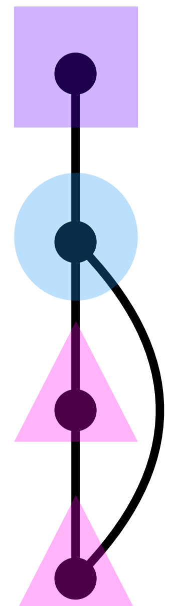

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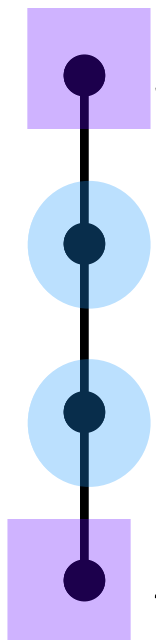

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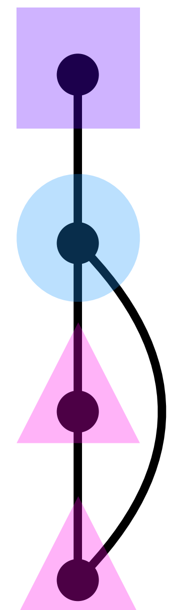

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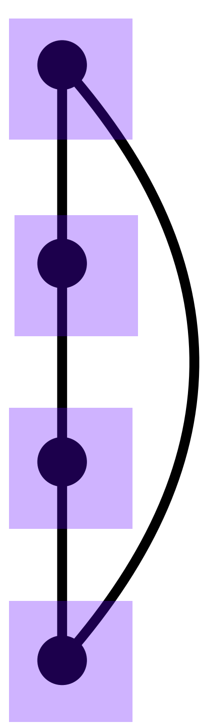

1 1 1
2 3 2
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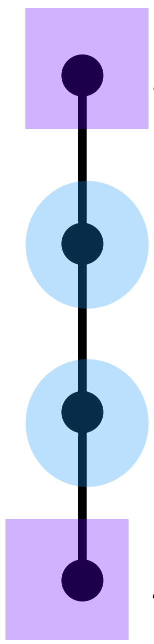

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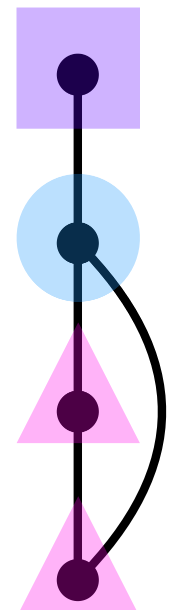

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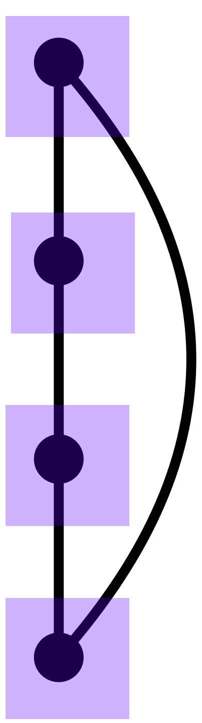

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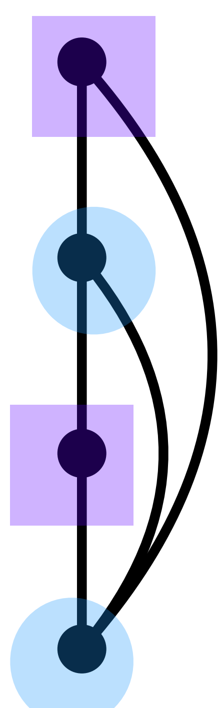

1 1 1 2 2 3
3 2 2 1 1 1
2 4 3 4 3 4
4 3 4 3 4 2

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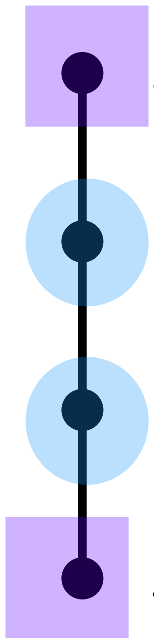

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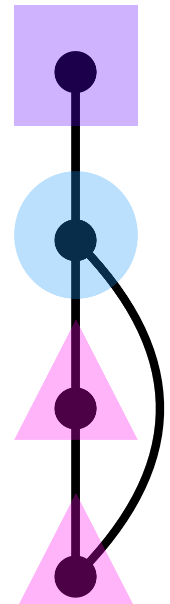

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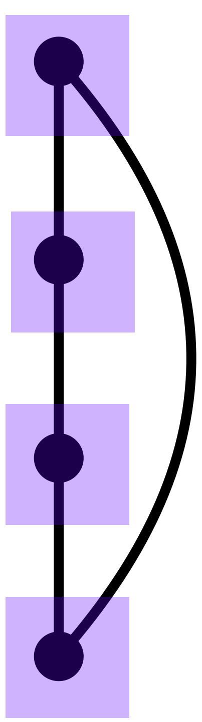

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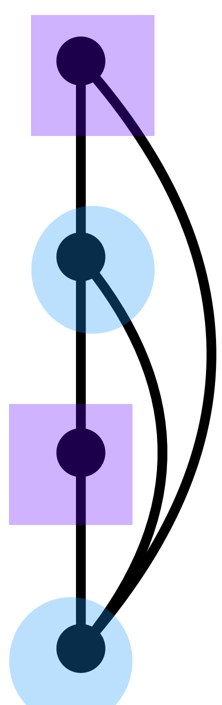

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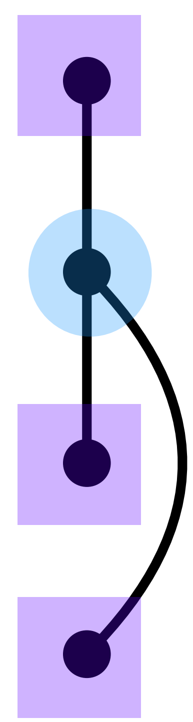

1 1 1 2
2 3 4 1
3 2 2 3
4 4 3 4

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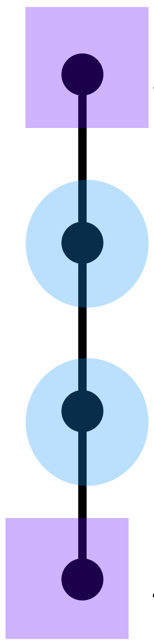

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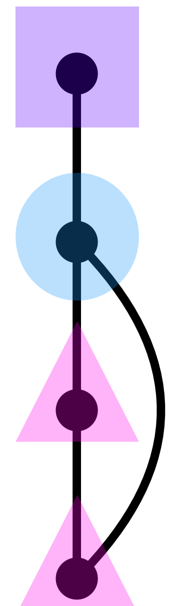

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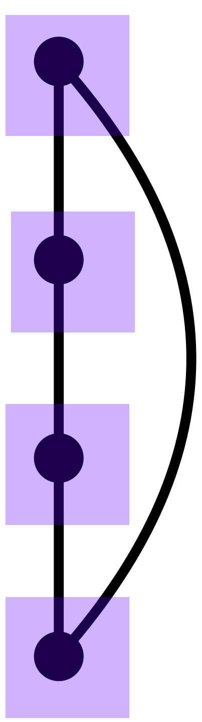

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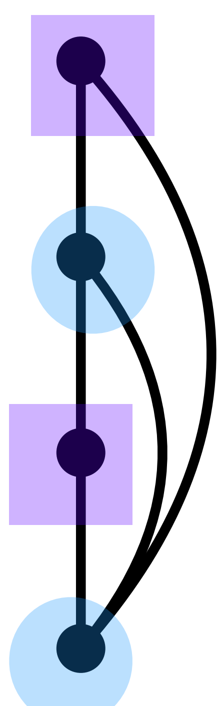

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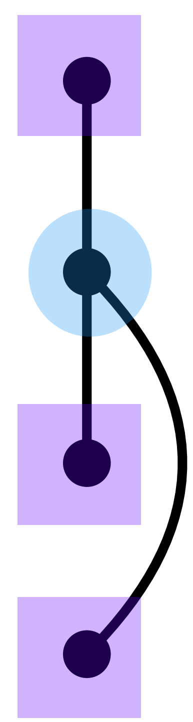

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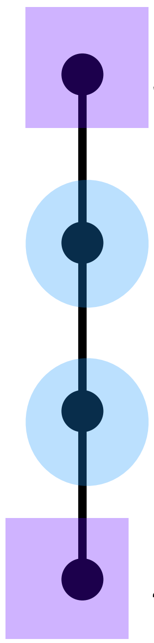

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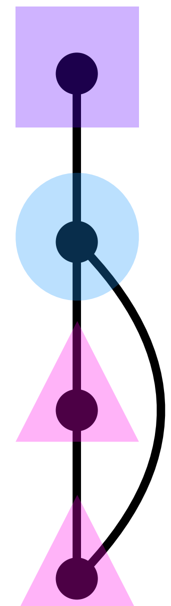

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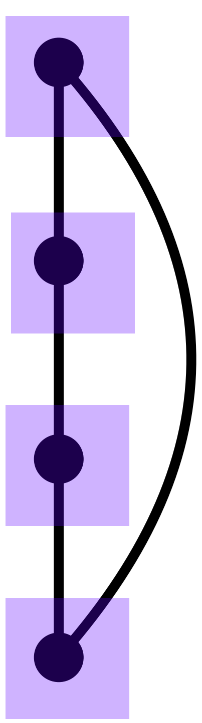

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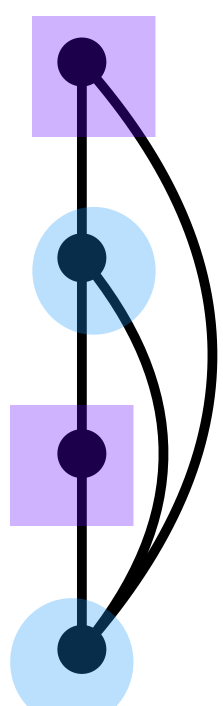

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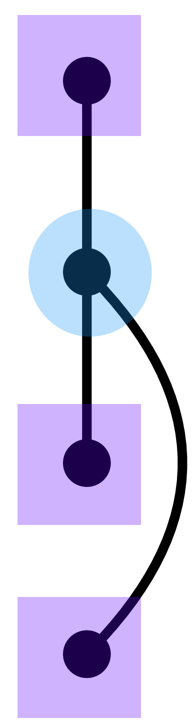

1 1 1 2

2 3 4 1

3 2 2 3

4 4 3 4

Different situations!

Result

Theorem [Drmota Kraus Panagiotou S. Stufler 26+] There exists a constant $c > 0$ such that the maximum degree Δ_n of a uniformly sampled n-vertex unlabelled series-parallel graph satisfies

$$\Delta_n = \frac{\log(n)}{\log(c)} - \frac{3 \log \log(n)}{2 \log(c)} + O_{\mathbb{P}}(1)$$

Reduction

[Stufler 2020] largest connected connected component has size $n - O(1)$ a.s., so it is sufficient to prove:

<u>Theorem</u> [Drmota Kraus Panagiotou S. Stufler 26+] There exists a constant $c > 0$ such that the maximum degree $\Delta_n^{(1)}$ of a uniformly sampled n-vertex unlabelled connected series-parallel graph satisfies

$$\Delta_n^{(1)} = \frac{\log(n)}{\log(c)} - \frac{3 \log \log(n)}{2 \log(c)} + O_{\mathbb{P}}(1)$$

Upper bound

First-moment method

Let $X_{n,\ell}$ = number of vertices of degree ℓ in the class $\mathscr{C}_n$ of uniform unlabelled connected SP graph of size n

First-moment method:

$$\mathbb{P}\left(\Delta_n^{(1)} > k\right) = \mathbb{P}\left(\sum_{\ell>k} X_{n,\ell} > 0\right) \leq \sum_{\ell>k} \mathbb{E}\, X_{n,\ell}$$

and

$$\mathbb{E}\, X_{n,\ell} = \frac{1}{|\mathscr{C}_n|} \underbrace{\sum_{G\in\mathscr{C}_n} \sum_{v\in G}}_{\neq \sum_{(G,v)\in\mathscr{C}_n^{\bullet}}} \mathbf{1}_{\deg_G(v)=\ell}$$

rooted class

First-moment method

Let $X_{n,\ell}$ = number of vertices of degree ℓ in the class $\mathscr{C}_n$ of uniform unlabelled connected SP graph of size n

First-moment method:

$$\mathbb{P}\left(\Delta_n^{(1)} > k\right) = \mathbb{P}\left(\sum_{\ell>k} X_{n,\ell} > 0\right) \leq \sum_{\ell>k} \mathbb{E}\, X_{n,\ell}$$

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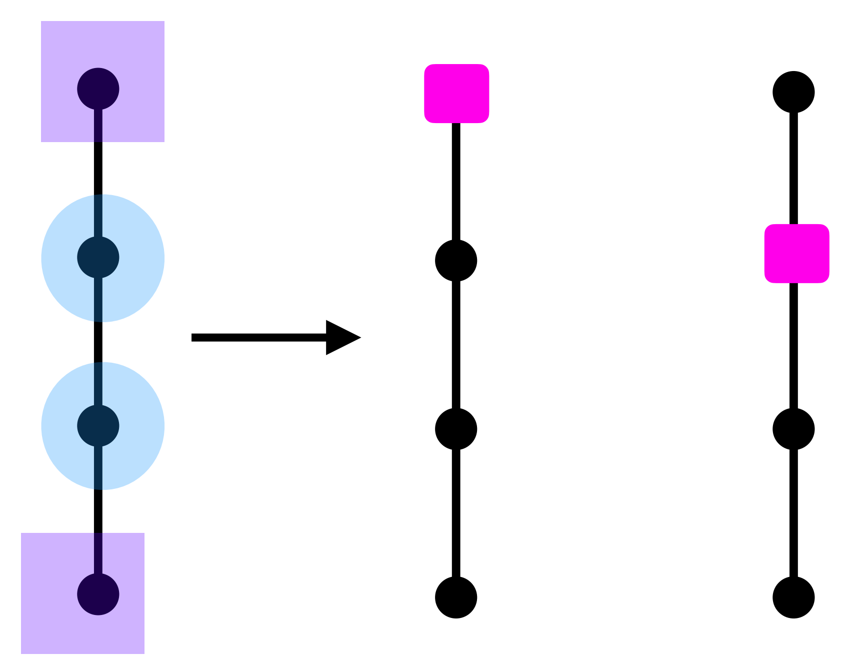
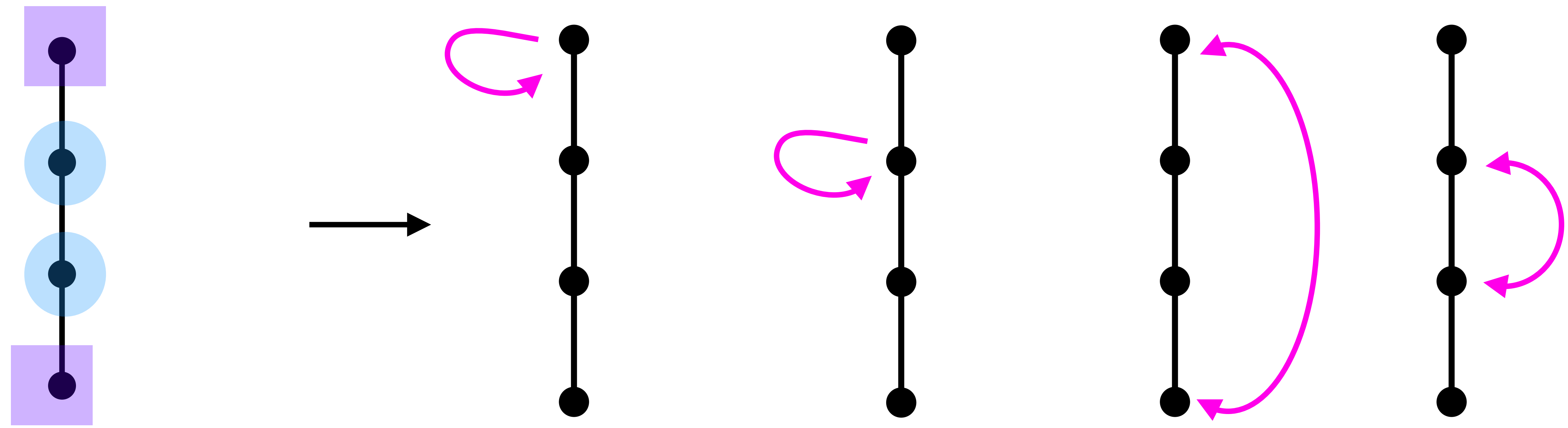
rooted class

Rooting $|\mathscr{C}_n^\bullet| \neq n|\mathscr{C}_n|$



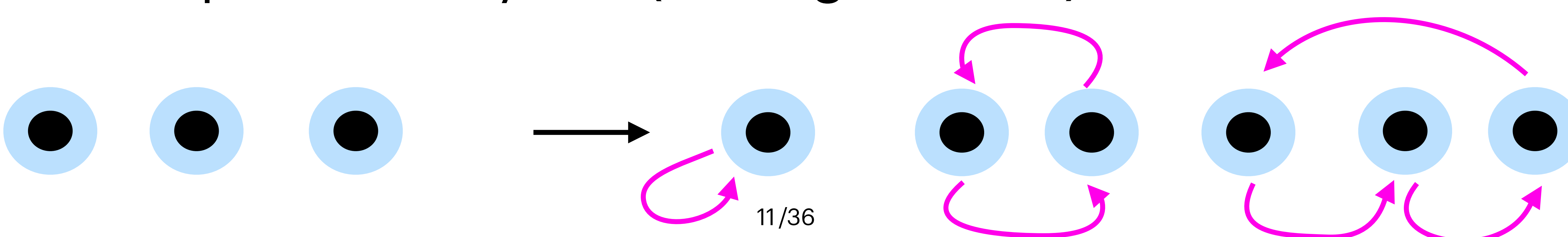
Cycle-rooting

[Bodirsky Fusy Kang Vigerske 2011]



Theorem [Bodirsky Fusy Kang Vigerske 2011] An unlabelled structure of size n has n non-"equivalent" cycles of automorphisms.

Idea each group of k equivalent vertices gives rises to k non-"equivalent" cycles (of length 1 to k)



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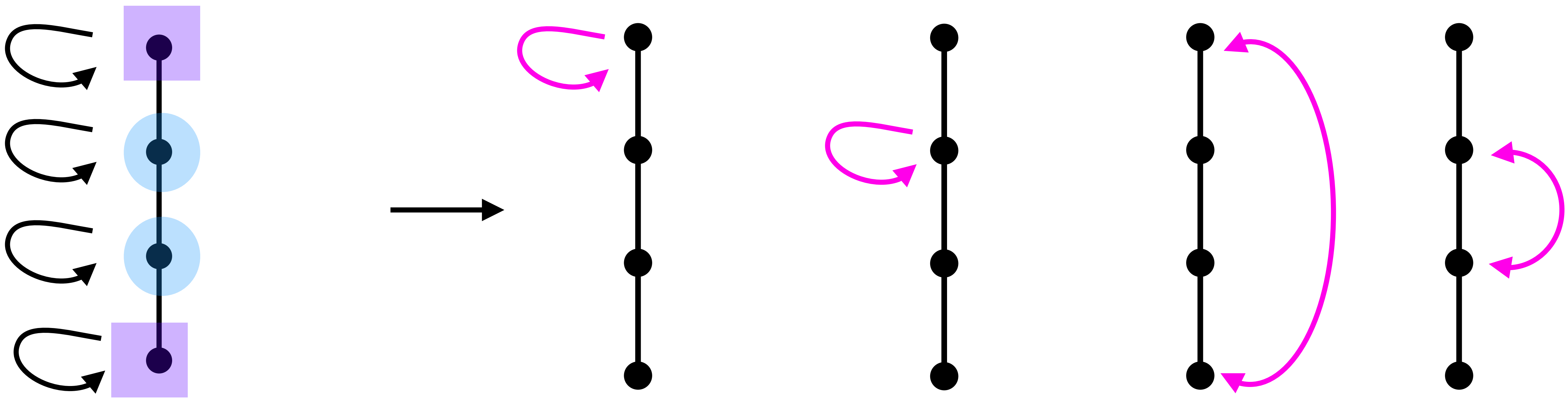

Theorem [Bodirsky Fusy Kang Vigerske 2011] An unlabelled structure of size n has n non-“equivalent” cycles of automorphisms.

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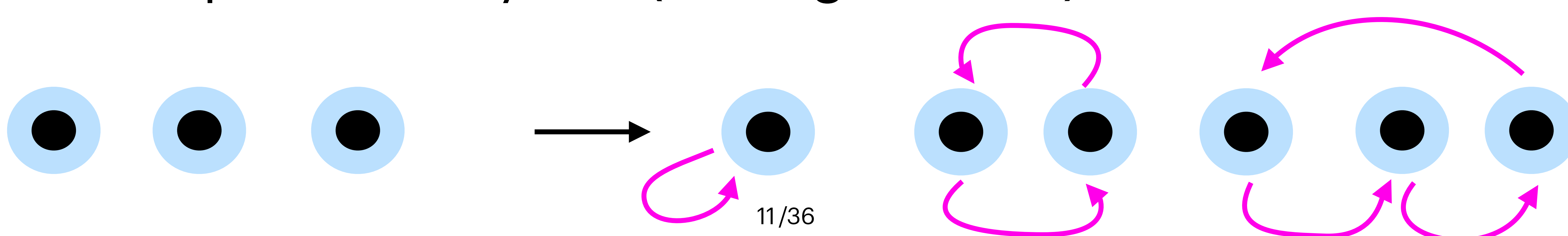

Cycle-rooting

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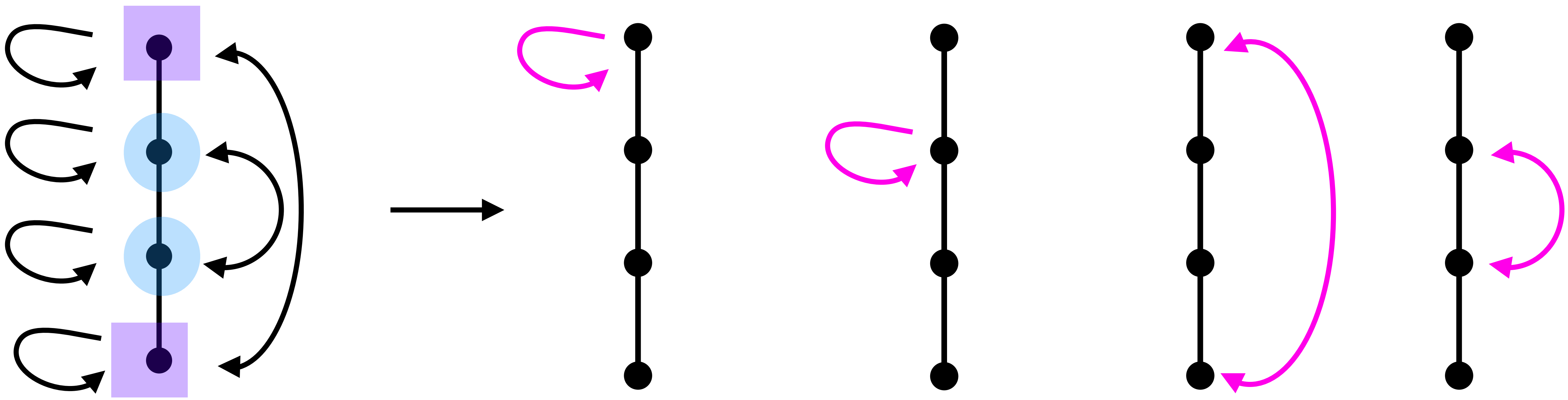

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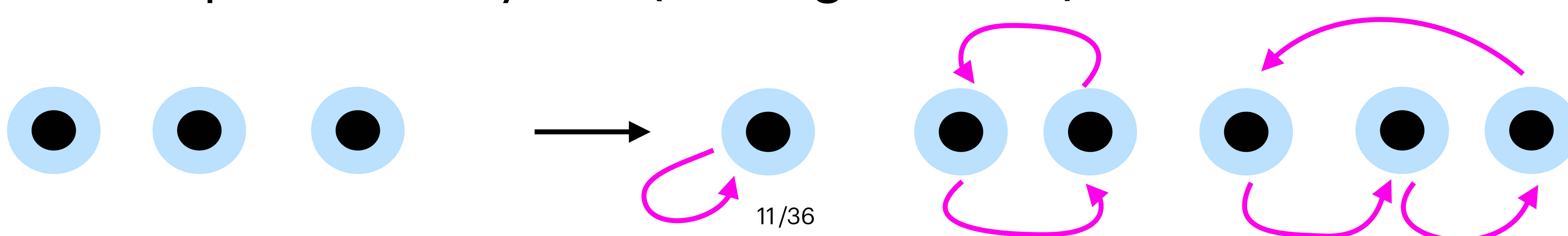
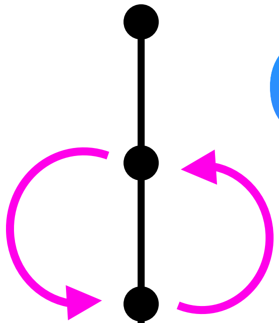


Cycle-rooted connected SP graphs



$\mathscr{C}_n^\circ$ = unlabelled cycle-rooted connected SP graph of size n

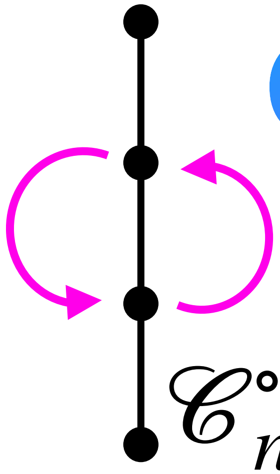
$C^\circ(x, w)$ = its GF, with x counting vertices and w the degree of the cycle-root (= degree of one of its vertices)

Then:

$$[x^n]C^\circ(x,1) = n\,|\mathscr{C}_n|$$

And

$$\mathbb{E}X_{n,\ell} = \frac{1}{|\mathscr{C}_n|}\sum_{G\in\mathscr{C}_n}\sum_{v\in G}\mathbf{1}_{\deg_G(v)=\ell} = n\frac{[x^n w^\ell]C^\circ(x,w)}{[x^n]C^\circ(x,1)}$$



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What is the asymptotic behaviour of the coefficients of $C^\circ(x, w)$?

Tools

The class of series-parallel graph is “block-stable”, so nice equation, which can track w

$C^{\bullet}(x)$ has finite radius of convergence $x_1 = 0.10655\ldots$ and has a square-root type singularity at $x = x_1$ [Drmota Fusy Kang Kraus Rué 2011]

Asymptotic expansion

We show SP networks have square-root type singularity "in x and w", so:

Lemma

$$C^\circ(x,w) = C^\circ(x) + C_2^\circ(x)\left(1 - \frac{w}{w_c(x)}\right) + C_3^\circ(x)\left(1 - \frac{w}{w_c(x)}\right)^{3/2} + \cdots$$

where $w_c(x)$ and $C_j^\circ(x)$ have a square-root type singularity at $x = x_1$, and $C_3^\circ(x_1) \neq 0$.

Asymptotic expansion of this kind studied by [Drmota Giménez Noy 2011]

Upper bound

So
$$\mathbb{E}\,\Delta_n^{(1)} = \sum_{k\geq 0} \mathbb{P}\{\Delta_n^{(1)} > k\} \leq k_0 + O\left(n w_c(x_1)^{-k_0} k_0^{-\frac{3}{2}}\right) + O(1)\,.$$
$= O(1)$ for well-chosen k_0

So $\mathbb{E}\,\Delta_n^{(1)} \leq \dfrac{\log n}{\log(w_c(x_1))} - \dfrac{3\log\log n}{2\log(w_c(x_1))} + O(1)\,.$

By Markov's inequality,

$$\Delta_n^{(1)} \leq \frac{\log n}{\log(w_c(x_1))} - \frac{3\log\log n}{2\log(w_c(x_1))} + O_{\mathbb{P}}(1)\,.$$

Lower bound

Reduction 2

[Stufler 2023] rooted connected unlabelled SP graph behaves like its unrooted counterpart, so it is sufficient to prove:

<u>Lemma</u> There exists a constant $c > 0$ such that the maximum degree $\Delta_n^{(1,r)}$ of a uniformly sampled n-vertex unlabelled vertex-rooted connected series-parallel graph satisfies

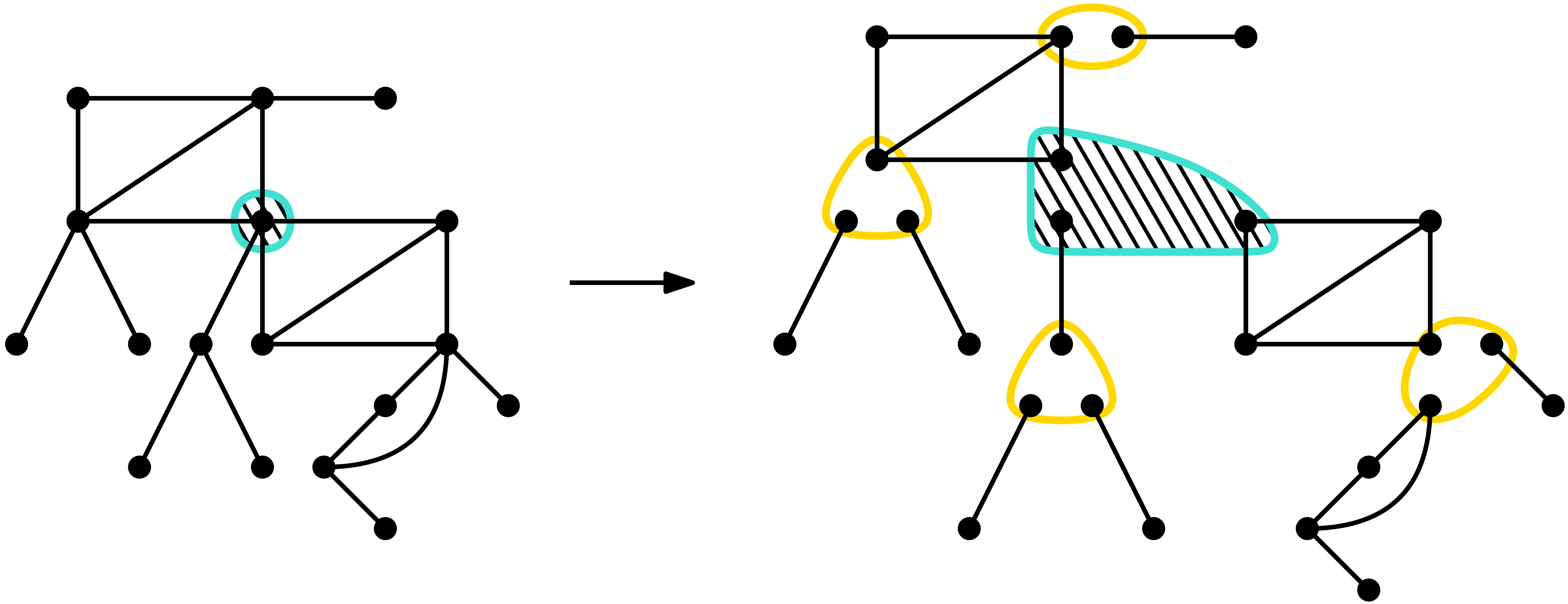
$$\Delta_n^{(1,r)} \geq \frac{\log(n)}{\log(c)} - \frac{3 \log \log(n)}{2 \log(c)} + O_{\mathbb{P}}(1)$$

[Drmota Fusy Kang Kraus Rué 2011]

Block-stable class equation

$\mathscr{C}^\bullet$ connected vertex-rooted (unlabelled) SP graphs

$\mathscr{B}$ = 2-connected SP graphs, $\mathscr{B}'$ = derived class



$$\mathscr{C}^\bullet = \mathscr{X} \cdot \mathrm{Set}(\mathscr{B}' \circ \mathscr{C}^\bullet)$$

Problem: unlabelled case → no easy translation to GS equation

Cycle-index sums

For a species $\mathscr{A}$, its cycle-index sum $Z_{\mathscr{A}}(s_1, s_2, \dots)$ is a generalisation of the notion of generating series:

$$Z_{\mathscr{A}}(s_1, s_2, \dots) = \sum_{\substack{a \in \mathscr{A} \\ \sigma \text{ automorphism of } a}} w_{(a,\sigma)}(s_1, s_2, \dots)$$

And:

$Z_{\mathscr{A}}(x, x^2, \dots) = A(x)$ = OGF of the unlabelled class

Boltzmann sampling

[Duchon Flajolet Louchard Schaeffer 2004]

Ordinary Boltzmann sampler $\Gamma A(x)$ = samples an element a of $\mathscr{A}$ such that

$$\mathbb{P}\left(\Gamma A(x) = a\right) = \frac{x^{|a|}}{A(x)}$$

=> uniform when fixed size

=> automatic translation of the decompositions of combinatorial classes

Boltzmann samplers originally for labelled classes [Duchon Flajolet Louchard Schaeffer 2004], extended for some operations to unlabelled [Fusy Flajolet Pivoteau 2007]

Pólya-Boltzmann Sampling

[Bodirsky Fusy Kang Vigerske 2011]

Pólya-Boltzmann sampler $\Gamma Z_{\mathscr{A}}(s_1, s_2, \ldots)$ = samples an element a of $\mathscr{A}$ with an automorphism σ such that

$$\mathbb{P}\left(\Gamma Z_{\mathscr{A}}(s_1, s_2, \ldots) = (a, \sigma)\right) = \frac{w_{(a,\sigma)}(s_1, s_2, \ldots)}{Z_{\mathscr{A}}(s_1, s_2, \ldots)}$$

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=> generalises ordinary Boltzmann samplers: choose $s_i = x^i$ to get $\Gamma A(x)$

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=> automatic translation of the decompositions of combinatorial classes

And now: the proof of the lower bound

PBS for rooted connected SP graphs

Algorithm 1 $\Gamma C^{\bullet}(x)$

Create a root vertex v
for $k = 1$ **to** ∞ **do**
  $m_k \leftarrow \text{Poisson}(Z_{\mathcal{B}'}(C^{\bullet}(x^k), C^{\bullet}(x^{2k}), \ldots)/k)$
  for $i = 1$ **to** m_k **do**
    $B \leftarrow \Gamma(B' \circ C^{\bullet})(x^k)$
    Attach k copies of B to v by identifying their root vertices with v
  end for
end for
return the resulting graph with root v

$$\mathscr{C}^{\bullet} = \mathscr{X} \cdot \mathrm{Set}(\mathscr{B}' \circ \mathscr{C}^{\bullet})$$

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$k = 1$, $m_k = 2$

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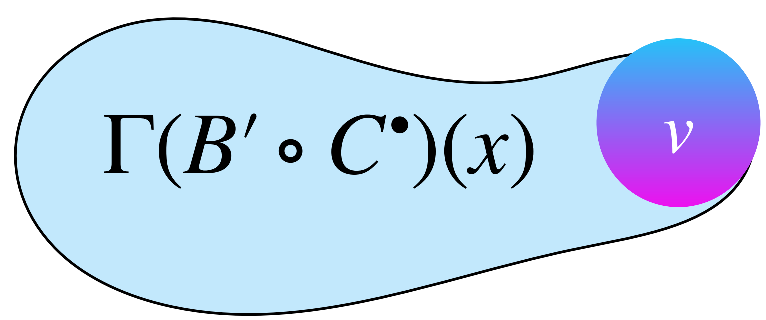

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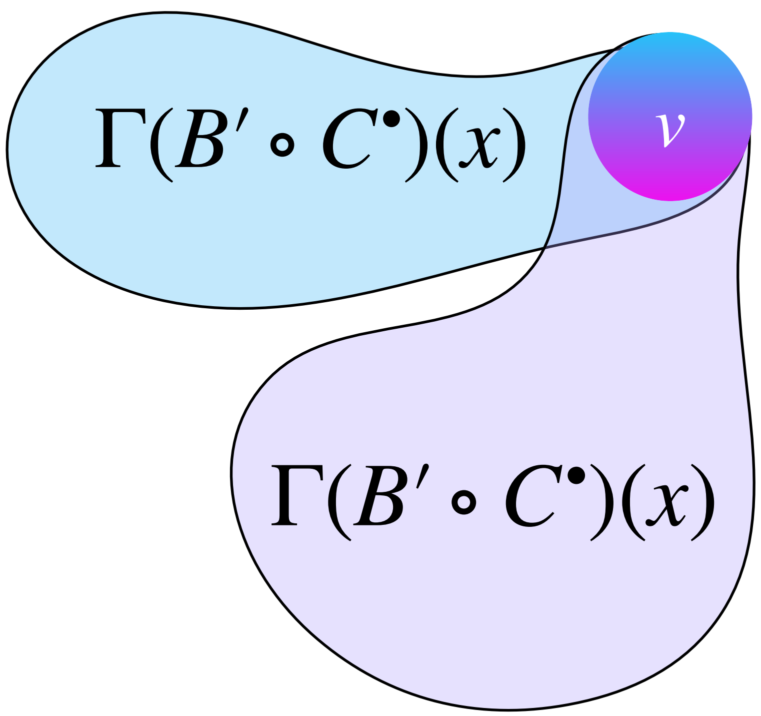

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$k = 1,\ m_k = 2$

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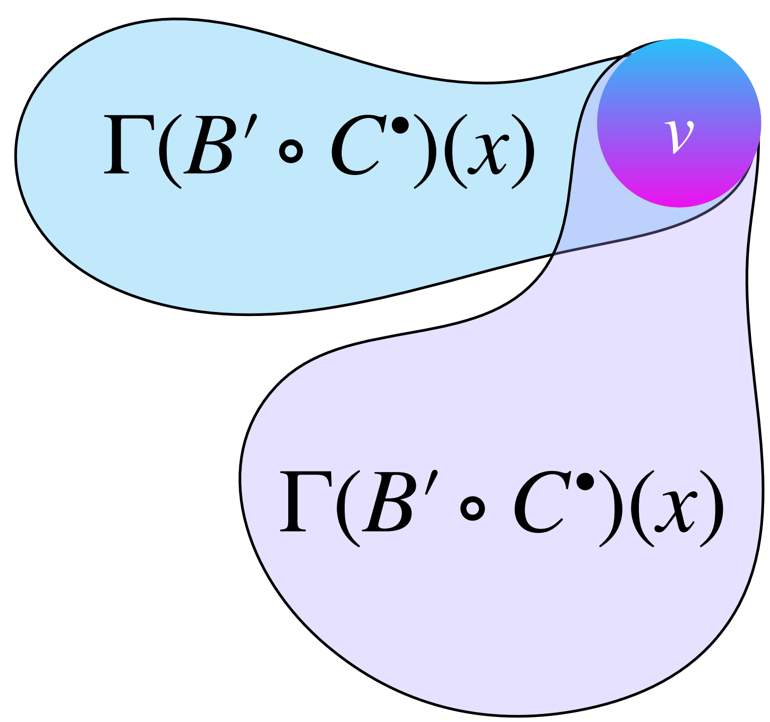

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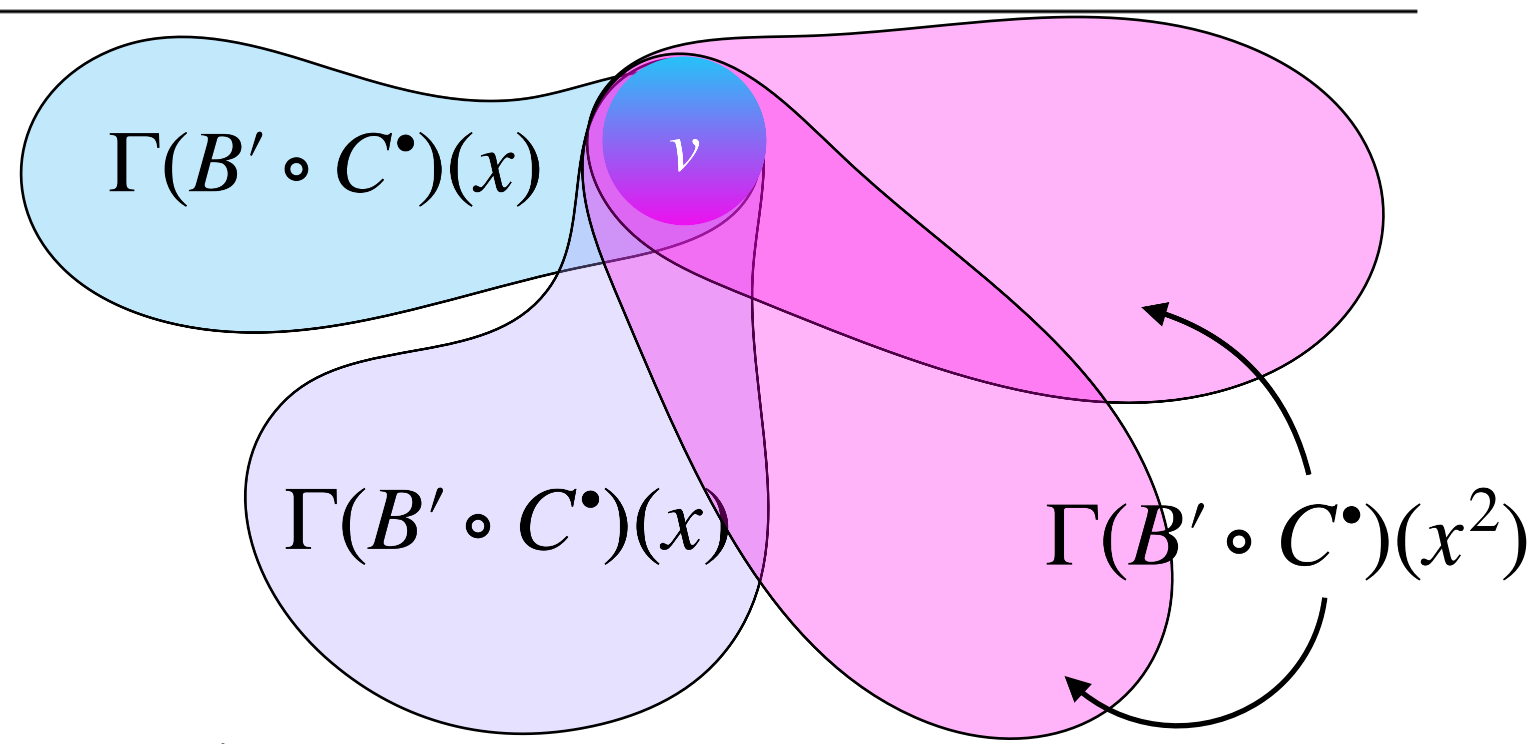

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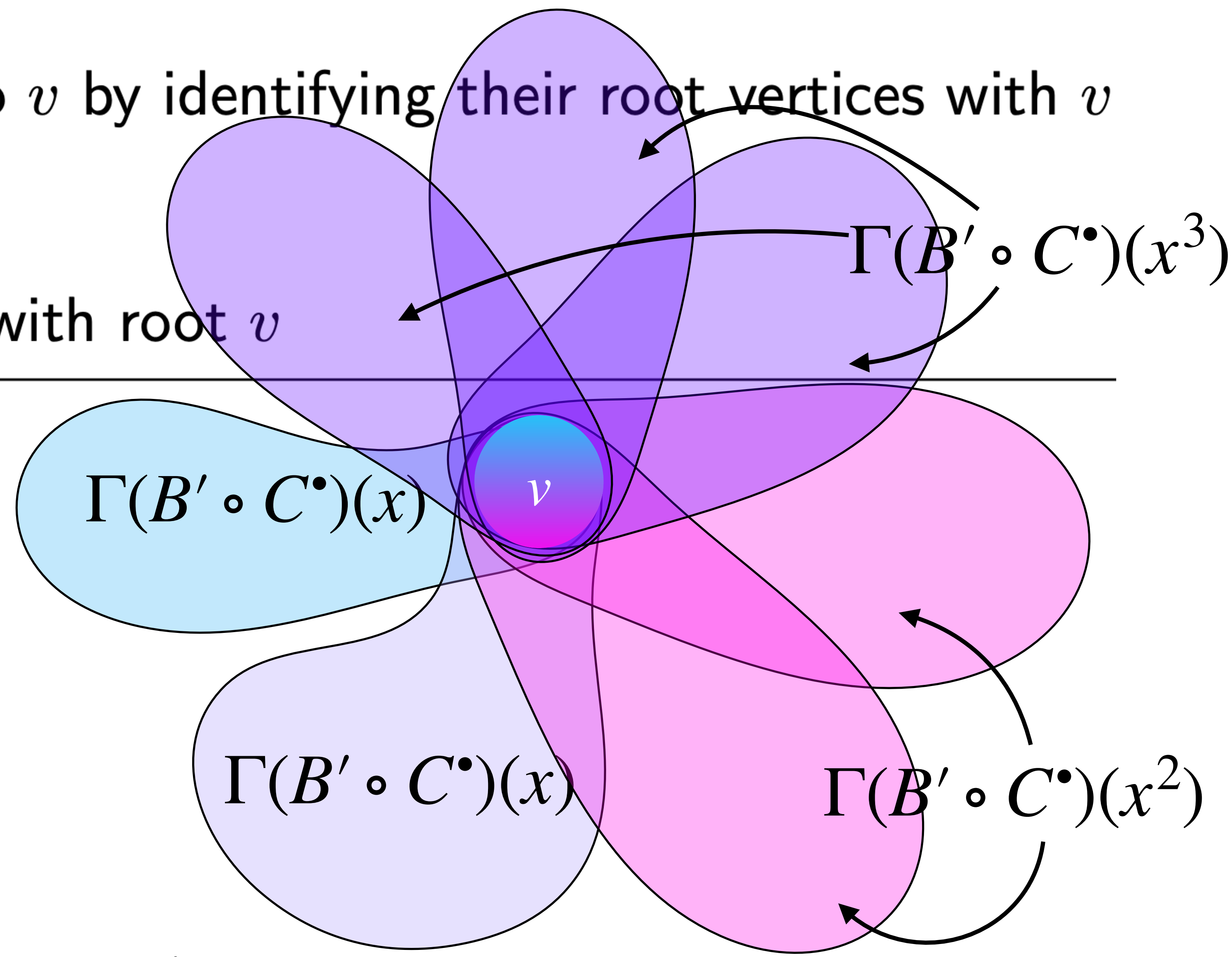

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$k = 1,\ m_k = 2$

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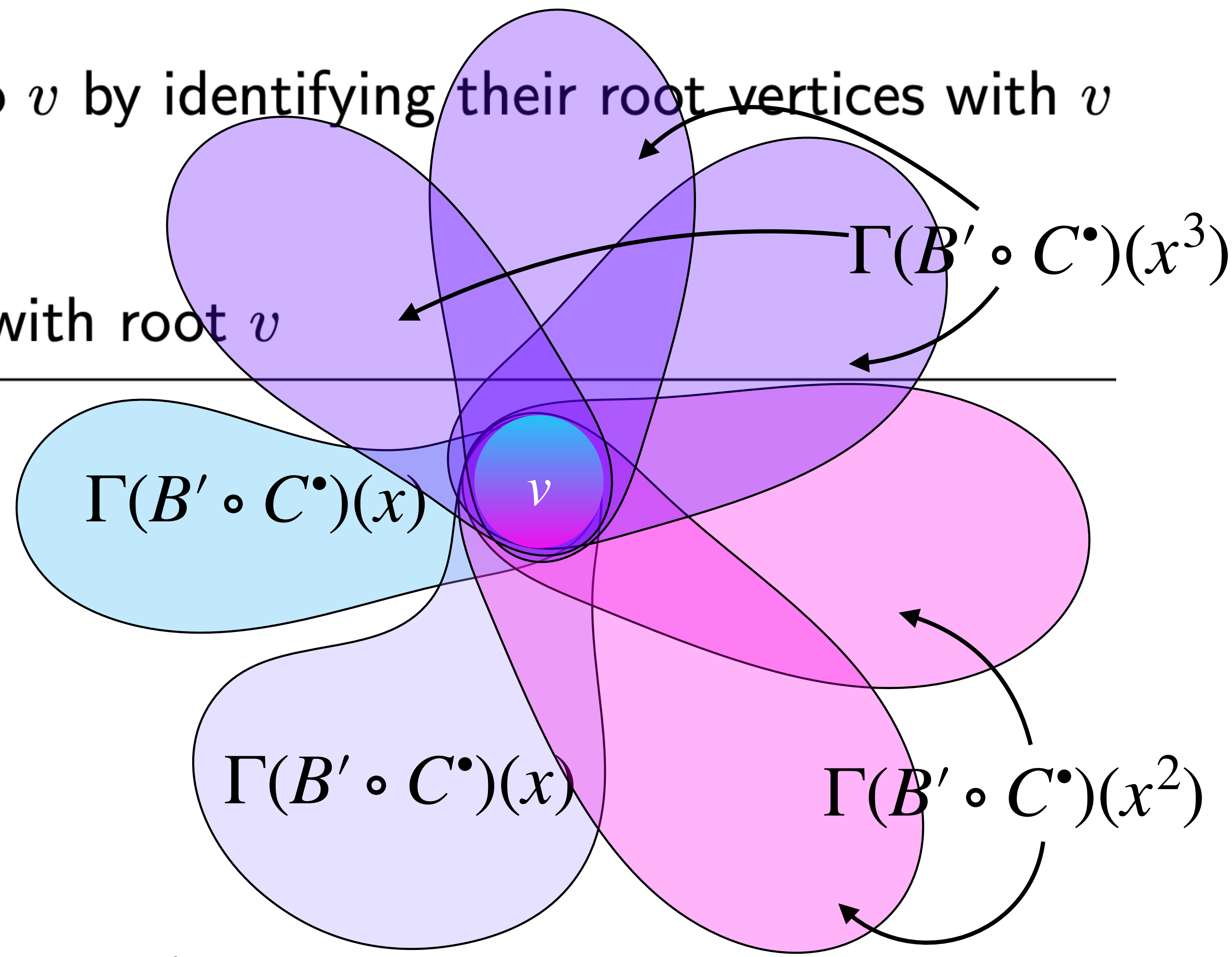
$k = 3,\ m_k = 1$

$k \geq 4,\ m_k = 0$

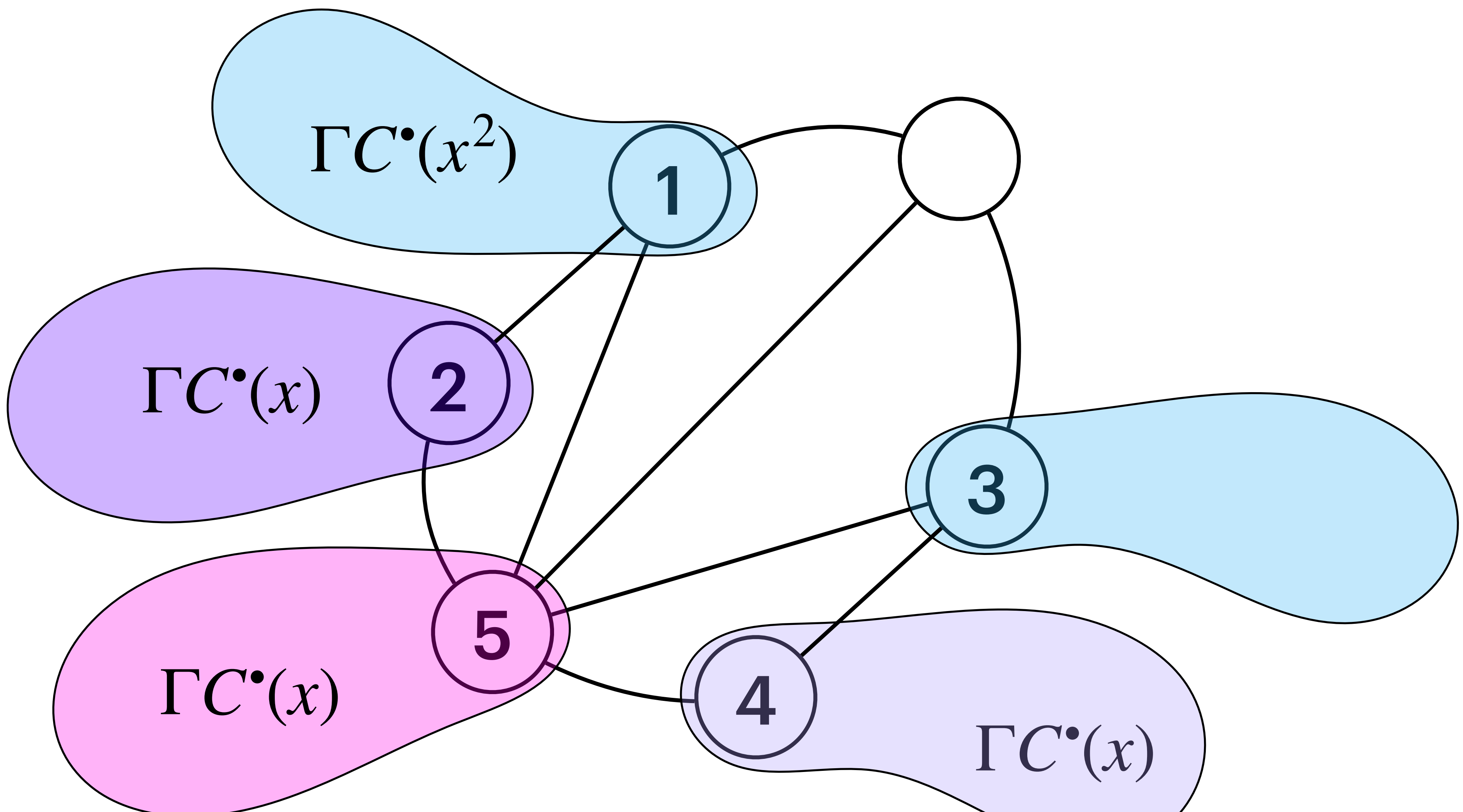


PBS for rooted connected SP graphs

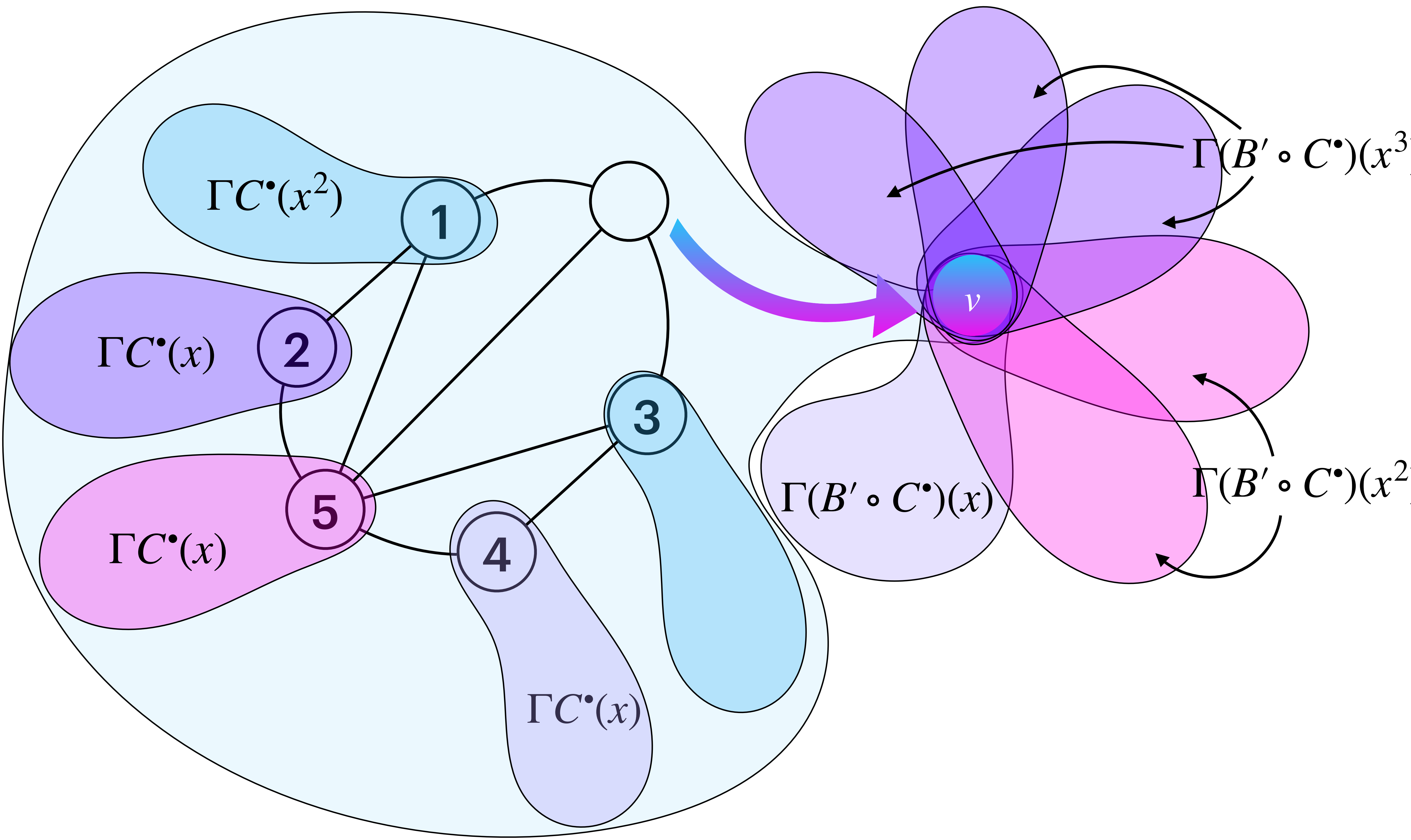
Algorithm 2 $\Gamma(B' \circ C^\bullet)(x)$

$(B, \sigma) \leftarrow \Gamma Z_{\mathcal{B}'}(C^\bullet(x), C^\bullet(x^2), \ldots)$
for each cycle τ of the permutation σ **do**
  $C \leftarrow \Gamma C^\bullet(x^{|\tau|})$
  Attach to each vertex v of B that is in τ a copy C_v of C by identifying its root vertex with v
end for
return an arbitrary relabelling of the resulting graph

$$\sigma = (1\ 3)(2)(4)(5)$$



PBS for rooted connected SP graphs



Recursive structure

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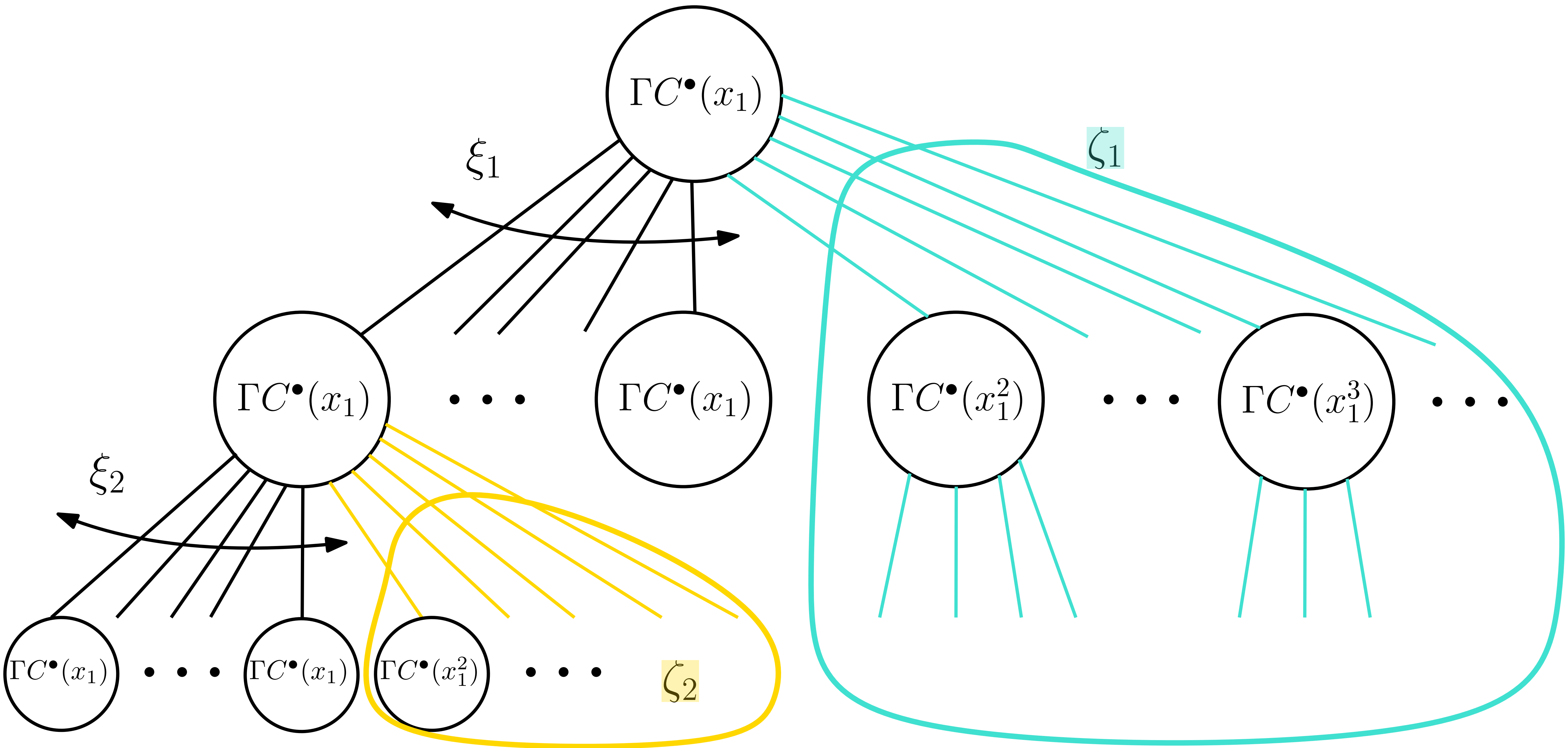
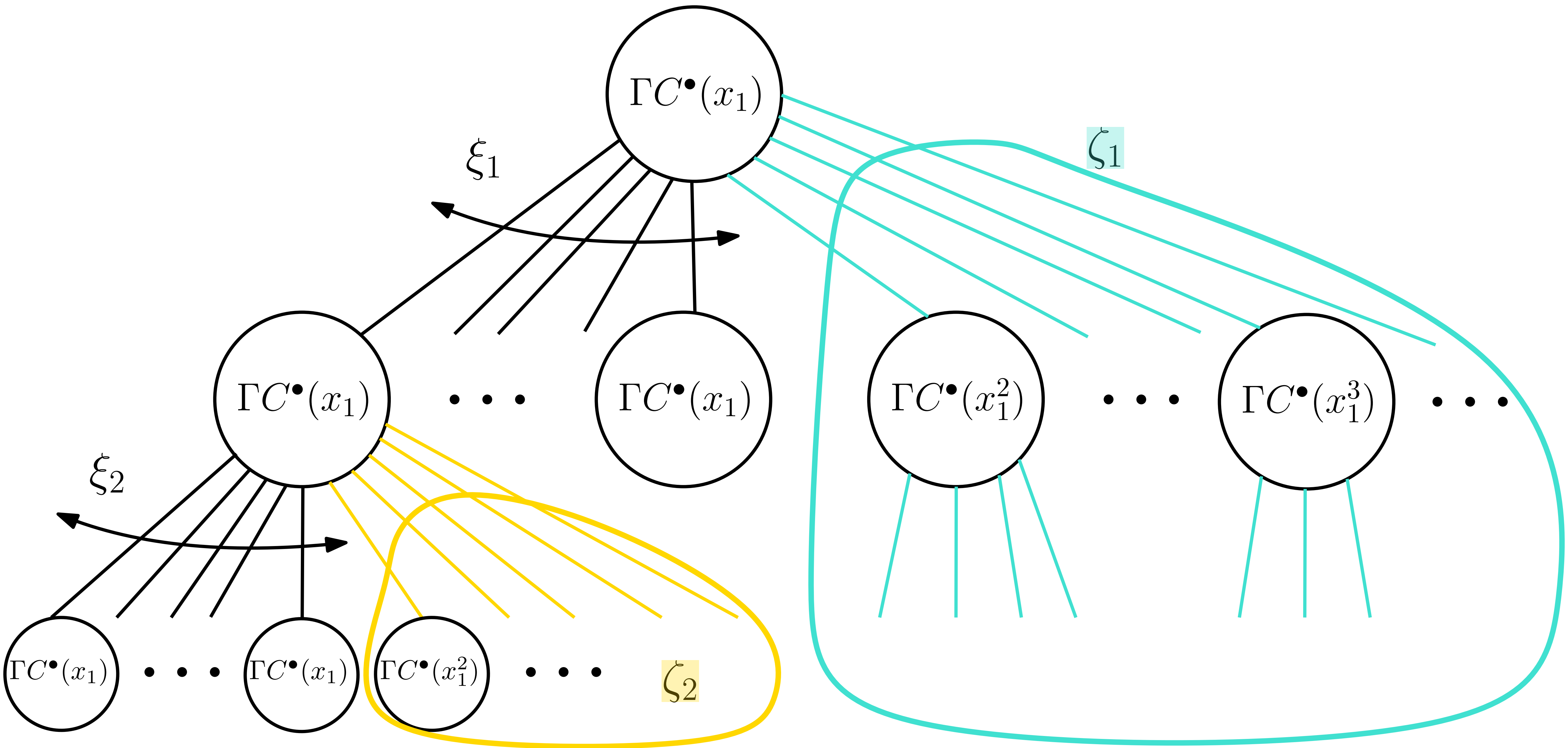
Execution tree



Execution tree

L = #total calls to $\Gamma C^{\bullet}(x_1)$ S = size of the output of $\Gamma C^{\bullet}(x_1)$



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$$L = \min \left\{ \ell \geq 1 \;\middle|\; \sum_{i=1}^{\ell} (\xi_i - 1) = -1 \right\}.$$

$$S = \sum_{i=1}^{L} (1 + \zeta_i)$$

Execution tree

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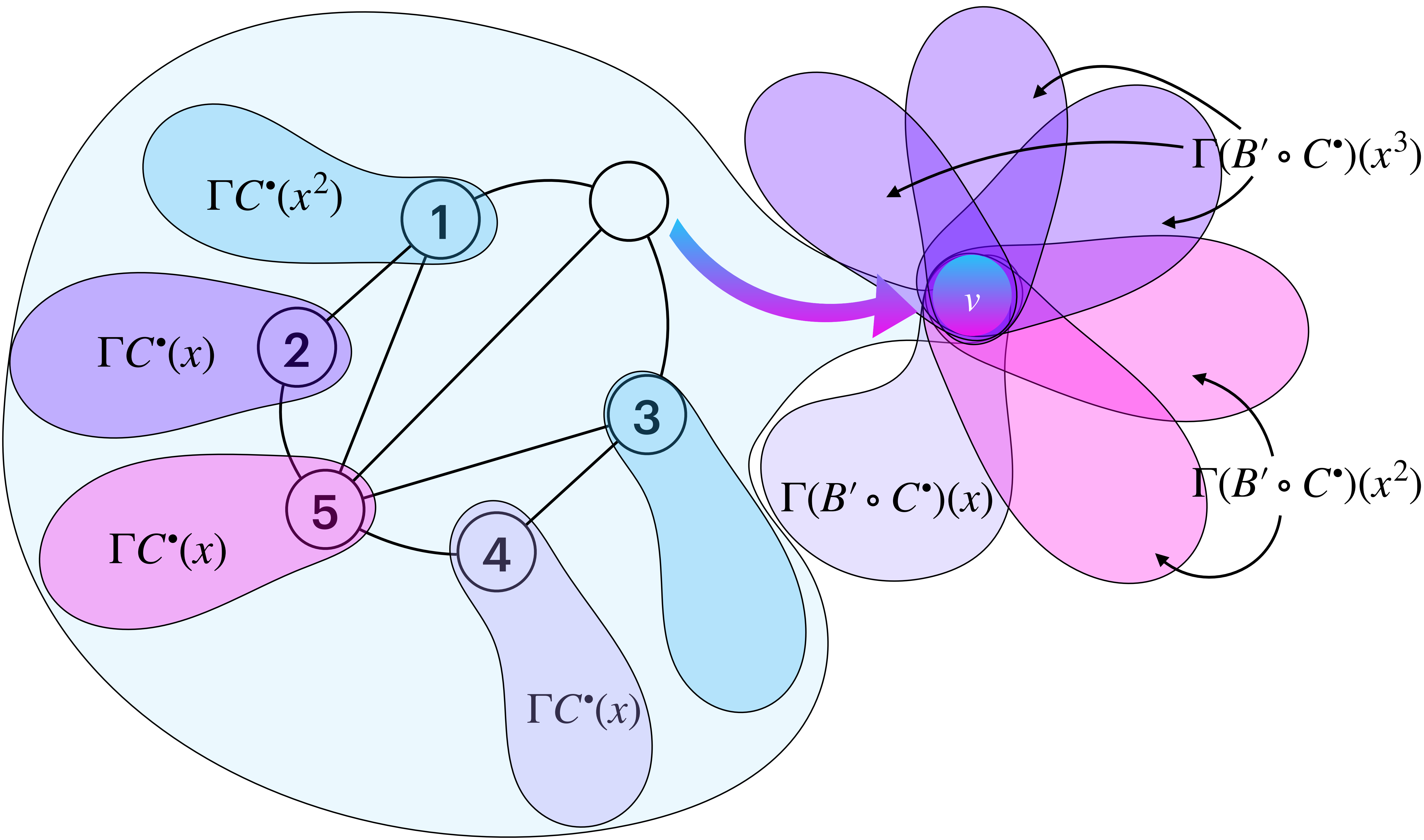
$$S = \sum_{i=1}^{L} (1 + \zeta_i)$$

L_n = #total calls to $\Gamma C^{\bullet}(x_1)$ conditioned on $S = n$

[Stufler 2018]:

$$\frac{L_n - \mu n}{\sigma \sqrt{n}} \xrightarrow[n \to \infty]{(d)} \mathcal{N}(0,1)$$

Focusing on calls to $\Gamma C^{\bullet}(x_1)$



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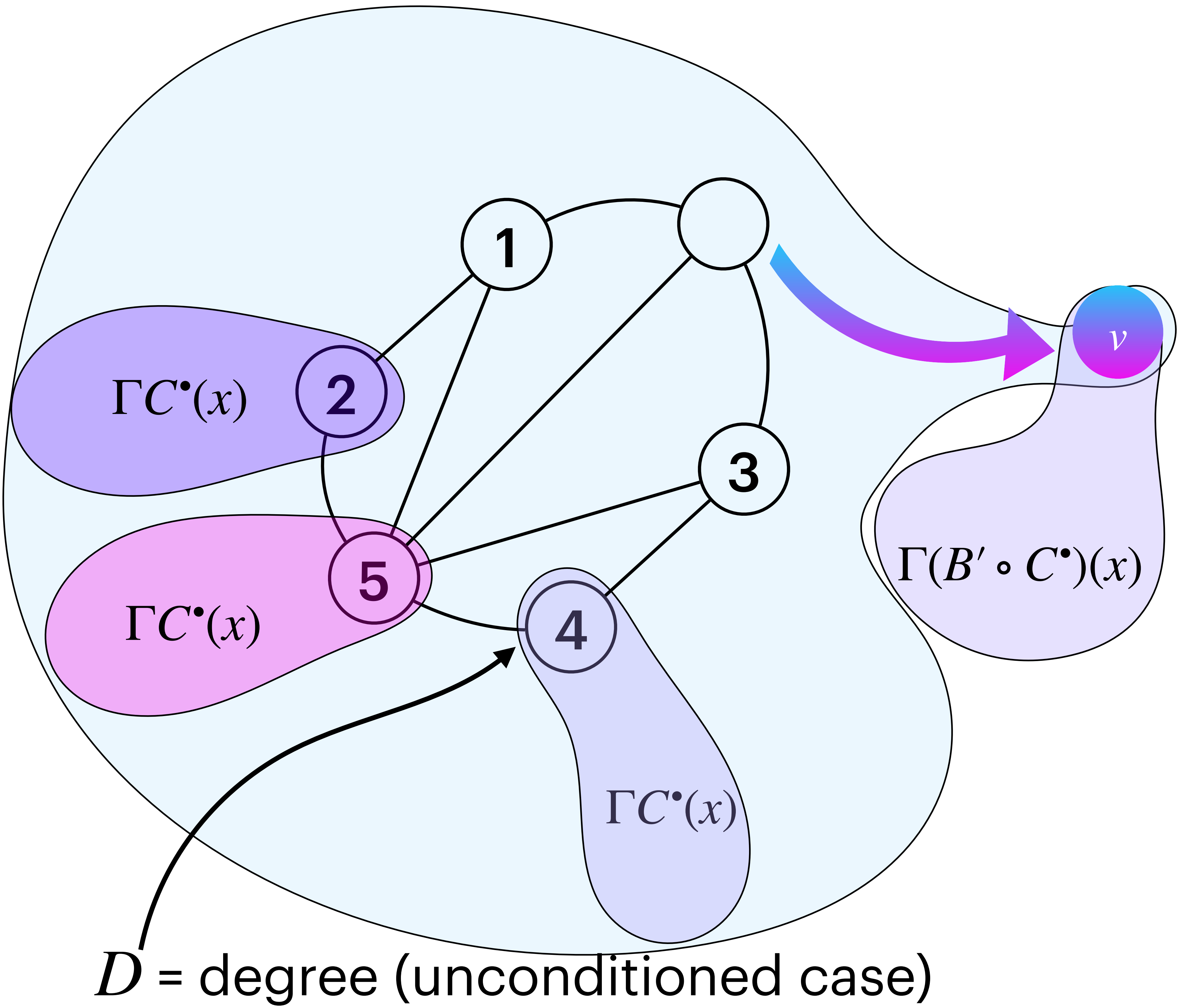
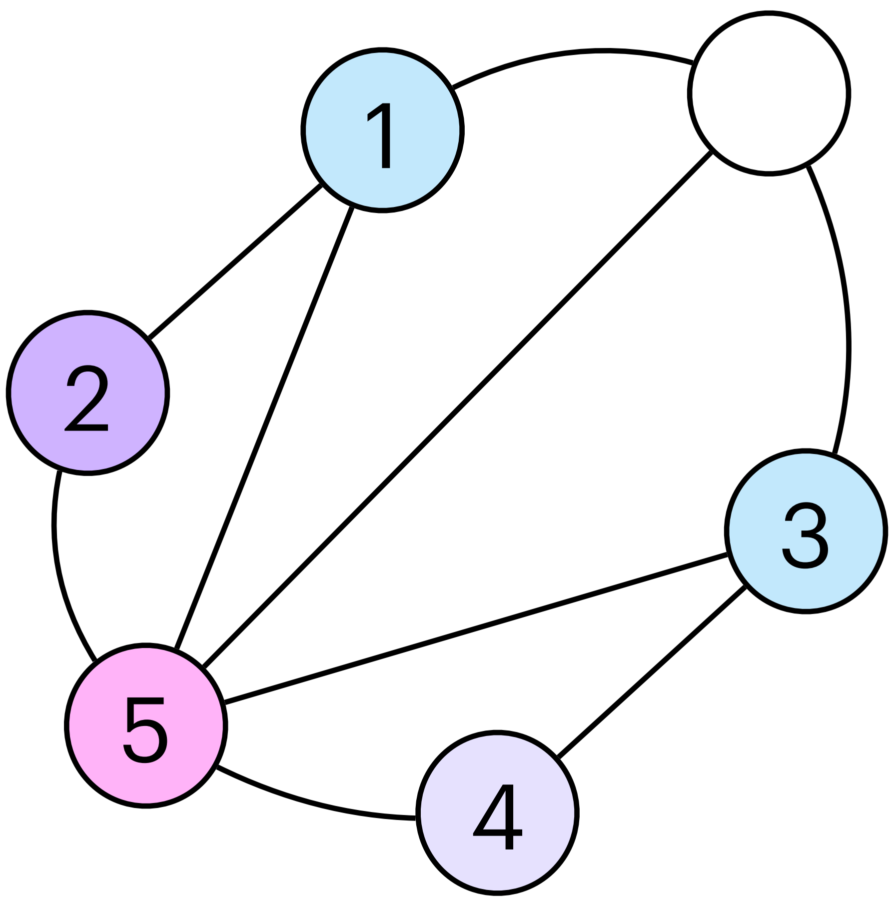


Degree of a "fixed point"

D = degree of a fixed point vertex in $(B', \sigma) \sim \Gamma Z_{\mathscr{B}'}(C^\bullet(x_1), \ldots)$

Rooting = distinguishing a cycle of length 1

So, $D \stackrel{(d)}{=}$ degree of the second root in $(B'', \sigma) \sim \Gamma Z_{\mathscr{B}''}(C^\bullet(x_1), \ldots)$



$\sigma = (1\,3)(2)(4)(5)$

By symmetry, $D \stackrel{(d)}{=}$ degree of the first root in $(B'', \sigma) \sim \Gamma Z_{\mathscr{B}''}(C^\bullet(x_1), \ldots)$

$$\mathbb{P}\left(D = k\right) = \frac{[w^k]B''(C^\bullet(x_1), w)}{B''(C^\bullet(x_1), 1)}$$

where B'' = SG of 2-c SP graphs where w marks the root degree

Degree of a "fixed point"

$$\mathbb{P}\left(D=k\right) = \frac{[w^k]B''(C^\bullet(x_1),w)}{B''(C^\bullet(x_1),1)}$$

We show

$$B''(C^\bullet(x_1),w) = B_0(C^\bullet(x_1)) + B_1(C^\bullet(x_1))\left(1-\frac{w}{w_c(x_1)}\right)^{1/2} + \cdots$$

So, by usual univariate transfert theorem [Flajolet Odlyzko 1990],

$$[w^k]B''(C^\bullet(x_1),w) \sim ck^{-3/2}w_c(x_1)^{-k}$$

as $k \to \infty$

So

$$\mathbb{P}\left(D>k\right) \sim ck^{-3/2}w_c(x_1)^{-k}$$

Conclusion: lower bound

Call to $\Gamma C^{\bullet}(x_1)$ conditioned on output of size n

=> L_n total (conditioned) calls to $\Gamma C^{\bullet}(x_1)$

=> $L_n \simeq \mu n$ calls to random variables behaving (approximately) like D, with

$$\mathbb{P}\left(D > k\right) \sim c k^{-3/2} w_c(x_1)^{-k}$$

Heuristics: max $m(n)$ is such that

$$\mathbb{P}\left(D > m(n)\right) \simeq \frac{1}{\mu n}$$

so

$$\Delta_n^{(1,r)} \geq m(n) \geq \frac{\log n}{\log(w_c(x_1))} - \frac{3 \log \log n}{2 \log(w_c(x_1))} + O_{\mathbb{P}}(1)\,.$$

Conclusion

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Upper bound

$$\Delta_n^{(1)} \leq \frac{\log n}{\log(w_c(x_1))} - \frac{3 \log \log n}{2 \log(w_c(x_1))} + O_{\mathbb{P}}(1)\,.$$

Lower bound

$$\Delta_n^{(1,r)} \geq \frac{\log n}{\log(w_c(x_1))} - \frac{3 \log \log n}{2 \log(w_c(x_1))} + O_{\mathbb{P}}(1)\,.$$

so

$$\Delta_n^{(1)} \geq \frac{\log n}{\log(w_c(x_1))} - \frac{3 \log \log n}{2 \log(w_c(x_1))} + O_{\mathbb{P}}(1)\,.$$

Results

computable

<u>Theorem</u> [Drmota Kraus Panagiotou S. Stufler 26+] The maximum degree $\Delta_n^{(1)}$ of a uniformly sampled n-vertex unlabelled connected series-parallel graph satisfies

$$\Delta_n^{(1)} = \frac{\log(n)}{\log(w_c(x_1))} - \frac{3\log\log(n)}{2\log(w_c(x_1))} + O_{\mathbb{P}}(1)$$

<u>Theorem</u> [Drmota Kraus Panagiotou S. Stufler 26+] The maximum degree Δ_n of a uniformly sampled n-vertex unlabelled series-parallel graph satisfies

$$\Delta_n = \frac{\log(n)}{\log(w_c(x_1))} - \frac{3\log\log(n)}{2\log(w_c(x_1))} + O_{\mathbb{P}}(1)$$

Comparison with the labelled case

We show

$$\frac{\Delta_n}{\log(n)} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{\log(w_c(x_1))} = 3.587582\ldots$$

Notice in the labelled case: [Drmota Giménez Noy 2011]

$$\frac{\Delta_n^{(l)}}{\log(n)} \xrightarrow[n\to\infty]{\mathbb{P}} c_l = 3.482774\ldots$$

2-connected unlabelled SP graphs

During the proof of the upper bound, we prove:

$$\Delta_n^{(2)} \leq \frac{\log n}{\log(c)} - \frac{3 \log \log n}{2 \log(c)} + O_{\mathbb{P}}(1)\,.$$

with $c = 1.2982941\ldots$ so $1/\log(c) = 3.830666\ldots$

We expect:

$$\Delta_n^{(2)} = \frac{\log n}{\log(c)} - \frac{3 \log \log n}{2 \log(c)} + O_{\mathbb{P}}(1)\,.$$

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$$\frac{\Delta_n^{(l,2)}}{\log(n)} \xrightarrow[n\to\infty]{\mathbb{P}} c_l = \boxed{3.679771\ldots}$$

Thank you!