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Étude unifiée des cartes planaires pondérées par blocs : propriétés combinatoires et probabilistes

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Unified study of block-weighted planar maps: combinatorial and probabilistic properties

Keywords. Combinatorics, Analytic Combinatorics, Probability, Planar maps, Random trees, Scaling limits.

Abstract. This thesis focuses on classes of planar maps with a weight $u > 0$ on certain components called *blocks*. In collaboration with Fleurat, we study the decomposition of generic planar maps into 2-connected components, revealing a phase transition between the universality classes of maps (converging to the Brownian sphere) and plane trees (converging to the Brownian tree), depending on the value of u. We identify a new class with the stable tree of parameter $3/2$ as the scaling limit in the critical case, and obtain precise results on block sizes in each phase. In a subsequent work, I show that it is possible to study many decomposition schemes along similar lines to shed light on a phase transition. I explain how to obtain enumerative results, block sizes and scaling limits for each phase. Indeed, the robustness of the method and its application in different cases is a main focus of my work. Finally, with Albenque and Fusy, we studied tree-rooted random planar maps decomposed into tree-rooted 2-connected blocks, where a spanning tree is drawn simultaneously with the map. This model, which is of interest in theoretical physics, shows new behaviours. Despite the complexity of dealing with generating series that are non algebraic, or non D-finite, we determine the asymptotic behaviour of 2-connected tree-rooted maps, reveal a phase transition, and study the properties of each phase.

Mots-clefs. Combinatoire, Combinatoire analytique, Probabilités, Cartes planaires, Arbres aléatoires, Limites d'échelle.

Résumé. Cette thèse étudie des classes de cartes planaires avec un poids $u > 0$ pour certaines composantes, que nous appelons *blocs*. Avec Fleurat, nous étudions la décomposition des cartes planaires génériques en composantes 2-connexes, révélant ainsi une transition de phase selon la valeur de u entre la classe d'universalité des cartes (convergeant vers la sphère brownienne) et celle des arbres plans (convergeant vers l'arbre brownien). Dans le cas critique, nous identifions une nouvelle classe d'universalité avec l'arbre stable de paramètre $3/2$ comme limite d'échelle. De plus, nous obtenons des résultats précis sur la taille des blocs dans chaque phase. Ensuite, je montre qu'il est possible d'étudier un grand nombre de schémas de décomposition de cartes planaires d'une manière similaire, afin de mettre en évidence une transition de phase dans chaque cas. J'explique comment obtenir des résultats énumératifs, les tailles des blocs et les limites d'échelle pour chaque phase. La robustesse de la méthode et son application dans différents cas sont au cœur de mon travail. Enfin, avec Albenque et Fusy, nous avons étudié les cartes boisées aléatoires décomposées en blocs 2-connexes boisés, où un arbre couvrant est tiré en même temps que la carte. Ce modèle, qui a un intérêt en physique théorique, présente de nouveaux comportements. Malgré la complexité introduite par le fait que les séries génératrices ne soient pas algébriques voire pas D-finies, nous arrivons à déterminer le comportement asymptotique des cartes boisées 2-connexes, et à mettre en évidence une transition de phase, dont nous étudions les propriétés.

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Chapter i

Résumé détaillé en français

This chapter is a French translation of Sections 0.1 and 0.2.

i.1 Une brève histoire des cartes planaires

Definition i.1. Une *carte planaire* est un plongement propre d'un multigraphe planaire fini connexe dans la sphère 2-dimensionnelle.

De manière informelle, cela signifie qu'une carte est le dessin (*plongement*) d'un graphe (autorisant les boucles et les arêtes multiples) sur la sphère de telle sorte que les arêtes ne se rencontrent qu'aux sommets (*propre*). Plus précisément, il s'agit de la classe d'équivalence des dessins propres qui ne diffèrent que par le fait qu'ils sont des déformations continues les uns des autres (*considérés à homéomorphismes près*), voir Fig. 1.

Les cartes planaires sont dessinées sur la sphère usuelle (qui est une surface 2-dimensionnelle habituellement représentée dans un espace tridimensionnel), mais, il est équivalent de les dessiner sur le plan 2-dimensionnel (en utilisant par exemple la *projection stéréographique*) par l'envoi d'un point à l'infini (*i.e.* le choix d'une face marquée pour jouer le rôle de face infinie) ; c'est donc ce qui sera fait dans le reste du manuscrit, voir Fig. 2.

L'étude des cartes, en particulier des cartes planaires, a évolué au cours des six dernières décennies et a fait l'objet d'une littérature abondante dans divers domaines d'étude. Elle a commencé avec les travaux combinatoires pionniers de Tutte. Les cartes ont ensuite été introduites en physique théorique, où elles constituent des modèles naturels pour les surfaces aléatoires et permettent de modéliser l'impact de la gravité quantique. Un point de vue probabiliste avec l'étude des limites d'échelle a émergé plus récemment.

i.1.1 Combinatoire

Dans les années 1850, Guthrie a remarqué que la carte (géographique) des comtés anglais pouvait être colorée avec 4 couleurs de telle sorte que deux comtés adjacents ne partagent pas la même couleur. C'est ainsi qu'a débuté une quête de plus d'un siècle visant à prouver ce résultat pour toutes les cartes (géographiques). Dans les années 1960, le problème était toujours ouvert et Tutte a introduit des cartes planaires pour tenter de prouver le théorème des 4 couleurs (qui n'était alors qu'une conjecture) [Tut62b, Tut62a, Tut62c, Tut63] (voir Fig. 3). Finalement, Appel et Hankel l'ont prouvé en utilisant des *chaînes de Kempe* [AH77, RSST97], mais les cartes combinatoires de Tutte sont restées un sujet très étudié. En effet, la simplicité inattendue des résultats de Tutte a suscité la curiosité et a conduit à une étude combinatoire plus approfondie, à la recherche d'une explication bijective.

Plus précisément, Tutte a réalisé un tour de force en obtenant des formules closes pour l'énumération de diverses familles de cartes planaires, telles que les triangulations, les quadrangulations et les cartes générales : en particulier, le nombre m_n° de cartes (enracinées) avec n arêtes est égal à

$$m_n^\circ = \frac{2(2n)!3^n}{(n+2)!n!}.$$

Pour cela, il a traduit les décompositions combinatoires de ces cartes en équations que satisfont leurs fonctions génératrices. Ce travail est détaillé à la Section 1.3.2. Ces équations ont tendance à être complexes, aussi est-il surprenant que les formules de comptage soient si simples. Cela a éveillé la curiosité et conduit à une étude combinatoire plus approfondie, à la recherche d'une explication bijective.

Ces recherches ont abouti à la bijection Cori–Vauquelin–Schaeffer [CV81, Sch98], qui relie les cartes planaires à un modèle d'arbres étiquetés et a été étendue en la bijection Bouttier–Di Francesco–Guitter [BDG04]. En outre, de nombreux autres objets mathématiques sont en bijection avec des (familles de) cartes : les (familles de) permutations [DGW96, BBMF08], les intervalles de Tamari (généralisés) [BB09, FPR17, FH19, FPR20], les poissons combattants [DH22], les lambda-termes [Fan23]...

La décomposition de Tutte est encore fructueuse à ce jour : il est apparu qu'elle peut être placée dans un cadre plus large appelé *recursion topologique* [Eyn16], qui permet de calculer des séries génératrices pour des cartes sur d'autres surfaces que la sphère. Les types de décompositions introduits par Tutte pour obtenir des résultats énumératifs sur les cartes planaires jouent également un rôle majeur dans l'étude énumérative des graphes planaires [GN09]. Les cartes sont des objets extrêmement riches et de nombreux autres points de vue permettent de les étudier. En plus du point de vue combinatoire, nous détaillons dans la suite les points de vue physique et probabiliste. Parmi les autres approches, les cartes sont exprimées comme des factorisations de permutations, ce qui permet de les étudier à travers la théorie des représentations du groupe symétrique [LZ04]. Elles sont également étudiées en géométrie algébrique, où les *dessins d'enfants* de Grothendieck sont des cartes biparties [Sch94] ; en géométrie informatique, elles représentent des structures gouvernant les incidences dans les maillages, ...

i.1.2 Physique théorique

Dans les années 1970, les cartes planaires ont été réintroduites, cette fois dans le domaine de la physique théorique. En effet, des chercheurs comme 't Hooft ont relié les problèmes liés à l'énumération des cartes planaires à des modèles d'intégrales de matrices, en s'appuyant sur le calcul d'intégrales gaussiennes sur l'espace des matrices hermitiennes ['t 74, BIPZ78, IZ80]. Comme l'explique Zvonkin, les intégrales de matrices sont « naturelles » en physique théorique car, du point de vue de la mécanique quantique, lorsqu'une particule se déplace, elle emprunte « toutes les trajectoires possibles à la fois », donc pour effectuer des calculs, on évalue une intégrale sur l'espace de toutes les trajectoires possibles. Dans la théorie des cordes, la particule n'est pas représentée par un point mais par un petit cercle, donc lorsqu'elle suit une trajectoire, on obtient une surface. Intégrer toutes les trajectoires revient alors à intégrer l'espace de ces surfaces 2-dimensionnelles [Zvo97].

Plus récemment, les cartes ont été utilisées dans le contexte de la gravité quantique, qui s'intéresse à la quantification de la relativité générale. Plusieurs théories, comme la théorie des cordes et la gravité quantique à boucles, tentent de développer une théorie cohérente de la gravité quantique, mais aucune n'y est encore parvenue. Compte tenu de la difficulté de la tâche, il est naturel de commencer par un cadre plus simple que celui d'un espace-temps à 4 dimensions contenant de la matière. Même sans matière (gravité pure), il faut considérer des géométries aléatoires, et, dans le cas à 2 dimensions, cela veut dire de surfaces aléatoires [ADJ97]. Les cartes sont des modèles naturels pour cette étude car elles sont la discrétisation de géométries aléatoires à 2 dimensions : en effet, une carte planaire a la topologie d'une sphère 2-dimensionnelle et est naturellement dotée d'une métrique, la distance de graphe. De plus, les *cartes décorées* permettent de fournir des modèles de gravité quantique 2-dimensionnelle couplée à la matière. Elles conduisent à de nouveaux comportements asymptotiques, et l'étude des limites d'échelle dans ce contexte est actuellement un sujet très stimulant pour les cartes aléatoires [GHS20]. Plus généralement, le développement d'outils mathématiques issus des cartes pour parvenir à une théorie de la gravité quantique est toujours un domaine de recherche d'actualité en physique [Nad23], et les travaux des physicien·ne·s ont contribué à une description précise des liens entre les cartes et la physique statistique, la géométrie algébrique et la théorie des représentations [LZ04, Bou05].

En outre, dans le contexte de la physique statistique, Knizhnik, Polyakov et Zamolodchikov ont préconisé l'étude de modèles sur des « treillis planaires aléatoires » soigneusement choisis, ce qui représente une perspective de gravité quantique. Le couplage avec un modèle (tel que le modèle d'Ising) permet alors d'étudier le modèle en présence de matière. Dans le contexte de la percolation, ces treillis se comportent comme des cartes planaires aléatoires [KPZ88]. Ensuite, la *formule KPZ* permet de retrouver les propriétés géométriques dans le cas euclidien. La formule KPZ est restée mystérieuse pendant très longtemps, et n'a notamment pas été prouvée de manière rigoureuse, mais Duplantier et Sheffield ont fait des progrès significatifs vers une meilleure compréhension de cette formule [DS09, Gar13].

Les modèles considérés dans cette thèse sont étroitement liés au monde de la physique : le principal modèle pondéré par blocs du Chapitre 2 a été introduit par Bonzom, Delepouve et Rivasseau dans le contexte de la physique statistique [BDR15], et dans le Chapitre 6, nous étudions un tel modèle dans le contexte des cartes décorées et considérons le cas emblématique des *cartes boisées*, *i.e.*, les cartes dotées d'un arbre couvrant.

i.1.3 Probabilités

Considérer les cartes aléatoires comme des surfaces aléatoires ouvre la voie aux contributions de la théorie des probabilités. En effet, les propriétés des cartes planaires aléatoires échantillonnées uniformément ou selon une loi de probabilité (dérivée de modèles de physique statistique) ont permis de mieux comprendre les propriétés métriques des cartes « génériques » [CS04, LG10]. Les travaux sur les cartes en théorie des probabilités cherchent souvent à répondre à la question « quelle est la forme d'une carte tirée au hasard lorsque sa taille tend vers l'infini ? ».

Développée dans les années 2000, une première façon d'étudier la forme de grandes cartes enracinées — *i.e.* dotées d'un demi-bord marqué — est la notion de *limite locale*, qui correspond informellement à l'étude de la limite du voisinage de la racine. Dans cette topologie, la limite est souvent une carte planaire infinie (une carte dont le graphe sous-jacent n'est pas fini). Ainsi, Angel et Schramm ont montré que les triangulations uniformes convergent au sens de la limite locale vers la *Triangulation Planaire Infinie Uniforme (UIPT)* [AS03]. D'autres limites locales ont ensuite été trouvées, par exemple pour les quadrangulations uniformes [Kri05] ou les cartes de Boltzmann biparties [BS14, Ste18]. La limite locale dépend fortement des propriétés de la carte : la limite locale d'une triangulation a des faces de degré 3 alors que celle d'une quadrangulation a des faces de degré 4, il ne peut donc pas y avoir une seule limite locale « universelle » (même si les différentes limites locales partagent de nombreuses propriétés).

Des développements plus récents ont étudié les *limites d'échelle* des cartes planaires, qui décrivent le comportement de grandes cartes aléatoires lorsque leur taille croît à l'infini. Ces limites (de cartes discrètes renormalisées par leur diamètre) sont des objets aléatoires continus, et leurs propriétés peuvent parfois donner des résultats qu'il serait difficile d'obtenir directement à partir du modèle discret. Cependant, les résultats des limites d'échelle découlent souvent de codages bijectifs puissants. Cela montre que le point de vue combinatoire et le comportement probabiliste sont profondément imbriqués, chaque aspect guidant la compréhension de l'autre.

L'étude des limites d'échelle de plusieurs familles a montré un *phénomène d'universalité* : différentes familles de cartes, *a priori* non liées, partagent la même limite d'échelle. Les quadrangulations uniformes ont été le premier modèle pour lequel la limite d'échelle — appelée *sphère brownienne* $\mathcal{S}$ (parfois carte brownienne, voir Fig. 4) — a été complètement caractérisée par Miermont [Mie13] et indépendamment par Le Gall [LG13]. Depuis, ces résultats ont été étendus à d'autres familles de cartes : la limite d'échelle est la sphère brownienne, toujours avec une renormalisation de $cn^{1/4}$ pour une constante $c > 0$ dépendant du modèle. C'est ainsi le cas pour les $2q$-angulations uniformes (pour $q \geq 2$) et les triangulations uniformes [LG13], les cartes planaires uniformes [BJM14], les triangulations simples uniformes et les quadrangulations simples uniformes [ABA17], les cartes planaires biparties avec une certaine séquence de degrés pour les faces [Mar18], les $(2q+1)$-angulations [ABA21] et les triangulations eulériennes [Car21]. Le lien entre la sphère brownienne et la théorie de la gravité quantique de Liouville relie les points de vue de la physique théorique et de la théorie des probabilités : Miller et Sheffield ont montré que la sphère brownienne est « en quelque sorte équivalente » à la gravité quantique de Liouville avec le paramètre $q = \sqrt{8/3}$ [MS21a, MS20, MS19, MS21b, MS21c].

A l'opposé de la carte brownienne, les classes de cartes « dégénérées », pour lesquelles le comportement de branchement domine, présentent un autre phénomène d'universalité : lors d'un changement d'échelle de $cn^{1/2}$, il y a convergence vers l'arbre brownien d'Aldous $\mathcal{T}^{(2)}$ (voir Fig. 5), qui est la limite d'échelle des arbres de Bienaymé–Galton–Watson critiques à variance finie [Ald93, LG06]. C'est le cas pour les classes de cartes ayant une décomposition arborescente, comme les triangulations en pile [AM08] ; les classes de cartes avec certaines conditions de bord, comme les quadrangulations d'un polygone [Bet15], les cartes dites *outerplanar* [Car16] ; ou, plus généralement, pour les classes « sous-critiques » [Stu20a] (voir [PSW16] pour le cas des graphes).

L'omniprésence de la carte brownienne (et de l'arbre brownien) en tant que limite d'échelle a incité à rechercher d'autres limites d'échelle en choisissant des distributions de cartes spécifiques, permettant d'obtenir une interpolation entre différents objets. Par exemple, des modèles interpolant entre l'arbre brownien et le cercle peuvent être obtenus en utilisant les *arbres de boucles* (looptrees) [CK13b]. Curien et Kortchemski ont considéré l'UIPT où chaque sommet est coloré (indépendamment) en blanc avec une probabilité $a \in (0, 1)$ et en noir sinon. Iels ont étudié la frontière des composantes connexes monochromatiques (*clusters de percolation*) et ont montré que si $a \in (0, 1/2)$, la limite d'échelle est l'arbre brownien, si $a \in (1/2, 1)$ c'est le cercle unité et si $a = 1/2$ c'est l'*arbre de boucles stable de paramètre* $3/2$. [CK13a], qui correspond à l'*arbre stable de paramètre* $3/2$ (voir Fig. 6, introduit par Duquesne [Duq03]) où chaque point de branchement est remplacé par un cercle. Richier [Ric18] a également montré que la frontière des cartes planaires critiques de Boltzmann avec des degrés pour les faces dans le domaine d'attraction d'une distribution stable de paramètre $\alpha \in (1, 2]$ présente une transition de phase similaire : si $\alpha \in (1, 3/2)$, la limite d'échelle est l'arbre de boucles stable de paramètre $(\alpha - 1/2)^{-1}$, et, avec Kortchemski, Richier a montré que c'est le cercle unité si $\alpha \in (3/2, 2]$ et a conjecturé que c'est aussi le cas pour $\alpha = 3/2$ [KR20]. Stefánsson et Stufler ont montré que les cartes *outerplanar* avec une pondération sur les faces présentent un diagramme de phase similaire : l'arbre de boucles stable de paramètre α étant la limite d'échelle lorsque leur $\alpha \in (1, 2)$, l'arbre brownien lorsque $\alpha = 2$ et le cercle unité lorsque $\alpha = 1$ [SS19]. Dans les trois cas, le paramètre du modèle permet d'ajuster le nombre de sommets de coupure apparaissant sur la frontière, passant ainsi d'une phase « ronde » à une phase « arborescente ».

Une autre façon d'obtenir différentes limites d'échelle est d'utiliser des modèles paramétrés et d'étudier la limite d'échelle lorsque la valeur du paramètre change. En effet, certains modèles naturels interpolent également entre la sphère brownienne et l'arbre brownien. Par exemple, considérons des quadrangulations aléatoires avec n faces et une frontière de longueur ℓ, où $\ell/\sqrt{n} \to \sigma$. Lorsque $\sigma = 0$, la limite d'échelle est la sphère brownienne, lorsque $\sigma = \infty$, c'est l'arbre brownien, et pour tout $\sigma \in (0, \infty)$ c'est le *disque brownien* avec une frontière de longueur σ [Bet15]. Un autre exemple est celui des cartes planaires biparties aléatoires avec des poids pour les faces normalisés: elles convergent vers l'arbre brownien lorsque la distribution sur les degrés de face a une espérance inférieure à 1 [JS15], et vers la carte brownienne lorsque l'espérance est 1 et que la variance est finie [Mar18]. De plus, lorsque l'espérance est 1 et que la distribution est dans le domaine d'attraction d'une loi stable de paramètre $\alpha \in (1, 2)$, ces cartes convergent, au moins le long de sous-séquences appropriées, vers une limite qui n'est pas la sphère brownienne, et on conjecture qu'il s'agit de la *carte stable* de paramètre α [LGM11].

i.2 Contributions de cette thèse

Une caractéristique fondamentale des cartes planaires utilisées par Tutte est qu'elles peuvent être décomposées, par exemple, en composantes de degré de connectivité plus élevé. De manière informelle, une carte est dite 2-*connexe* s'il est nécessaire d'enlever au moins 2 sommets pour déconnecter le graphe sous-jacent ; cette notion est définie plus en détail dans la Section 2.1.1. Une telle décomposition relie généralement une famille de cartes planaires à une autre et produit une équation entre leurs séries génératrices, voir par exemple [BFSS01]. C'est fondamental dans ce travail et je le détaille dans le Chapitre 2. Au cours de ma thèse, j'ai publié trois articles, qui sont listés dans cette section.

[FleuratSalvy24] Dans un travail avec William Fleurat [FS24], j'ai étudié un modèle de cartes planaires pondérées par leur nombre de composantes 2-connexes. Notre étude a permis de relier deux classes d'universalité : celles des cartes (qui convergent, comme expliqué plus haut, vers la sphère brownienne) et celles des arbres plans, qui convergent vers l'arbre brownien [Ald93, LG06]. Ceci éclaire d'un jour nouveau le fait que certaines familles de cartes qui ressemblent à des arbres sont dans la classe d'universalité des arbres et non des cartes. De plus, entre ces deux comportements, nous avons montré une nouvelle classe d'universalité où la limite d'échelle est l'*arbre stable de paramètre* $3/2$ (voir Fig. 6). Il s'agit du premier modèle d'interpolation entre ces deux situations qui n'utilise pas la face racine pour jouer un rôle spécial, *i.e.* la transition dans notre modèle ne se manifeste pas à travers la frontière.

Plus précisément, nous introduisons une loi de probabilité (de Boltzmann) indexée par un paramètre $u > 0$ où la probabilité d'obtenir une certaine carte fixée dépend de son nombre de composantes 2-connexes (appelées « blocs »). En utilisant le cadre de la combinatoire analytique [FS09], nous mettons en évidence un point critique u_C et une transition de phase en fonction de la valeur de u par rapport à u_C. L'utilisation de l'algébricité de la série génératrice simplifie nos calculs et permet d'obtenir des formules closes élégantes.

Notre méthodologie consiste à utiliser une décomposition de chaque carte en un bloc auquel sont attachées des sous-cartes. Itérer cette décomposition révèle une structure arborescente sous-jacente, qui a été rendue explicite par Addario-Berry [AB19]. Nous montrons que de nombreuses propriétés de la carte sont directement encodées dans cet arbre, qui est beaucoup plus facile à analyser. C'est notamment le cas de la taille des blocs de la carte, dont nous obtenons explicitement la distribution jointe.

Ces arbres s'avèrent également utiles pour obtenir les limites d'échelle dans les cas critique et surcritique. Le résultat de convergence était déjà connu dans le cas surcritique [Stu20a], mais nous fournissons une nouvelle preuve, plus générale, qui nous permet de traiter à la fois le cas surcritique (moments exponentiels) et le cas critique (queue lourde), pour lesquels aucune méthode n'existait auparavant. Dans les deux cas, nous montrons que la géométrie de la carte est très proche de la géométrie de l'arbre associé, ce qui nous permet de conclure qu'ils ont la même limite d'échelle (à un facteur constant près).

Dans le cas sous-critique, l'analyse de la taille des blocs montre un bloc de taille linéaire et tous les autres de taille sous-linéaire. Nous montrons que lorsque nous renormalisons la carte, seul ce bloc linéaire est encore visible. Par conséquent, c'est sa géométrie qui donne la géométrie de la limite. Ceci est similaire à ce qui se passe pour les quadrangulations *uniformes* à n faces et leur cœur simple [ABW17].

[Salvy23] Dans [Sal23], j'ai montré que la méthodologie développée avec Fleurat pour les tailles de blocs n'est pas limitée au cas particulier des cartes décomposées en composantes 2-connexes, mais peut être appliquée à de nombreuses décompositions de familles de cartes : cartes sans boucles décomposées en composantes simples, triangulations simples décomposées en composantes irréductibles, ... Pour des considérations de dénombrement et de génération aléatoire (voir *e. g.* [Sch99]), Banderier, Flajolet, Schaeffer et Soria dressent une longue liste de schémas de décomposition pour des familles de cartes [BFSS01]. Iels analysent la taille du plus grand bloc dans le cas uniforme en utilisant l'analyse de singularité. Grâce à des techniques de combinatoire analytique, je montre une transition de phase pour la plupart de leurs schémas de décomposition. Je combine ces outils avec la décomposition en arbre de l'article précédent pour dériver des résultats probabilistes sur la taille des plus grands blocs. De plus, je montre que les résultats énumératifs (obtenus par Bonzom pour le cas de la décomposition en composantes 2-connexes [Bon16]) peuvent également être obtenus d'une manière unifiée. La robustesse de la méthode développée [FS24] est encore renforcée par les résultats unifiés obtenus pour tous ces schémas de décomposition.

[AlbenqueFusySalvy24] Dans [AFS24], j'ai travaillé avec mes directeurices de thèse Marie Albenque et Éric Fusy sur les cartes planaires *boisées* aléatoires (comme dit à la Section i.1.2, une carte planaire boisée est une carte planaire munie d'un arbre couvrant distingué). Notez que nous échantillonnons la carte et l'arbre couvrant simultanément (au lieu d'échantillonner d'abord une carte puis un arbre couvrant de cette carte), ce qui implique que la carte planaire sous-jacente n'est pas tirée uniformément au hasard, mais selon une distribution de probabilité pondérée par son nombre d'arbres couvrants. Du point de vue de la physique théorique, cela signifie quitter le cas de la « gravité pure » (où il n'y a qu'un champ mais pas de matière qui interagisse avec lui) pour des cas plus intéressants. Les modèles bien connus de cartes qui sortent de la « pure gravité » sont par exemple les cartes dotées d'un modèle d'Ising [BDG07], ou, plus généralement, d'un modèle de Potts [ZJ00]. Comme le prévoit notamment la physique théorique, ces cartes boisées présentent de nouveaux comportements, et leur étude peut donc conduire à des résultats novateurs.

En combinatoire analytique, certaines séries génératrices ont une propriété d'*algébricité*, qui donne automatiquement de nombreux résultats et simplifie les calculs. Les séries génératrices des travaux précédents ont toutes cette propriété, mais elle n'est plus vérifiée dans le cas des séries génératrices des cartes boisées (2-connexes). Au lieu de cela, ces séries génératrices sont

D-*finies* ou même seulement D-*algébriques*, qui sont des propriétés plus faibles (l'algébricité implique la D-finitude qui implique la D-algébricité). Ces propriétés sont définies dans la Definition 1.3. Cela rend les méthodes de la combinatoire analytique plus difficiles à mettre en œuvre, mais elles restent des alliées fiables.

En utilisant ces méthodes ainsi qu'un logiciel de calcul formel, nous effectuons une étude énumérative détaillée des cartes boisées décomposées en composantes 2-connectées. Au passage, nous obtenons le comportement asymptotique du nombre de cartes arborescentes planaires 2-connexes, qui était inconnu auparavant. Une fois de plus, ce qui montre la robustesse de la méthode utilisée, cela nous permet de montrer que le schéma de décomposition présente une transition de phase et nous obtenons la taille des plus grands blocs avec précision, ainsi que des résultats de limite d'échelle pour les cas critique et surcritique.

Contents

Chapter 0

Introduction

This thesis is one of a growing number of works on *maps*, and more specifically on *planar* maps. Let me begin with a brief history of this field, then, once this context has been introduced, I present my results and the organisation of this thesis.

0.1 A brief history of maps

Definition 0.1. A *planar map* is a proper embedding of a connected planar finite multigraph into the two-dimensional sphere.

Informally, this means that a map is the drawing (*embedding*) of a graph (allowing for loops and multiple edges) on the sphere such that edges meet only at vertices (*proper*). More precisely, it is the class of proper drawings which differ only in so that they are continuous deformations of each others (same as *considering up to homeomorphisms*), see Fig. 1.

Planar maps are drawn on the usual sphere (which is a two-dimensional surface usually represented in a three-dimensional space), but, up to sending a point at infinity (*i.e.* up to the choice of a marked face to play the role of infinite face), it is equivalent to draw them on the two-dimensional plane (using for example *stereographic projection*) so this is what will be done in the remaining of the manuscript, see Fig. 2.

The study of maps, particularly planar maps, has evolved over the past six decades and has been the focus of an extensive literature from various fields of study. It started with the

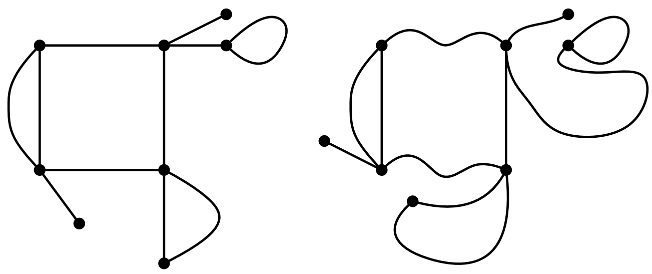

Figure 1: Two representations of the same map. Continuous deformations can change the shape or the place of the vertices and edges, but not the cyclic ordering of edges around a vertex.

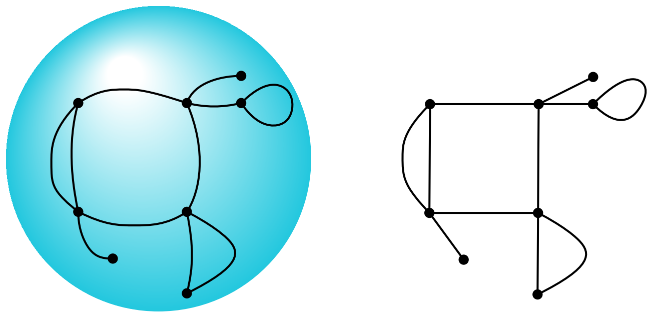

Figure 2: Two representations of the same map, one drawn on the sphere and the other on the plane.

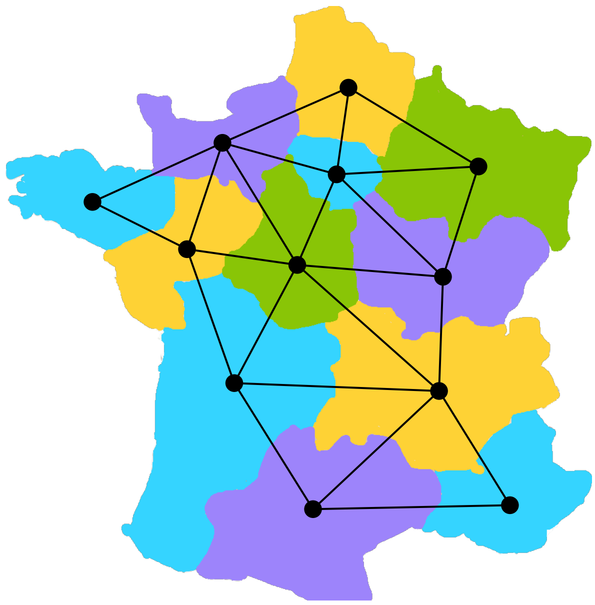

Figure 3: Four-coloured geographical map of continental Hexagonal France, with its underlying planar map.

groundbreaking combinatorial work of Tutte. Maps were later introduced in theoretical physics, where they are natural models for random surfaces and enable to model the impact of quantum gravity. More recently, a probabilistic point of view with the study of scaling limits has gained traction.

0.1.1 Combinatorics

In the 1850's, Guthrie noticed that the (geographical) map of English counties could be coloured with 4 colours such that no two adjacent counties shared the same colour. This started a more than century-long quest to prove this result for all (geographical) maps. In the 1960's, the problem was still open and Tutte introduced planar maps in an attempt to prove the 4-colour theorem (then a conjecture) [Tut62b, Tut62a, Tut62c, Tut63] (see Fig. 3). In the end, this was proved by Appel and Hankel using *Kempe chains* [AH77, RSST97], but Tutte's combinatorial maps remained a hot topic. Indeed, the unexpected simplicity of Tutte's results aroused curiosity and led to a more in-depth combinatorial study, in search of a bijective explanation.

More precisely, Tutte achieved a *tour de force*, obtaining closed enumerative formulae for various families of planar maps, such as triangulations, quadrangulations, and general maps: in particular, the number m_n° of (rooted) maps with n edges is equal to

$$m_n^\circ = \frac{2(2n)!3^n}{(n+2)!n!}.$$

To that end, he translated combinatorial decompositions of these maps into equations that their generating functions satisfy. This work is further detailed in Section 1.3.2. These equations tend to be intricate, so it came as a surprise that the counting formulae were so simple. This aroused curiosity and led to a more in-depth combinatorial study, in search of a bijective explanation.

This research culminated in the Cori–Vauquelin–Schaeffer bijection [CV81, Sch98], which links planar maps to a model of labelled trees and was further extended into the Bouttier–Di Francesco–Guitter bijection [BDG04]. Furthermore, many other mathematical objects are in bijection with (families of) maps: (families of) permutations [DGW96, BBMF08], (generalised) Tamari intervals [BB09, FPR17, FH19, FPR20], fighting fish [DH22], lambda-terms [Fan23]...

Tutte's decomposition is still fruitful to this day: it has come to light that it can be put into a broader framework called *topological recursion* [Eyn16], which enables the computation of generating series for maps on other surfaces than the sphere. The types of decompositions Tutte introduced to obtain enumerative results about planar maps also play a major role in the enumerative study of planar graphs [GN09]. Maps are extremely rich objects and many other points of view allow to study them. In addition to the combinatorial point of view, we detail in the following the physical and the probabilistic ones. Other approaches include maps being expressed as factorizations of permutations, allowing for their investigation through the theory of representations of the symmetric group [LZ04]. They are also studied in algebraic geometry where Grothendieck's *dessins d'enfants* are bipartite maps [Sch94]; and in computational geometry as the structure giving combinatorial incidences in meshes, ...

0.1.2 Theoretical physics

In the 1970's, planar maps were introduced again, this time in the domain of theoretical physics. Indeed, researchers like 't Hooft connected problems related to the enumeration of planar maps with matrix integral models, relying on the computation of Gaussian integrals over the space of Hermitian matrices ['t 74, BIPZ78, IZ80]. As explained by Zvonkin, matrix integrals come "naturally" in theoretical physics since, from a quantum mechanics perspective, when a particle moves it takes "all possible paths at once", so to make computations one evaluates an integral over the space of all possible trajectories. In string theory, the particle is not represented by a point but by a small circle, so when it follows a path, one obtains a surface. Integrating over all trajectories then amounts to integrating over the space of these two-dimensional surfaces [Zvo97].

More recently, maps have been used in the context of *quantum gravity*, which is concerned with the quantisation of general relativity. Several theories, like string theory and loop quantum gravity, are attempting to develop a coherent theory of quantum gravity, but none have fully succeeded yet. Given the difficulty of the task, it is natural to begin by a simpler framework than that of 4-dimensional space-time containing matter. Even without matter (pure gravity), one must consider random geometries, and in the two-dimensional case, these are random surfaces [ADJ97]. Maps are natural models for this study as they are the discretisation of random two-dimensional geometries: indeed, a planar map has the topology of a two-dimensional sphere and is naturally endowed with a graph distance. Moreover, *decorated maps* are instrumental to provide models of two-dimensional quantum gravity coupled with matter. They lead to new asymptotic behaviours, and the study of scaling limits in that context is currently a very challenging topic in random maps [GHS20]. More generally, developing mathematical stemming from maps tools to arrive at a theory of quantum gravity is still a topical area of research in physics [Nad23], and physicists' works have contributed to a precise description of the links between maps and statistical physics, algebraic geometry and representation theory [LZ04, Bou05].

Moreover, in the context of statistical physics, Knizhnik, Polyakov, and Zamolodchikov advocated studying models on carefully chosen "random planar lattices", representing a quantum gravity perspective. *Coupling* with a model (such as the Ising model) then enables to study the model in the presence of matter. In the context of percolation, these lattices behave as random planar maps [KPZ88]. Then, the *KPZ formula* allows to retrieve the geometric properties in the Euclidean case. The KPZ formula remained mysterious for a very long time, and in particular was not proved rigorously, but Duplantier and Sheffield made significant steps towards a better understanding of it [DS09, Gar13].

The models considered in this thesis are closely related to the physics world: the main block-weighted model of Chapter 2 was introduced by Bonzom, Delporte and Rivasseau in the context of statistical physics [BDR15], and in Chapter 6, we study such a model in the context of decorated maps and consider the emblematic case of *tree-rooted maps*, *i.e.*, maps endowed with a spanning tree.

0.1.3 Probability theory

Considering random maps as random surfaces opens the possibility for contributions from probability theory. Indeed, the properties of random planar maps sampled uniformly or according to a probability (*e.g.* derived from statistical physics models) has led to a deeper understanding of the metric properties of "generic" maps [CS04, LG10]. Work on maps in probability theory often seeks to answer the question "what is the shape of a map drawn at random when its size tends to infinity?".

Developed in the 2000's, a first way of studying the shape of large rooted *i.e.* endowed with a marked half-edge — maps is the notion of *local limit*, which informally corresponds to looking at the limit of the neighbourhood of the root. In this topology, the limit is often an infinite planar map (a map whose underlying graph is not finite). Indeed, Angel and Schramm showed that uniform triangulations converge in the sense of the local limit towards the *Uniform*

Figure 4: Approximation of the Brownian sphere by a simple quadrangulation of size $50\ 000$, using a generator by Éric Fusy.

Infinite Planar Triangulation (UIPT) [AS03]. Other local limits were then found, for example for uniform quadrangulations [Kri05] or Boltzmann bipartite maps [BS14, Ste18]. The local limit depends strongly on the properties of the map: the local limit of a triangulation has faces of degree 3 while that of a quadrangulation has faces of degree 4, so there cannot be a single "universal" local limit (even though the different local limits share many properties).

More recent developments have studied the *scaling limits* of planar maps, which describe the global behaviour of large random maps as the size grows to infinity [LG07, LGM11, Mie13, LG13]. These limits (of renormalised discrete maps) are random continuous objects, and sometimes their properties can be translated into results that might be challenging to obtain directly from the discrete model. However, the scaling limit results often stem from powerful bijective encodings. This shows that the combinatorial point of view and probabilistic behaviour are deeply intertwined, as each side guides the understanding of the other.

The study of scaling limits of several families showed a *universality phenomenon*: different families of maps, *a priori* unrelated, share the same scaling limit. Uniform quadrangulations was the first model for which the scaling limit — the so-called *Brownian sphere* $\mathcal{S}$ (also called Brownian map, see Fig. 4) — was completely characterised by Miermont [Mie13] and independently by Le Gall [LG13]. Since then, these results have been extended to other families of maps: the scaling limit is the Brownian sphere, always with a rescaling by $cn^{1/4}$ for some model-dependent $c > 0$. In particular, uniform $2q$-angulations ($q \geq 2$) and uniform triangulations [LG13], uniform planar maps also converge towards the Brownian sphere [BJM14], uniform simple triangulations and uniform simple quadrangulations [ABA17], bipartite planar maps with a prescribed face-degree sequence [Mar18], $(2q+1)$-angulations [ABA21] and Eulerian triangulations [Car21]. The connection between the Brownian sphere and Liouville quantum gravity theory bridges the theoretical physics and probability theory points of view: Miller and Sheffield showed that the Brownian sphere is "somehow equivalent" to the Liouville quantum gravity with parameter $q = \sqrt{8/3}$ [MS21a, MS20, MS19, MS21b, MS21c].

"Opposite" the Brownian map, "degenerate" classes of maps, for which the branching behaviour dominates, exhibit another universality phenomenon: upon rescaling by $cn^{1/2}$, there is a convergence to Aldous's *Brownian tree* $\mathcal{T}^{(2)}$ (see Fig. 5), which is the scaling limit of critical Bienaymé–Galton–Watson trees with finite variance [Ald93, LG06]. This is the case for classes of maps with a tree-decomposition such as stack triangulations [AM08] (see Fig. 5); classes of maps with some particular boundary conditions, such as quadrangulations of a polygon [Bet15], outerplanar maps [Car16]; or, more generally for "subcritical" classes [Stu20a] (see [PSW16] for the case of graphs).

The ubiquity of the Brownian map (and the Brownian tree) as the scaling limit prompted the research of alternative scaling limits using adjusted map distributions, allowing to obtain interpolation between different objects. For example, models interpolating between the Brownian tree and the circle can be obtained by using *looptrees* [CK13b]. Curien and Kortchemski considered the UIPT where each vertex is coloured (independently) white with probability $a \in (0,1)$ and black otherwise and studied the boundary of monochromatic connected components (*percolation clusters*). They showed that if $a \in (0, 1/2)$, the scaling limit is the Brownian tree, if $a \in (1/2, 1)$ it is the unit circle and if $a = 1/2$ it is the *stable looptree of parameter* $3/2$ [CK13a], which corresponds to the *stable tree of parameter* $3/2$ (see Fig. 6, introduced by Duquesne [Duq03]) where each branching point is replaced by a circle. Richier [Ric18] also showed that the boundary of critical Boltzmann planar maps with face-degrees in the domain of attraction of a stable distribution with parameter $\alpha \in (1, 2]$ exhibit a similar phase transition: if $\alpha \in (1, 3/2)$, the scaling limit is the stable looptree of parameter $(\alpha - 1/2)^{-1}$, and, with Kortchemski, Richier showed that it is the circle of unit length if $\alpha \in (3/2, 2]$ and conjectured that this holds also for $\alpha = 3/2$ [KR20]. Stefánsson and Stufler showed that face-weighted outerplanar maps have a similar phase diagram, with the α-stable looptree being the scaling limit when their $\alpha \in (1, 2)$, the Brownian tree when $\alpha = 2$ and the deterministic circle of unit length when $\alpha = 1$ [SS19]. In all three cases, the parameter of the model allows the number of cut vertices appearing on the boundary to be adjusted, thus changing from a "round" to a "tree" phase.

Another way of showing different scaling limits is to use parametrised models and study the scaling limit when the value of the parameter changes. Indeed, some natural models also interpolate between the Brownian sphere and the Brownian tree. For example, consider random quadrangulations with n faces and a boundary of length ℓ, where $\ell/\sqrt{n} \to \sigma$. When $\sigma = 0$, the scaling limit is the Brownian sphere, when $\sigma = \infty$ it is the Brownian tree, and for all $\sigma \in (0, \infty)$ it is the *Brownian disk* with boundary length σ [Bet15]. Another example is random bipartite planar maps with properly normalized face-weights, which converge towards

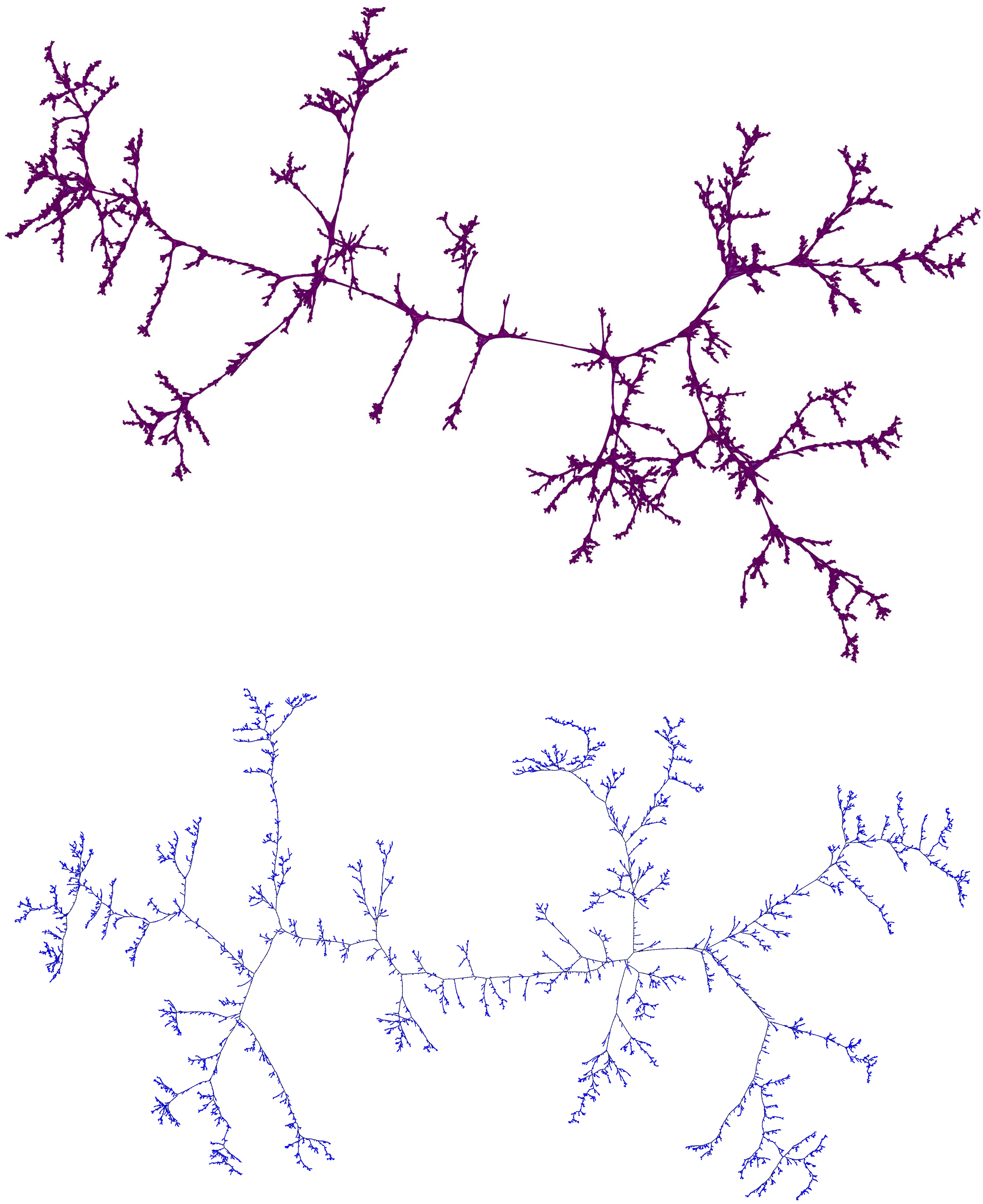

Figure 5: (Top) A uniform stack-triangulation with $50\ 000$ faces by Jérémie Bettinelli. (Bottom) Approximation of the Brownian tree by a binary tree of size approximately $70\ 000$.

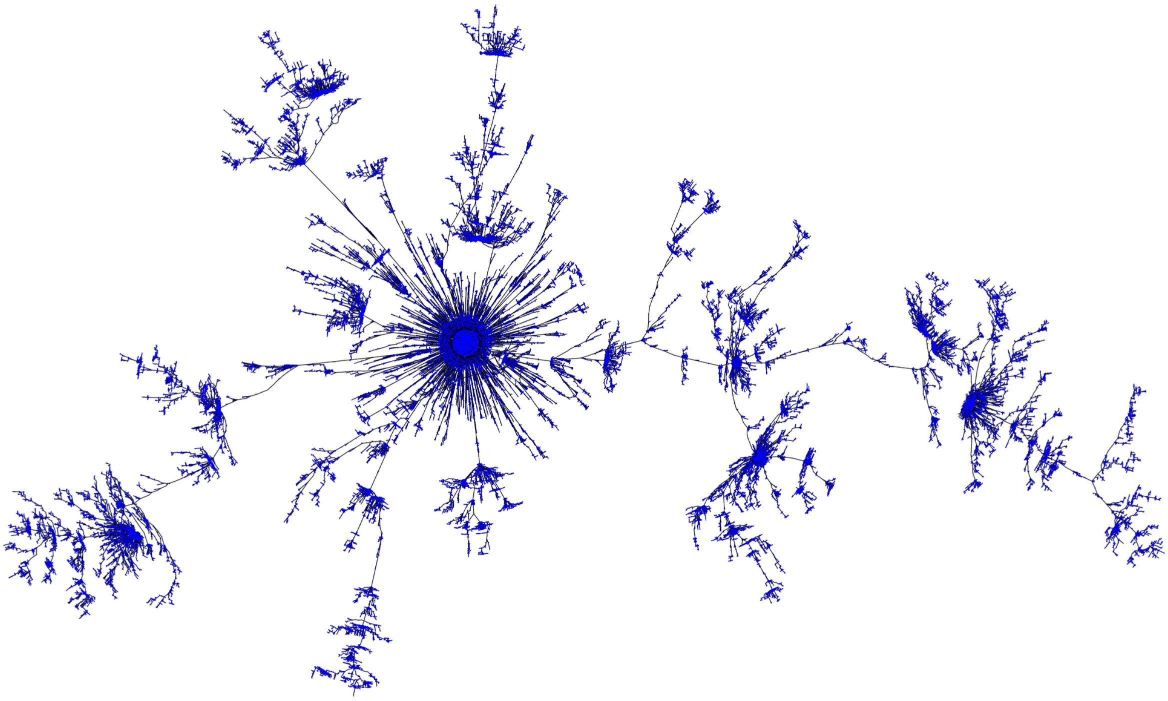

Figure 6: Approximation of the $3/2$-stable tree by a tree of size approximately $150\ 000$.

the Brownian tree when the weighted distribution on face-degrees has expected value smaller than 1 [JS15], and towards the Brownian map when the expected value is 1 and the variance is finite [Mar18]. Moreover, when the expected value is 1 and the distribution is in the domain attraction of a stable law of parameter $\alpha \in (1, 2)$, these maps converge, at least along suitable subsequences, towards a limit which is not the Brownian sphere, and is conjectured to be the *stable map* of parameter α [LGM11].

0.2 Contributions of the thesis

A fundamental feature of planar maps used by Tutte is that they can be decomposed, *e.g.* into components of higher connectivity degree. Informally, a map is said 2-*connected* if it is necessary to remove at least 2 vertices to disconnect the underlying graph; this notion is defined in more details in Section 2.1.1. Such a decomposition usually relates one family of planar maps to another one and yields an equation between their generating series, see *e.g.* [BFSS01], and this is cardinal in this work and is detailed in Chapter 2. During my thesis, I published three articles, which are listed in this section.

[FleuratSalvy24] In a joint work with William Fleurat [FS24], I have studied a model of general planar maps weighted by their number of 2-connected components. Our study has linked two classes of universality: those of maps (which converge, as explained above, towards the Brownian sphere) and those of plane trees, which converge towards the Brownian tree [Ald93, LG06]. This sheds new light on the fact that certain families of maps that resemble trees are in the universality class of trees and not of maps. Furthermore, between these two behaviours, we have shown a new universality class where the scaling limit is the *stable tree of parameter* $3/2$ (see Fig. 6). This is the first interpolation model between these two situations which does not use the root face to play a special role, *i.e.* the transition in our model does not manifest through the boundary.

More precisely, we introduce a (Boltzmann) probability law indexed by a parameter $u > 0$ where the probability of drawing a certain fixed map depends on its number of 2-connected components (called "blocks"). Using the framework of analytic combinatorics [FS09], we highlight a critical point u_C and a phase transition as a function of the value of u relative to u_C. Using the algebraicity of the generating series simplifies our computations and allows for elegant closed formulae.

Our methodology consists of using a decomposition of every map into a block with submaps attached to it. Iterating this decomposition reveals an underlying tree structure, which was made explicit by Addario-Berry [AB19]. We show that many properties of the map are directly encoded in this tree, which is much easier to analyse. This is particularly true of the size of the map's blocks, for which we explicitly obtain the joint distribution.

These trees also prove useful for obtaining the scaling limits in the critical and supercritical cases. The convergence result was already known in the supercritical case [Stu20a], but we provide a new, more general proof which allows us to deal with both the supercritical case (exponential moments) and the critical case (heavy tail), for which no method existed before. In both cases, we show that the geometry of the map is very close to the geometry of the associated tree, allowing us to conclude that they have the same scaling limit (up to a constant factor).

In the subcritical case, the block size analysis shows one block of linear size and all the others of sub-linear size. We show that when we renormalise the map, only this linear block is still visible. Consequently, it is its geometry that gives the geometry of the limit. This is similar to what happens for *uniform* quadrangulations with n faces and their simple core [ABW17].

[Salvy23] In [Sal23], I have shown that the methodology developed with Fleurat for block sizes is not limited to the special case of maps decomposed into 2-connected components, but can be applied to numerous decompositions of families of maps: maps without loops decomposed into simple components, simple triangulations decomposed into irreducible components, ... Motivated by considerations of enumeration and random generation (see *e.g.* [Sch99]), Banderier, Flajolet, Schaeffer and Soria make a long list of decomposition schemes for families of maps [BFSS01]. They analyse the size of the largest block in the uniform case using singularity analysis. Using analytic combinatorics, I shed light into a phase transition for most of their decomposition schemes. I combine these tools with the tree decomposition framework of the previous article to derive probabilistic results on the size of largest blocks. Moreover, I show that enumerative results (obtained by Bonzom for the case of the decomposition into 2-connected components [Bon16]) can also be derived in a unified way. The robustness of the method developed [FS24] is further enhanced by the unified results obtained for all decomposition schemes.

[AlbenqueFusySalvy24] In [AFS24], I worked with my PhD advisors Marie Albenque and Éric Fusy on *tree-rooted* random planar maps (recall from Section 0.1.2 that a *tree-rooted* planar map is a planar map with a distinguished spanning tree). Note that we are sampling the map and the spanning tree simultaneously (instead of sampling first a map and then a tree spanning that map), which implies that the underlying planar map is not drawn uniformly at random, but rather according to a probability distribution weighted by its number of spanning trees. From the point of view of theoretical physics, this means leaving "pure gravity" (where there is only a field but no object to interact with it) for more interesting cases. Well-known models of maps which are not "pure gravity" are for example maps endowed with an Ising model [BDG07], or, more generally, with a Potts model [ZJ00]. As predicted among others by theoretical physics, these tree-rooted maps exhibit new behaviours, and thus their investigation can lead to exciting results.

In analytic combinatorics, certain generating series have a property of *algebraicity*, which automatically gives many results and simplifies computations. The generating series in the previous works all have this property, but it is no longer verified in the case of the generating series of (2-connected) tree-rooted maps. Instead, these generating series are D*-finite* or even only D*-algebraic*, which are weaker properties (algebraicity implies D-finiteness which implies D-algebraicity). These properties are defined in Definition 1.3. This makes the methods of analytic combinatorics more difficult to implement, but they are still reliable allies.

By using these methods as well as computer algebra software, we carry out a detailed enumerative study of tree-rooted maps decomposed into 2-connected components. In passing, we obtain the asymptotic behaviour of the number of planar 2-connected tree-rooted maps, that was unknown before. Once again demonstrating the robustness of the method used, this allows us to show that the decomposition scheme exhibits a phase transition and we obtain the size of the largest blocks with precision, as well as scaling limit results for the critical and supercritical cases.

0.3 Organisation of the manuscript

In order to demonstrate the generality of the methods used in the article with William Fleurat, we are adopting a "cross-sectional" presentation of the methods.

We begin by introducing in Chapter 1 fundamental notions of analytic combinatorics and combinatorics of maps in order to provide a precise scientific context for the work being carried out. Readers familiar with these subjects can easily skip this chapter.

In Chapter 2, we present in detail the notion of block decomposition, which is central to our study. We begin by recalling the decomposition of general maps into 2-connected components described by Tutte; as well as its reformulation for quadrangulations decomposed into simple components. The two models are those studied in the article with Fleurat [FS24]. We then present a general decomposition framework that allows seven other cases to be treated in a unified way (Theorem 2.15), as discussed in [Sal23]. We introduce a Boltzmann probability law with a weight u per block, and show that it has 3 regimes depending on how u compares to a critical value u_C (Theorem 2.19). Finally, we explain how to build a random generator to sample maps according to the Boltzmann probability law.

In Chapter 3, we explore the phase transition in the context of enumeration. We shed light on a phase transition in the asymptotic behaviour of the enumeration: for $u < u_C$ (the subcritical case), the polynomial correction is that of planar maps: $n^{-5/2}$. Conversely, for $u > u_C$ (the supercritical case), the correction is that of plane trees: $n^{-3/2}$. Additionally, at $u = u_C$, a novel asymptotic behaviour emerges, characterized by a polynomial correction of $n^{-5/3}$ (Theorem 3.2 and Proposition 3.5). In the case of maps decomposed into 2-connected blocks, this study had already been carried out by Bonzom [Bon16] and therefore not discussed in [FS24]. However, for the other models, it constitutes a new result from [Sal23], in which the proof was not provided due to space constraints. An interesting aspect of this new work is that we provide a unified proof for all the decomposition schemes considered. We finish by analysing the complexity of the random generator, and show that it is linear in the supercritical case (Theorem 3.10).

In Chapter 4, we investigate the size of the largest blocks of maps drawn according to the law $\mathbb{P}_{n,u}$ at size n with weight u per block, when the size $n \to \infty$. We analyse all decomposition schemes using a unified proof, except for the last one where special care is needed. We prove that if $u < u_C$, a condensation phenomenon occurs, resulting in the largest block being of size $\Theta(n)$ (Theorem 4.2). For $u > u_C$, the largest block is of size $\Theta(\ln(n))$ (Theorem 4.7). At the critical value $u = u_C$, the largest block is of size $\Theta(n^{2/3})$ (Theorem 4.4). These results are proved in [FS24, Sal23] and those not concerning the main decomposition model are present here in greater detail than in [Sal23].

In Chapter 5, we reveal a phase transition regarding the metric properties. Unlike in previous chapters, we do not provide a unified treatment for all models but focus on our primary example of general maps decomposed into 2-connected blocks and their reformulation as quadrangulations decomposed into simple blocks. In the final section, we discuss how these results could be extended to other decomposition schemes; and emphasize the technical difficulties that have yet to be overcome. Specifically, the first section is an excerpt from [FS24], in which we provide a unified proof of convergence towards the *Brownian Continuum Random Tree* $\mathcal{T}^{(2)}$, renormalizing distances by $n^{1/2}$, in the supercritical case $u > u_C$, and towards the $3/2$*-stable tree* $\mathcal{T}^{(3/2)}$, renormalizing distances by $n^{1/3}$, in the critical case $u = u_C$ (Theorem 5.4). These results hold both for maps and their 2-connected components, as well as for quadrangulations and their simple components. Finally, when $u < u_C$, we show that quadrangulations converge towards the Brownian sphere when distances are renormalized by $n^{1/4}$ (Theorem 5.22) and explain why we anticipate this result to hold for general maps as well.

Chapter 6 studies a more complex combinatorial class compared to those previously examined: tree-rooted maps decomposed into 2-connected blocks. Using the tools introduced in the previous chapters, we analyse the phase transition undergone by the model. We start by obtaining the asymptotic equivalent for the number of 2-connected tree-rooted maps. Then, we consider the law $\mathbb{P}_{n,u}$ at size n with weight u per block and exhibit 3 regimes depending on how u compares to an explicit critical value u_C (Theorem 6.9) and show a phase transition for asymptotic the enumeration (Theorem 6.11). Afterwards, we identify the presence of a linear-sized giant block when $u < u_C$, mesoscopic blocks of order $\Theta(n^{1/2})$ when $u = u_C$, and smaller blocks of order $\Theta(\ln(n))$ when $u > u_C$ (Theorem 6.14). Additionally, we show that the block-weighted maps converge to the Brownian tree $\mathcal{T}^{(2)}$ for all $u \geq u_C$, with a discontinuity at u_C in the scaling magnitude: it is $\sqrt{n/\ln n}$ at $u = u_C$, and $\sqrt{n}$ for $u > u_C$ (Theorem 6.17). The scaling limit result for $u > u_C$ also follows from Stufler [Stu20a].

Finally, Chapter 7 offers a conclusion and perspectives for future work.

Chapter 1

Prelude

This chapter recalls basic notions in two cardinal areas of this thesis: analytic combinatorics and the study of planar maps. Section 1.1 gives a reminder of concepts of probability. Then, Section 1.2 presents the basics of analytic combinatorics: symbolic dictionary (Section 1.2.1), singularity analysis (Section 1.2.3) and Boltzmann generation (Section 1.2.4). Finally, we present general definitions regarding maps in Section 1.3: we introduce general vocabulary as well as several families of maps which are studied in this work (Section 1.3.1) and Tutte's method for enumeration (Section 1.3.2). Readers familiar with these areas can move on to the next chapter without loss.

In the whole manuscript, the set of whole numbers is written $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Integer intervals are written $[\![k, l]\!] = \{k, k+1, \dots, l\}$.

1.1 Probabilistic notions and notation

Many of the results in this thesis are probabilistic, so we introduce here some usual notions of probability which allow to express our results formally. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilistic space. The expectation of a law μ is denoted by $\mathbb{E}[\mu]$, its variance $\mathbb{V}(\mu)$ and its standard deviation $\sigma(\mu) = \sqrt{\mathbb{V}(\mu)}$. We begin by introducing two notions of convergence.

Definition 1.1. A sequence $(A_n)_{n\in\mathbb{N}_0}$ of real-valued random variables *converges in distribution* towards the real-valued random variable A, which is denoted

$$(A_n)_{n\in\mathbb{N}_0} \xrightarrow[n\to\infty]{(d)} A,$$

if for all bounded continuous function f,

$$\mathbb{E}[f(A_n)] \xrightarrow[n\to\infty]{} \mathbb{E}[f(A)].$$

The sequence $(A_n)_{n\in\mathbb{N}_0}$ *converges in probability* towards A if

$$\forall \varepsilon > 0, \qquad (\mathbb{P}(|A_n - A| \geq \varepsilon)) \xrightarrow[n\to\infty]{} 0,$$

in which case one writes

$$(A_n)_{n\in\mathbb{N}_0} \xrightarrow[n\to\infty]{\mathbb{P}} A.$$

An event is said to happen *almost surely* (abbreviated as *a.s.*) if it happens with probability 1. The sequence $(A_n)_{n\in\mathbb{N}_0}$ *converges almost surely* towards A if

$$\mathbb{P}\left(\lim_{n\to\infty} A_n = A\right) = 1,$$

in which case one writes

$$(A_n)_{n\in\mathbb{N}_0} \xrightarrow[n\to\infty]{\text{a.s.}} A.$$

Informally, convergence in distribution is concerned only with the distributions of A_n and A, and does not prejudge a priori a link between what is represented by one or the other, unlike convergence in probability. The latter indicates that, for any $\varepsilon > 0$, one can find a set of measures at least $1-\varepsilon$ where convergence does indeed take place. For almost sure convergence, the set where convergence does not occur is negligible. This last notion of convergence is therefore the strongest: it implies convergence in probability, which in turn implies convergence in distribution.

We also introduce the *total variation distance* d_{TV} between two probability distributions μ and ν

$$d_{TV}(\mu,\nu) = \sup_{F\in\mathcal{F}} |\mu(F) - \nu(F)|.$$

By abuse of notation, if A and B are random variables of respective probability laws μ and ν, we define

$$d_{TV}(A,B) = d_{TV}(\mu,\nu).$$

If the random variable sequences $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ satisfy

$$d_{TV}(A_n,B_n) \xrightarrow[n\to\infty]{} 0,$$

we write

$$A_n \overset{(d)}{\approx} B_n.$$

Notice that if $A_n \overset{(d)}{\approx} B_n$, then, if they converge in distribution, they have the same limit.

Now, we extend asymptotic notation to probabilistic convergence; for a sequence $(A_n)_{n\in\mathbb{N}_0}$ of random variables and a positive sequence a_n, one writes:

- $A_n = O_{\mathbb{P}}(a_n)$ to mean that, for all sequences (u_n) of positive numbers that converge towards $+\infty$,

$$(\mathbb{P}(|A_n| \leq a_n u_n))_{n\in\mathbb{N}_0} \xrightarrow[n\to\infty]{} 1;$$

- $A_n = \Theta_{\mathbb{P}}(a_n)$ to mean that, for all sequence (u_n) of positive numbers that converges towards $+\infty$,

$$(\mathbb{P}(a_n/u_n \leq |A_n| \leq a_n u_n))_{n\in\mathbb{N}_0} \xrightarrow[n\to\infty]{} 1.$$

1.2 Analytic combinatorics

The initial goal in *combinatorics* is counting, be it the ways of taking k distinct elements among n, or the number of *binary trees* — trees where each internal node has exactly two children — with n internal nodes. For these two simple examples, it is possible obtain a closed formula: there are

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

ways to take k distinct elements among n, and the number of binary trees with n internal nodes is called the *Catalan number*, and is equal to:

$$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}.$$

Analytic combinatorics provide tools to obtain information about the sequence $(c_n)_n$ giving the number of objects of size n in a combinatorial class, even when there is no closed formula for the enumeration as in the two examples above.

Definition 1.2. A *combinatorial class* $\mathcal{C}$ is a (potentially infinite) set of objects with a function of size $|.|_{\mathcal{C}} : \mathcal{C} \to \mathbb{N}_0$ (or simply $|.|$), and such that for each $n \in \mathbb{N}_0$, the number c_n of objects of size n is finite.

A key idea in analytic combinatorics is to represent $\mathcal{C}$ by its *(ordinary) generating series*

$$C(z) = \sum_{n\in\mathbb{N}_0} c_n z^n = \sum_{\gamma\in\mathcal{C}} z^{|\gamma|}.$$

By convention, the combinatorial class is represented by a calligraphy capital letter, the series by a capital letter and the coefficients by a lower case letter, and $[z^n]C(z)$ denotes c_n, the coefficient of z^n in $C(z)$. Generating series allow to study sequences of numbers using analytical tools *via* the generating series representation.

For more details on analytic combinatorics, we refer to the book by Flajolet and Sedgewick [FS09].

1.2.1 Symbolic dictionary and generating series

A crucial point is to obtain information about the generating series, with the aim of using analytical combinatorics tools to obtain information about the combinatorial class of interest. And a first method for doing this is to obtain an equation for the generating series, expressing it either in terms of itself or in terms of other already known generating series.

To that end, we have an automatic method which translates usual combinatorial constructions into operations on generating series, called *symbolic dictionary*. The idea behind is to decompose any object in the class into objects in other classes, for which we know the generating series; or in terms of objects of the same class and study the recursive equation.

In particular, we define two special classes: the *neutral class* $\mathcal{E}$: the combinatorial class containing only one object of size 0 and the *atomic class* $\mathcal{Z}$: the combinatorial class containing only one object of size 1. Their generating series are written respectively

$$E(z) = 1 \qquad \text{and} \qquad Z(z) = z.$$

Now consider any combinatorial classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and let us introduce basic operations on them.

Disjoint union If $\mathcal{C} = \mathcal{A} \sqcup \mathcal{B}$ (also denoted $\mathcal{C} = \mathcal{A} + \mathcal{B}$), then

$$C(z) = A(z) + B(z),$$

considering that the size of an object in $\mathcal{C}$ is its size in $\mathcal{A}$ or $\mathcal{B}$.

Cartesian product We define the size of a pair $(\alpha, \beta), \alpha \in \mathcal{A}, \beta \in B$ by $|(\alpha, \beta)| = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}$.
Then, if $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, then

$$C(z) = A(z) \times B(z).$$

With these two operations, we can already write an equation for the generating series $T(z)$ of binary trees counted by their internal nodes: such a binary tree is either made up of a single leaf, so it has size 0 and is represented by the neutral class $\mathcal{E}$. Otherwise, it is an internal node to which two sub-trees are attached (potentially reduced to one leaf). Each of these sub-trees is itself a binary tree. This internal node is represented by the atomic class $\mathcal{Z}$ and so we can write the following relation for the combinatorial class $\mathcal{T}$ of binary trees:

$$\mathcal{T} = \mathcal{E} + \mathcal{Z} \times \mathcal{T} \times \mathcal{T}$$

which translates immediately into the following generating series equation for

$$T(z) = 1 + zT(z)^2.$$

Solving this quadratic equation yields

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2z}, \tag{1.1}$$

which, using Newton expansion on $(1 - 4z)^{1/2}$ yields

$$[z^n]T(z) = \frac{1}{n+1}\binom{2n}{n}, \tag{1.2}$$

and we recognize the famous Catalan numbers. This example shows how, from elementary and simple operations, one can already obtain strong results on the combinatorial class. The expression for $T(z)$ shows that it ceases to be differentiable when $z = 1/4$, which is therefore an *singularity* of the generating series. A cardinal principle of analytic combinatorics is to obtain *singular expansion* of the generating series at its singularities, and deduce *asymptotic expansion* of the coefficients. Before detailing this, we now list some other operations that will be useful later on.

Sequence construction The class *sequence* $\mathrm{Seq}(\mathcal{A})$ corresponds to finite sequences of elements of $\mathcal{A}$. It is formally defined by the infinite sum

$$\mathrm{Seq}(\mathcal{A}) = \mathcal{E} + \mathcal{A} + \mathcal{A} \times \mathcal{A} + \mathcal{A} \times \mathcal{A} + \dots,$$

in other words

$$\mathrm{Seq}(\mathcal{A}) = \{(a_1, \dots, a_\ell) \mid \ell \geq 0, a_i \in \mathcal{A}\}.$$

The size of $(a_1, \dots, a_\ell)$ is $\sum_{i=1}^{\ell} |a_i|$ by the definition of the size of a Cartesian product. The number of elements of size n is therefore finite if and only if $\mathcal{A}$ contains no elements of size 0. In this case, $\mathrm{Seq}(\mathcal{A})$ is indeed a combinatorial class; and, if $\mathcal{C} = \mathrm{Seq}(\mathcal{A})$, then:

$$C(z) = \frac{1}{1 - A(z)}.$$

Pointing and derivation. Combinatorial structures are made up of a certain number of atoms, which determines their size. Pointing an element consists in distinguishing one of these atoms (the atoms are assumed to be distinguishable) and is denoted with a $\bullet$ in exponent. If $\mathcal{C} = \mathcal{A}^\bullet$, then:

$$C(z) = zA'(z).$$

Deriving a structure consists of pointing to an atom and no longer counting it in the size. There is then, if $\mathcal{C} = \mathcal{A}'$,

$$C(z) = A'(z).$$

Substitution The substitution $\mathcal{B} \circ \mathcal{A} = \mathcal{B}(A)$ consists in replacing each atom of an element of $\mathcal{B}$ by an element of $\mathcal{A}$, without changing the global structure of the element of $\mathcal{B}$. Formally,

$$\mathcal{B} \circ \mathcal{A} = \mathcal{B}[\mathcal{A}] = \sum_{n \geq 0} \langle \mathcal{B}_n \rangle \times \mathcal{A}^n;$$

where $\langle \mathcal{B}_n \rangle$ denotes the set of objects of size n of $\mathcal{B}$ whose size is turned to 0 (because we want the final size to be the sum of the sizes of the $\mathcal{A}$-elements, and not add again the size of the $\mathcal{B}$-element). Then, if $\mathcal{C} = \mathcal{B} \circ \mathcal{A}$ and $\mathcal{A}$ does not contain elements of size 0, $\mathcal{C}$ is a combinatorial class and

$$C(z) = B(A(z)).$$

1.2.2 Properties of series

The previous section highlights the importance that series will play in our work, which is why we introduce here a few properties that we will encounter. We describe them in terms of formal series (*i.e.* without asking questions about the convergence of the series), but the notions immediately extend to analytic functions.

Definition 1.3. A formal power series $F(x)$ is *algebraic* if it satisfies a polynomial equation, *i.e.* if there exists a non-zero polynomial $P(X, Y)$ in two variables:

$$P(F(x), x) = 0.$$

It is *D-finite* (or *holonomic*) if there exist $k \in \mathbb{N}_{>0}$ and $k+1$ polynomials $P_0(X), \dots, P_k(X)$ such that P_0 is non-zero and:

$$P_0(x)F^{(k)}(x) + \cdots + P_{k-2}(x)F'(x) + P_{k-1}(x)F(x) + P_k(x) = 0.$$

It is *D-algebraic* if there exist $k \in \mathbb{N}_{>0}$ and a non-zero polynomial P in $k+2$ variables such that:

$$P\left(F^{(k)}(x), \dots, F'(x), F(x), x\right) = 0.$$

In general (and in this thesis), the polynomials involved in the definition have rational coefficients. Moreover, algebraic power series are D-finite, and D-finite power series are D-algebraic. In Chapter 6, series that are dealt with are only D-finite while some others are not D-finite but are D-algebraic but most of the generating series we study in this thesis are algebraic and thus have many nice properties. Moreover, most of them will even satisfy a *Lagrangean parametrization*.

Definition 1.4. A power series $F(x)$ satisfy a *Lagrangean parametrization* if there exists a power series $\phi(y) = \sum_{k \in \mathbb{N}_0} \phi_k y^k$ such that $\phi_0 \neq 0$ and

$$F(x) = x\phi(F(x)).$$

When this is the case, the coefficients of F, its powers and its image by an arbitrary function can be computed using the Lagrange-Bürmann inversion formula.

Proposition 1.5 (Lagrange inversion theorem — Bürmann form). *If $F(x)$ satisfies a Lagrangean parametrization with the notation of Definition 1.4, then for H an arbitrary function*

$$[x^n]H(F(x)) = \frac{1}{n}[y^{n-1}]H'(y)\phi(y)^n.$$

To conclude this brief summary of power series techniques and properties, we describe *polynomial elimination*. This method allows, given two polynomials $P(X, Y)$ and $Q(X, Y)$ in two variables such that

$$P(F, G) = 0 \qquad \text{and} \qquad Q(G, H) = 0,$$

to obtain a new polynomial R satisfying

$$R(F, H) = 0.$$

This can be used for example to eliminate auxiliary quantities between two equations. This can be done algorithmically using *resultants* or *Gröbner bases*, but this is not the focus of this thesis. Instead, we will simply resort to the implementations of `Groebner:-Basis` and `resultant` of the `Maple` computer algebra software. For more details, we refer to [FS09, §B.1] and the references therein.

1.2.3 Singularity analysis

One key idea of analytic combinatorics is to look at generating series not only as formal power series but also as functions analytic at 0.

Definition 1.6. A function f defined over an open connected domain $\Omega \subseteq \mathbb{C}$ is *analytic at a point* $z_0 \in \Omega$ if there exists an open disc $D \subset \Omega$ centred at z_0 where f is representable by a convergent power series expansion, *i.e.* there exists a sequence $(c_n)_{n\in\mathbb{N}_0}$ such that for all $z \in D$, there

$$f(z) = \sum_{n\in\mathbb{N}_0} a_n (z - z_0)^n.$$

We say that f is *analytic on* Ω if it is analytic at every point of Ω.

This framework was pioneered by Flajolet and Odlyzko [FO90] and has opened the way for the use of complex analysis techniques such as singularity analysis to study combinatorial problems. More precisely, one of the principles of analytic combinatorics is to transform information on the asymptotic expansion of the coefficients using the singular expansion of the generating series near its *dominant singularities*. These "transforms" are called *transfer* rules. Before diving deeper into these notions, let us recall a few basic properties of complex analysis.

Analytic functions

First, the property of being analytic remains true under common mathematical operations. From the definition, we see that functions analytic at a point $z = z_0$ stay analytic under addition, multiplication, quotient (as long as the denominator is not zero), and differentiation. Moreover, they also remain analytic under composition: if $f(z)$ is analytic at $z = z_0$ and $g(w)$ is analytic at $f(z_0)$, then $g \circ f(z)$ is analytic at $z = z_0$. This is important because these operations appear when we apply automatic transformations to combinatorial class operations detailed in Section 1.2.1. In other words, if a class $\mathcal{C}$ is derived from a class $\mathcal{A}$ (and possibly a class $\mathcal{B}$) using any of the operations mentioned earlier, and $\mathcal{A}$ (and possibly $\mathcal{B}$) has an analytic generating series, then the generating series of $\mathcal{C}$ is also analytic.

On the other hand, another central element is the points at which a function ceases to be analytical, called *singularities*. More formally, a singularity of a function f is a point where it cannot be continued analytically, as expressed in the following definition.

Definition 1.7. Let f be an analytic function on an open disc D and z_0 on the boundary of D. We say that z_0 is a *regular point* of f when there is a disc D' centred at z_0 and

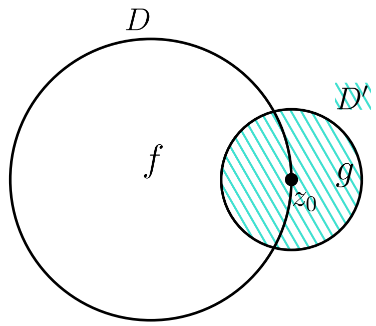

Figure 1.1: If $f = g$ on $D \cap D'$, then z_0 is a regular point of f.

a analytic function g on D' such that $f = g$ on $D \cap D'$ (see Fig. 1.1). Otherwise, it is a *singularity* of f.

To understand where singularities can occur, we define the disc and radius of convergence.

Definition 1.8. For a function f that is analytic at a point z_0, there is a disc of maximal radius (possibly infinite) such that the series representing $f(z)$ converges inside the disc. This disc is called the *disc of convergence* at z_0, and its radius is the *radius of convergence* of $f(z)$ at z_0.

If no z_0 is explicitly specified, it means z_0 is chosen to be equal to 0. Indeed, the origin plays a special role in combinatorics because that is where the expansion coefficients correspond to the series of interest. In particular, the radius of convergence of a function f at 0 is denoted by $\mathrm{RCV}(f)$.

By definition, a function analytic at 0 is analytic inside the interior of its disc of convergence, so it has no singularities within this disc. Moreover, it is a theorem that it must have at least one singularity on the boundary of the disc. Moreover, since the analytic functions used in combinatorics usually have coefficients that are whole numbers (because they count objects), they meet the conditions of Pringsheim's theorem, which makes finding singularities easier.

Proposition 1.9 (Pringsheim's theorem)**.** *If $f(z)$ is analytic at the origin and its series expansion in 0 has non-negative coefficients; then the point $z = \mathrm{RCV}(f)$ is a singularity of $f(z)$.*

In analytic combinatorics, we are interested in a particular type of singularity, *dominant* singularities.

Definition 1.10. A singularity ρ of a function f analytic at 0 is said *dominant* if no other singularity has smallest modulus: for all s singularity of the function, $|\rho| \leq |s|$.

Therefore, by Pringsheim's theorem, to find the radius of convergence of "combinatorial" generating functions, one only needs to check where the function stops being analytic along the positive real line.

Transfer rules

The main method of analytic combinatorics can be summarised as follows. Let $C(z) = \sum_{n\in\mathbb{N}_0} c_n z^n$ be a series whose coefficients are to be studied. Suppose that C admits a unique dominant singularity.

1. Locate the dominant singularity of $C(z)$;
2. Obtain the singular expansion of $C(z)$ in a neighbourhood of its dominant singularity;
3. Apply a transfer theorem to deduce the asymptotic expansion of c_n.

If there are more than one dominant singularity, simply apply the process for each dominant singularity; the asymptotic expansion of c_n is the sum of all the contributions of the singular expansions. This case often arises when the function is *periodic*.

Definition 1.11. Let $F(x) = \sum_{k\in\mathbb{N}_0} f_k x^k$ be a formal power series. Its period p is defined as

$$p = \gcd\{i - j \mid f_i \neq 0, f_j \neq 0\}.$$

If $p = 1$, the series is said *aperiodic*, otherwise it is *periodic of period* p.

This method is very robust and applies to a wide variety of functions. In particular, this method relies only on *local* properties of the functions at its singularities, which makes it still applicable for functions which are only known through functional equations.

For example, let us go back to the generating series $T(z)$ of binary trees which was shown in (1.1) to be equal to

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Therefore, the only singularity of T is $1/4$, and it is the only dominant singularity. Moreover, when z is in a neighbourhood of $1/4$,

$$T(z) = 2 - 2\sqrt{1 - 4z} + o(1 - 4z). \tag{1.3}$$

This is what is called *square-root type singularity*, and one can apply the following transfer rule. This requires a Δ-*analyticity* property, which is used to check that the error terms in the singular expansion of $f(z)$ do not behave too badly when coefficients are extracted. Fortunately, to do this, all one needs to do is check that the expansion is valid in a Δ-*domain*.

Definition 1.12. Let $R > 1$ (a radius), $\varphi \in (0, \pi/2)$ (an angle) and define the open domain $\Delta(R, \varphi)$ as

$$\Delta(R, \varphi) = \{z \mid |z| < R,\ z \neq 1,\ |\arg(z - 1)| > \varphi\}.$$

A domain is a Δ-*domain* at 1 if it can be written as a $\Delta(R, \varphi)$ (see Fig. 1.2). It is a Δ-domain at $\rho \neq 0$ if it is the image by the mapping $z \mapsto \rho z$ of a Δ-domain at 1.

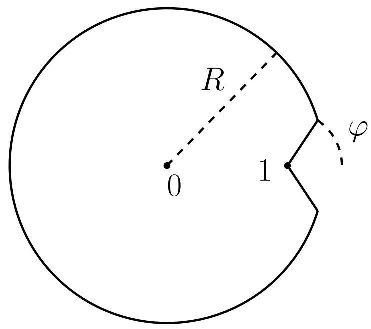

Figure 1.2: A Δ-domain at 1.

A function is Δ-analytic if it is analytic on some Δ-domain.

Proposition 1.13 (Transfer rule). *[FO90] Let $\rho \in \mathbb{C}^*$. Then, if the function f admits ρ as its unique dominant singularity, is analytic in a Δ-domain at ρ, and admits the following behaviour when z is in a neighbourhood of ρ:*

$$f(z) \underset{z \to \rho}{\sim} \left(1 - \frac{z}{\rho}\right)^{-\alpha}, \qquad \textit{for} \quad \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0};$$

it holds that

$$[z^n]f(z) \underset{n \to \infty}{\sim} \rho^{-n} \frac{n^{\alpha-1}}{\Gamma(\alpha)}$$

where Γ denotes Euler's Gamma function.

In the case of $T(z)$, the function is analytic on $\mathbb{C} \setminus [1/4, +\infty]$, so that the expression stated in (1.3) implies by the transfer rule:

$$[z^n]T(z) \underset{n \to \infty}{\sim} -2 \cdot \left(-\frac{(1/4)^{-n}\, n^{-3/2}}{2\sqrt{\pi}}\right) = \frac{4^n\, n^{-3/2}}{\sqrt{\pi}},$$

which is exactly what one obtains when using Stirling's result on the closed formula for $[z^n]T(z)$ in (1.2).

In fact, the transfer rules apply to a whole variety of behaviours, in particular to all functions whose singular behaviour, in a neighbourhood of a dominant singularity ρ, can be written as

$$\left(1 - \frac{z}{\rho}\right)^{-\alpha} \left(\ln \frac{1}{1 - \frac{z}{\rho}}\right)^{\beta}.$$

We refer to [FS09, Chapter 6] for extensive transfer rules.

1.2.4 Boltzmann generation

Let us now resume our discussion of the symbolic decomposition of combinatorial classes. We showed how they can be automatically translated into functional equations for the generating series. Now, we discuss how they can be translated into random generators for elements of the combinatorial class: Boltzmann random generators were introduced by Duchon, Flajolet, Louchard, and Schaeffer [DFLS03]. They can be obtained simply and systematically for combinatorial classes that are expressed using decompositions [FFP07, Fus09, BFKV11].

Definition 1.14. An *(ordinary) Boltzmann generator* ΓC for a combinatorial class $\mathcal{C}$ of generating series $C(t) = \sum_{n \in \mathbb{N}_0} c_n t^n$ is an algorithm which, given $t > 0$ such that $0 < C(t) < \infty$, draws an element $\mathfrak{c}$ from $\mathcal{C}$ according to the probability law

$$\mathbb{P}\left(\Gamma C(t) = \mathfrak{c}\right) = \frac{t^{|\mathfrak{c}|}}{C(t)},$$

where we recall that $|\mathfrak{c}|$ denotes the size of $\mathfrak{c}$.

Boltzmann generators allow to randomly generate a combinatorial object without fixing its size, and the probability to get an element of size n is

$$\mathbb{P}\left(|\Gamma C(t)| = n\right) = \frac{c_n t^n}{C(t)}. \tag{1.4}$$

Moreover, conditioning on a fixed size n, the generator ΓC outputs an element according to the uniform distribution over all the elements of $\mathcal{C}$ of size n. On average, the size of the object generated is:

$$\mathbb{E}\left[|\Gamma C(t)|\right] = \frac{tC'(t)}{C(t)}. \tag{1.5}$$

In the same way that decompositions of a combinatorial class $\mathcal{C}$ in terms of other classes $\mathcal{A}$ and $\mathcal{B}$ can be automatically translated into equations on generating series giving $C(z)$ in function of $A(z)$ and $B(z)$, they can also be automatically translated into a Boltzmann generator $\Gamma C(z)$ in terms of $\Gamma A(z)$ and $\Gamma B(z)$, thanks to the rules given below [DFLS03, Fus09].

We recall that a random variable X follows a *Bernoulli law* $\mathrm{Bern}(p)$ *with parameter* p if $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$; it follows a *geometric law* $\mathrm{Geom}(p)$ *with parameter* p if for $k \in \mathbb{N}_0$, $\mathbb{P}(X = k) = (1 - p)^{k-1} p$.

Algorithm 1 $\Gamma C(z)$ for the atomic class $\mathcal{C} = \mathcal{Z}$

```
return Z
```

Algorithm 2 $\Gamma C(z)$ for disjoint union $\mathcal{C} = \mathcal{A} \sqcup \mathcal{B}$

```
if Bern(A(z)/(A(z)+B(z))) then
    return ΓA(z)
else
    return ΓB(z)
end if
```

Algorithm 3 $\Gamma C(z)$ for Cartesian product $\mathcal{C} = \mathcal{A} \times \mathcal{B}$

```
return (ΓA(z), ΓB(z))
```

Algorithm 4 $\Gamma C(z)$ for the sequence construction $\mathcal{C} = \mathrm{Seq}(\mathcal{A})$

```
k = Geom(A(z))
return (ΓA(z))_{1≤i≤k}  // k independent calls
```

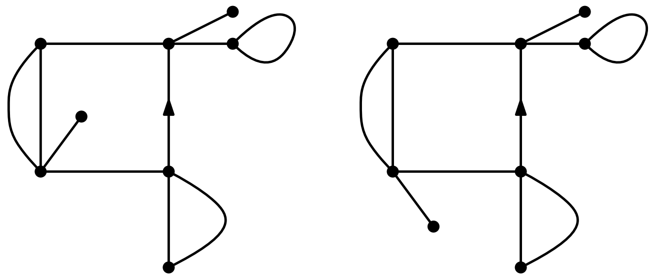

Figure 1.3: Two distinct (rooted planar) maps with the same underlying graph.

Algorithm 5 $\Gamma C(z)$ for substitution $\mathcal{C} = \mathcal{B}[\mathcal{A}]$

```
y = A(z)
γ = ΓB(y)
for i = 1 to |γ| do
    γ_i = ΓA(z)
end for
return (γ, (γ_i)_{1≤i≤|γ|})
```

This will allow us to generate large random maps under a block-weighted model.

1.3 Combinatorics on maps

As expressed in Definition 0.1, a planar map is the proper embedding of planar finite multigraph into the sphere, considered up to orientation-preserving homeomorphisms. Embedding a graph onto the sphere fixes the cyclical order of neighbours around each vertex, so maps have an additional structure compared with (non-embedded) graphs, and, in general, a planar graph has several embeddings, see Fig. 1.3. This section starts by vocabulary on maps. Then, we take a quick look at the usual enumerative methods used on maps.

1.3.1 Terminology on maps

For $\mathfrak{m}$ a planar map, let $V(\mathfrak{m})$ be the set of its vertices, $E(\mathfrak{m})$ the set of its edges and $F(\mathfrak{m})$ the set of its faces. The size of the map $\mathfrak{m}$ is generally defined as its number of edges (unless explicitly stated otherwise). In all cases, we use the notation $|\mathfrak{m}|$ for the size of $\mathfrak{m}$.

A *half-edge* e is an edge (possibly a loop) and an orientation on this edge (each edge yields two half-edges). The orientation allows to define the *starting vertex*, denoted by e^-, and the *end vertex*, denoted by e^+. The half-edge is represented as half of an edge starting from e^-. Let $\overrightarrow{E}(\mathfrak{m})$ be the set of half-edges of $\mathfrak{m}$.

A *corner* is the angular sector between two consecutive half-edges in anticlockwise order around a vertex. Each half-edge is canonically associated to the corner that follows it in anticlockwise order around its starting vertex. The *degree* of a face is the number of corners incident to it.

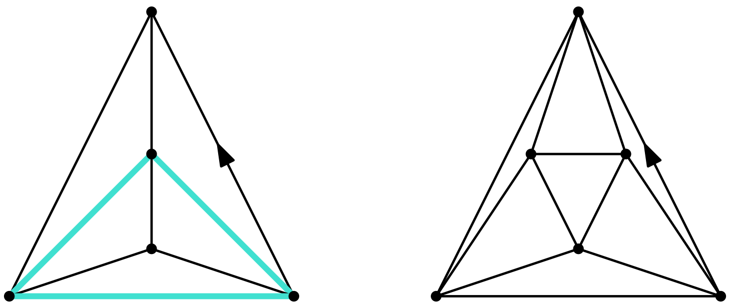

Figure 1.4: A triangulation not irreducible on the left (with the 3-cycle not bounding a face in blue), an irreducible triangulation on the right.

All the maps considered in this thesis are *rooted*, meaning that one of their half-edges (or, equivalently, one of their corners) is distinguished. Rooting simplifies the study by avoiding symmetry problems, however we expect our results on undecorated maps remain true in the non-rooted setting due to the general results of [RW95]. The face containing the corner associated to the root is called the *root face*. By convention, we set $m_0^\circ = 1$ which corresponds to the *vertex map* of $\mathcal{E}$: the map reduced to a single vertex. Similarly, define the *edge map* as the map reduced to a single edge between two vertices.

Finally, for Section 2.2.1, the following definitions are needed.

Definition 1.15. A map is called *loopless* if it has no loops. A map is called *simple* if it is loopless and has no multiple edges.

Definition 1.16. A *quadrangulation* is a map with all faces of degree 4. A *triangulation* is a map with all faces of degree 3.

Definition 1.17. A simple triangulation (resp. quadrangulation) is *irreducible* if every 3-cycle (resp. 4-cycle) bounds a face, see Fig. 1.4.

1.3.2 Tutte's equation for the enumerative study of generic maps

The enumerative study of rooted planar maps was initiated by Tutte in the 60's. Tutte's idea is to write the decomposition equation for maps, obtained from the deletion of their root edge. More precisely, fix $\mathfrak{m} \in \mathcal{M}^\circ \setminus \mathcal{E}$; and delete its root edge. Two possibilities can occur:

- There are two disjoint maps, so that the initial map can be described by a pair of maps and an edge (as in the case of binary plane trees), which corresponds to the combinatorial class $\mathcal{Z} \times \mathcal{M}^\circ \times \mathcal{M}^\circ$;
- We obtain a new planar map $\mathfrak{m}'$.

Unfortunately, the latter case cannot simply be expressed in terms of the class $\mathcal{M}^\circ$, since there is *a priori* no canonical way to add an edge to a map. Moreover, several elements

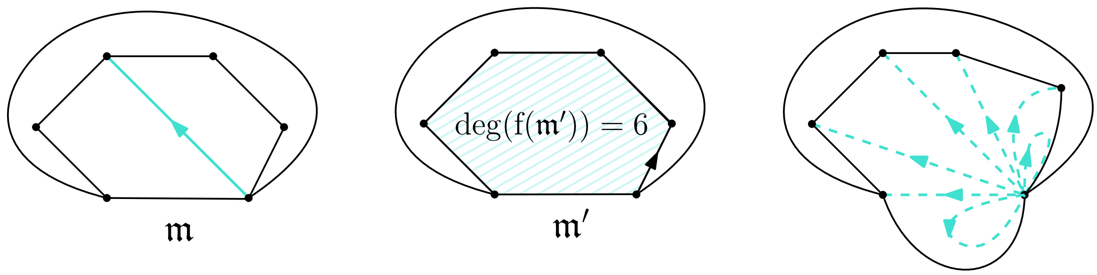

Figure 1.5: A map $\mathfrak{m}$ (left) whose root edge (blue) is deleted, creating the map $\mathfrak{m}'$ (middle), and on the right, the $\deg(\mathrm{f}(\mathfrak{m}'))+1$ preimages of $\mathfrak{m}'$ via Tutte's construction.

$\mathfrak{m} \in \mathcal{M}^\circ$ produce the same $\mathfrak{m}'$, and it is not difficult to convince oneself (see Fig. 1.5) that the number of preimages of $\mathfrak{m}'$ is equal to $\deg(\mathrm{f}(\mathfrak{m}'))+1$, where the root face of a map $\mathfrak{m}$ is denoted by $\mathrm{f}(\mathfrak{m})$. Therefore, we use a *catalytic variable* to refine our decomposition by keeping track of a new parameter, in this case the degree of the root face. More precisely, let

$$M_\circ(z,s) = \sum_{\mathfrak{m}\in\mathcal{M}^\circ} z^{|\mathfrak{m}|} s^{\deg(\mathrm{f}(\mathfrak{m}))}.$$

Notice that $M_\circ(z) = M_\circ(z,1)$. The easiest way to write the series equation is to attach a weight to a map, and consider the new weight that can be obtained when *adding* an edge to a map. When this is done on a map of weight $z^n s^k$, it creates a map of weight $z^n s^d$ where $d \in [\![1, k+1]\!]$. The generating function for the last case can therefore be written

$$\sum_{\substack{\mathfrak{m}\in\mathcal{M}^\circ\\ n=|\mathfrak{m}|\\ k=\deg(\mathrm{f}(\mathfrak{m}))}} z^{n+1}\left(s+\cdots+s^{k+1}\right) = sz \sum_{\substack{\mathfrak{m}\in\mathcal{M}^\circ\\ n=|\mathfrak{m}|\\ k=\deg(\mathrm{f}(\mathfrak{m}))}} z^n \frac{s^{k+1}-1}{s-1}.$$

Finally, this gives the following equation, referred to as *Tutte's equation*:

$$M_\circ(z,s) = 1 + zs^2 M_\circ(z,s)^2 + zs\frac{sM_\circ(z,s) - M_\circ(z,1)}{s-1}. \tag{1.6}$$

The goal is to obtain $M_\circ(z,1)$, but simply specialising s to 1 makes (1.6) degenerate. This equation is non-linear and involves the *discrete derivative* in 1 (which is the quotient term); such equation is usual of classes of maps. Tutte's original work consisted in guessing an expression for $M(z,s)$ based on its first coefficients, checking that it was a solution of (1.6), and then concluding by a uniqueness argument. Since then, the resolution of equations with one catalytic variable has been systematised, and an automatic general framework was obtained by Bousquet-Mélou and Jehanne to solve such polynomial equations [BMJ06]: informally the idea is to perform a substitution $s := S(z)$ in (1.6) for a well-chosen formal series in order to cancel out certain terms. We start by regrouping terms of (1.6):

$$zs^2(s-1)M_\circ(z,s)^2 + (zs^2 - s + 1)M_\circ(z,s) - zsM_\circ(z,1) + s - 1 = 0;$$

then differentiate with respect to the catalytic variable s:

$$\begin{aligned}&\left(s-1-zs^2-2zs^2(s-1)M_\circ(z,s)\right)\frac{\partial}{\partial s}M_\circ(z,s)\\&-zs(3s-2)M_\circ(z,s)^2+(1-2zs)M_\circ(z,s)+zM_\circ(z,1)-1=0.\end{aligned} \tag{1.7}$$

The terms that one wants to cancel is the factor of $\frac{\partial}{\partial s}M_\circ(z,s)$, called the *kernel*. Indeed, if $S(z)$ is such that

$$S(z)-1-zS(z)^2-2zS(z)^2(S(z)-1)M_\circ(z,S(z))=0, \tag{1.8}$$

then by (1.7), it also holds that

$$-zS(z)(3S(z)-2)M_\circ(z,S(z))^2+(1-2zS(z))M_\circ(z,S(z))+zM_\circ(z,1)-1=0. \tag{1.9}$$

Then, (1.6) where $s=S(z)$ and Equations (1.8) and (1.9) are three *polynomial* equations in the variables $S(z)$, $M_\circ(z,S(z))$ and $M_\circ(z,1)$. They can thus be solved by polynomial elimination techniques gives the following equation satisfied by $M_\circ(z,1)=M_\circ(z)$:

$$27z^3M_\circ(z)^3+z(1-45z)M_\circ(z)^2+(16z^2+17z-1)M_\circ(z)-16z+1=0. \tag{1.10}$$

We deduce that the generating series of maps is *algebraic*.

The polynomial equation (1.10) has three solutions, and only one of them is a generating series in z with non-negative coefficients, so one can deduce that:

$$M_\circ(z)=\frac{18z-1+(1-12z)^{3/2}}{54z^2}. \tag{1.11}$$

In particular, the first terms of the expansion of $M_\circ$ are

$$M_\circ(z)=1+2z+9z^2+54z^3+378z^4+2916z^5+\dots. \tag{1.12}$$

Using this explicit formula (1.11), one can immediately extract a closed formula for m_n°.

Proposition 1.18. *[Tut63] The number m_n° of maps of size $n\in\mathbb{N}_0$ is equal to*

$$m_n^\circ=\frac{2(2n)!3^n}{(n+2)!n!}\underset{n\to\infty}{\sim}\frac{2}{\sqrt{\pi}}12^n n^{-5/2}. \tag{1.13}$$

So that, in particular, writing $\rho_{M_\circ}$ for the radius of convergence of the series $M_\circ$, one has

$$\rho_{M_\circ}=\frac{1}{12}\qquad\text{and}\qquad M_\circ(\rho_{M_\circ})<\infty.$$

The fact that the formula obtained is so simple was the cause of much interest, and a bijective proof was subsequently established [CV81, Sch97, Sch98].

Tutte's method can (and have) be extended to other families of maps such as *triangula-*

tions [Tut62b] or more generally maps with face degree constraints [Gao93], bipartite maps, or even coloured maps. However, it becomes more intricate for families of maps which are not stable under root deletion, *e.g.* families of maps with higher connectivity constraints (even if is still doable in some cases [Bro63]) or higher girth[1] constraints (such as simplicity).

To study these other cases, Tutte developed the *core decomposition method*, which is central to the work in this thesis and is detailed in Section 2.1.1. In particular, Tutte used it to enumerate *2-connected maps*, defined later. As this work is essential for the rest of my thesis, I describe it in detail in Section 2.1.1).

[1]The *girth* of map is the length of its smallest cycle.

Chapter 2

Weighted block decompositions of families of planar maps

A key aspect of planar maps is that they can be decomposed, *e.g.* into components of higher connectivity degree. Work described in this thesis relies heavily on the decomposition of planar maps into smaller components, called "blocks". Such decompositions typically relate one family of planar maps to another and give an equation between their generating series. Understanding the behaviour of these components and their structural arrangement leads to numerous results about maps. This is not limited to the class of all planar maps: subfamilies of maps can also be expressed through such decomposition schemes.

Section 2.1 describes a first block decomposition: that of general planar maps into 2-connected components (Section 2.1.1). This serves as a guiding example throughout the thesis, and we will use it to detail our method. Reflecting its importance, throughout the manuscript, $\mathcal{M}^\circ$ (resp. $\mathcal{B}^\circ$) denotes the class of general (resp. 2-connected) planar maps, while $\mathcal{M}$ denotes any class of maps and $\mathcal{B}$ denotes the class of components of the decomposition scheme considered. We then introduce a Boltzmann weight for the number of blocks (Section 2.1.2), which allows to put into light a phase transition (Section 2.1.3). This model was first introduced in the context of statistical physics by Bonzom, Delepouve and Rivasseau [BDR15]. This decomposition of general planar maps into 2-connected components is intimately related to the decomposition of quadrangulations into simple components (Section 2.1.4).

Section 2.2 introduces various map decomposition schemes — initially listed by Banderier, Flajolet, Schaeffer and Soria [BFSS01] — for which we develop a systematic approach, using tools from both analytic combinatorics and probability theory. Formulating the generating series equations in Lagrangean form "naturally" reveals the underlying block tree structure (Section 2.2.1). Finally, we introduce the foundations for studying these families decomposed into blocks, either from a probabilistic perspective (Section 2.2.2) or a combinatorial one (Section 2.2.3).

Notation. Recall from Section 1.3.1 that $\mathcal{M}^\circ$ is the class of (rooted planar) maps; m_n° the number of maps of size $n \in \mathbb{N}_0$, *i.e.* with n edges, and $M_\circ(z) = \sum_{n\in\mathbb{N}_0} m_n^\circ z^n$ is the associated generating series.

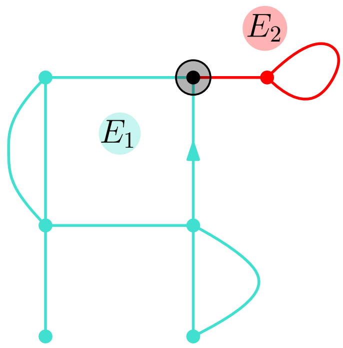

Figure 2.1: Example of a separable map. The circled black dot is a cut vertex.

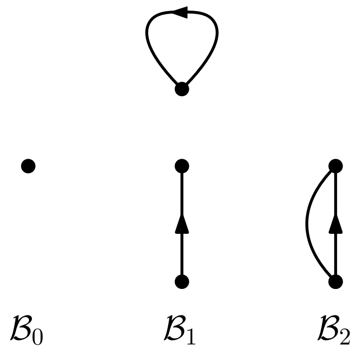

Figure 2.2: The sets $\mathcal{B}_0^\circ$, $\mathcal{B}_1^\circ$ and $\mathcal{B}_2^\circ$ of 2-connected maps with respectively 0, 1 and 2 edges.

2.1 A first block decomposition

2.1.1 Decomposition of maps into 2-connected blocks

We start by turning our attention to a first decomposition scheme: that of (general) maps into 2-connected components.

Definition 2.1. A map $\mathfrak{m} \in \mathcal{M}^\circ$ is said to be *separable* if it is possible to partition its edge-set $E(\mathfrak{m})$ into two non-empty sets E_1 and E_2 such that there is exactly one vertex (called a *cut vertex*) incident to both a member of E_1 and a member of E_2. The map $\mathfrak{m}$ is said to be 2-*connected* otherwise, see Fig. 2.1.

Note that, by definition, the vertex map is 2-connected. For $n \in \mathbb{N}_0$, we write $\mathcal{B}_n^\circ$ for the set of 2-connected maps of size n, and $b_n^\circ = |\mathcal{B}_n^\circ|$. From Fig. 2.2, we see that $b_0^\circ = 1$, $b_1^\circ = 2$ and $b_2^\circ = 1$. Notice in particular that the only 2-connected map with a loop is the map reduced to a loop-edge[1].

The decomposition of maps into 2-connected components was introduced by Tutte in the 60's and dubbed the "block decomposition of maps" [Tut63]. Roughly speaking, it corresponds to cutting the map at all cut-vertices, and is illustrated on Fig. 2.3 (this is known for graphs as well and called *block-cut tree*, see *e.g.* [Har69]). In Section 2.2, we extend this notion of "block decomposition" to other decomposition schemes, changing the family of maps which are decomposed and the family into which the decomposition is done.

Definition 2.2. A *block* of a planar map $\mathfrak{m}$ is a maximal 2-connected submap *of positive size*. The number of blocks of $\mathfrak{m}$ is denoted by $b(\mathfrak{m})$.

We describe here Tutte's decomposition drawing inspiration from Addario-Berry's presentation [AB19, §2]. Let $\mathfrak{m}$ be a map and let $\mathfrak{b}$ be the block containing its root. For each half-edge e of $\mathfrak{b}$, let c be the corner of $\mathfrak{b}$ to the left of e and define the *pendant submap* $\mathfrak{m}_e$ of e as the maximal submap of $\mathfrak{m}$ disjoint from $\mathfrak{b}$ except at e^- and located in the region of c (it might be reduced to the vertex map). If $\mathfrak{m}_e$ has at least one edge, we root it at the half-edge of $\mathfrak{m}$

[1] Contrary to Tutte [Tut63], we choose $m_0^\circ = b_0^\circ = 1$ (and not $m_0^\circ = b_0^\circ = 0$) and express the results accordingly.

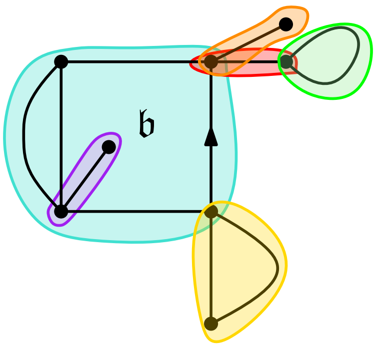

Figure 2.3: Decomposition of a map into blocks: the block $\mathfrak{b}$ containing the root is in blue.

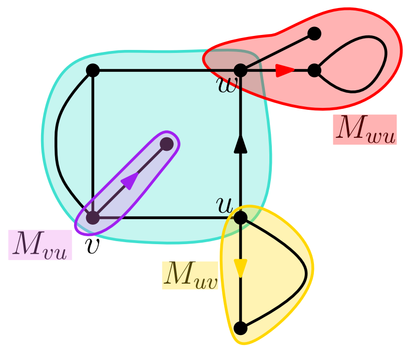

Figure 2.4: Pendant submaps: the block to which the half-edges uv, vu and wu belong is in blue; M_{uw} is the vertex map.

following e in anticlockwise order around e^- (see Fig. 2.4).

From $\mathfrak{b}$ and the $2|E(\mathfrak{b})|$ pendant submaps $\{\mathfrak{m}_e, e \in \overrightarrow{E}(\mathfrak{b})\}$, it is possible to reconstruct the map $\mathfrak{m}$: for each $\mathfrak{m}_e$ rooted at the half-edge ρ, insert $\mathfrak{m}_e$ in the corner associated to e in such a way that ρ is the first edge after e in anticlockwise order and merge ρ^- and e^-. Thus, a map can be encoded as a block where each edge is decorated by two maps. This decomposition induces an identity of generating series, thanks to the symbolic method (see Section 1.2.1):

Proposition 2.3. *[Tut63]*

$$M_\circ(z) = B_\circ(zM_\circ(z)^2). \tag{2.1}$$

This enabled Tutte to obtain the following enumerative results for 2-connected maps and their generating series $B_\circ(y) = \sum_{n\in\mathbb{N}_0} b_n^\circ y^n$. More precisely, Tutte guesses the following rational parametrization of $M_\circ(z)$ [Tut63, (5.2)][2]:

$$\eta = -3(1-\eta)^2 z, \qquad M_\circ(z) = \frac{1}{3}(1-\eta)(3+\eta)$$

and proves it using the Lagrange-Bürmann inversion formula (recalled in Proposition 1.5) and checking that it gives the coefficients m_n° of (1.13). Writing $y = zM_\circ^2(z)$, this gives

$$\eta = -y\frac{27}{(3+\eta)^2} \qquad \text{and} \qquad B_\circ(y) = M_\circ(z) = \frac{1}{3}(1-\eta)(3+\eta).$$

Then, polynomial elimination (introduced in Section 1.2.2) gives an algebraic equation for $B_\circ$

$$B_\circ^3(y) - B_\circ^2(y) - 18B_\circ(y)y + 27y^2 + 16y = 0$$

which can then be explicitly solved. Tutte does not use this method, but rather the Lagrange-Bürmann inversion formula which gives immediately the following.

[2] Up to presentation changes.

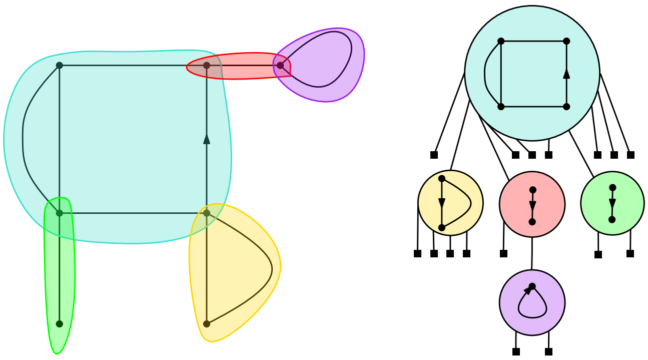

Figure 2.5: Block tree corresponding to a planar map.

Proposition 2.4. *[Tut63] The number b_n° of 2-connected maps of size n is*

$$b_0^\circ = 1, \qquad \text{and for } n \geq 1,\ b_n^\circ = \frac{2(3n-3)!}{n!(2n-1)!} \sim \sqrt{\frac{3}{\pi}}\frac{2}{27}\left(\frac{27}{4}\right)^n n^{-5/2}, \quad n \to \infty. \quad (2.2)$$

Moreover, writing $\rho_{B_\circ}$ for the radius of convergence of the series $B_\circ$, one has

$$\rho_{B_\circ} = \frac{4}{27}, \qquad B_\circ(\rho_{B_\circ}) = 4/3 \qquad \text{and} \qquad \rho_{B_\circ} \cdot B_\circ'(\rho_{B_\circ}) = \sum_{n \in \mathbb{N}_0} n b_n^\circ \rho_{B_\circ}^n = 4/9. \quad (2.3)$$

In particular, the first terms of the expansion of $B_\circ$ are

$$B_\circ(y) = 1 + 2y + y^2 + 2y^3 + 6y^4 + 22y^5 + 91y^6 + 408y^7 + \dots$$

The enumeration of 2-connected planar maps of size n can also be obtained bijectively through a bijection with so-called *left ternary trees* [LRP00, JS98].

Tutte's block decomposition can also be applied recursively, *i.e.* we consider first the root block and then apply the block decomposition to each of the pendant submaps. By doing so, for any map $\mathfrak{m}$ we can obtain a decomposition tree $T_\mathfrak{m}$, which was first explicitly described by Addario-Berry [AB19, §2]. More precisely:

1. Let $\mathfrak{b} = (\mathfrak{b}, \rho)$ be the maximal 2-connected submap containing the root ρ. The root v_ρ of $T_\mathfrak{m}$ represents $\mathfrak{b}$, and has $2|E(\mathfrak{b})|$ children (in particular, if $\mathfrak{b}$ is of size 0, v_ρ is a leaf);

2. List the half-edges of $\mathfrak{b}$ as $a_1, \dots, a_{2|E(\mathfrak{b})|}$ according to an arbitrarily fixed deterministic order on half-edges (*e.g.* the order in a left-to-right depth first search). Then, the subtree of $T_\mathfrak{m}$ attached to its i-th child is the tree encoding $T_{\mathfrak{m}_{a_i}}$.

An example of such a correspondence is described in Fig. 2.5. This decomposition has three essential properties, which stem directly from its definition and are summarized in the following proposition.

Proposition 2.5. *[Tut63, AB19] The block tree $T_\mathfrak{m}$ of a map $\mathfrak{m}$ satisfies the following properties:*

- *The edges of $T_{\mathfrak{m}}$ correspond to the half-edges of $\mathfrak{m}$;*
- *The internal nodes of $T_{\mathfrak{m}}$ correspond to the blocks of $\mathfrak{m}$: if an internal node v of $T_{\mathfrak{m}}$ has k children, then the corresponding block $\mathfrak{b}_v$ of $\mathfrak{m}$ has size $k/2$;*
- *The map $\mathfrak{m}$ is entirely determined by $(T_{\mathfrak{m}}, (\mathfrak{b}_v, v \in T_{\mathfrak{m}}))$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$ if v is an internal node; else, by convention, $\mathfrak{b}_v$ is the vertex map.*

By abuse of language, we might refer to $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ as the family of blocks (even if blocks necessarily have positive size). A direct consequence of this proposition is that to study the block sizes of a map $\mathfrak{m}$, it is sufficient to study the degree distribution of $T_{\mathfrak{m}}$. This is precisely the strategy developed by Addario-Berry in [AB19]. This allows him to study the block sizes of a uniform random map $\mathbf{M}_n$ of size n, by describing $T_{\mathbf{M}_n}$ as a Bienaymé–Galton–Watson tree with an explicit degree distribution conditioned to have $2n$ edges, and one of the contributions of this work is to extend Addario-Berry's result to a block-weighted model and to other decomposition schemes.

2.1.2 Weighting at blocks

In addition to decomposing maps into blocks, we want to introduce a weighting: a Boltzmann weight $u > 0$ assigned to each block in the map. In the following, we show that depending on the value of u, the behaviour changes, transitioning from a "generic map" phase to a tree-like phase; and, at the critical point, we highlight a new behaviour not previously known for maps.

Namely, we consider maps enumerated by both their number of edges and their number of blocks, and introduce the following bivariate series:

$$M_\circ(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}^\circ} z^{|\mathfrak{m}|} u^{b(\mathfrak{m})}, \tag{2.4}$$

where, as introduced before, $b(\mathfrak{m})$ is the number of blocks of $\mathfrak{m}$ (Definition 2.2) and $|\mathfrak{m}|$ is its number of edges (Section 1.3.1). Tutte's decomposition of a map into blocks translates into the following refined version of (2.1):

$$M_\circ(z,u) - 1 = u\left(B_\circ(zM_\circ(z,u)^2) - 1\right),$$

where both subtractions by 1 account for the fact that the vertex map has no block by Definition 2.2 (even if it is 2-connected). This equation can be rewritten

$$M_\circ(z,u) = uB_\circ(zM_\circ(z,u)^2) + 1 - u. \tag{2.5}$$

For $u \geq 0$, denote by $\rho_\circ(u) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ the radius of convergence of $z \mapsto M_\circ(z,u)$.

For $z \geq 0$ and $u \geq 1$, it holds that

$$M_\circ(z,u) \leq \sum_{\mathfrak{m}\in\mathcal{M}^\circ} z^{|\mathfrak{m}|}u^{|\mathfrak{m}|} = M_\circ(uz),$$

so if $|uz| < \rho_{M_\circ} = 1/12$, then $M_\circ(z,u)$ is a converging sum. Hence, for $u \geq 1$, $\rho_\circ(u) \geq \frac{1}{12u} > 0$. On the other hand, since $\rho_\circ(u)$ is decreasing, for $u \leq 1$ we have $\rho_\circ(u) \geq \rho_\circ(1) = \rho_{M_\circ} = 1/12 > 0$. Therefore, for all $u > 0$,

$$\rho_\circ(u) > 0.$$

In view of the form of the equation (2.5) and in particular that it is non-linear in $M_\circ(z,u)$, it holds that

$$M_\circ(\rho_\circ(u),u) < \infty.$$

Indeed, since $B_\circ(y) \geq 1 + 2y$ for all $y \geq 0$, we get $M_\circ(z,u) \geq 1 + 2uzM_\circ(z,u)^2$. This shows that it is impossible that $M(z,u) \xrightarrow[z\to\rho_\circ(u)^-]{} +\infty$.

2.1.3 Phase transition

We demonstrate now how this weighted decomposition can be used to reveal a phase transition. To do this, we introduce the following probability distributions. For $u > 0$, $n \in \mathbb{N}_0$ and $\mathfrak{m} \in \mathcal{M}^\circ$, set

$$\mathbb{P}_u(\mathfrak{m}) = \frac{\rho_\circ(u)^{|\mathfrak{m}|}u^{b(\mathfrak{m})}}{M_\circ(\rho_\circ(u),u)} \qquad \text{and} \qquad \mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}}{[z^n]M_\circ(z,u)}\mathbb{1}_{|\mathfrak{m}|=n}. \tag{2.6}$$

The question we ask is as follows: *for a fixed u, what is the behaviour of $\mathbf{M}$ drawn according to $\mathbb{P}_{n,u}$ as n tends to $+\infty$?* This model was first introduced by Bonzom, Delepouve and Rivasseau in the context of statistical physics [BDR15].

A crucial idea that allows to obtain a wide range of results is to study not directly the map $\mathbf{M}$ itself but to look at the corresponding block tree [Stu20a]. Indeed, many properties of the map are directly readable in the block tree, which is much simpler to study. In particular, it follows a *Bienaymé–Galton–Watson distribution* with a certain offspring distribution μ, *i.e.*, each node of the tree draws independently its number of children according to the law μ. This distribution is well-known and we can readily use results from the literature to obtain information on block trees. For μ a probability distribution on $\mathbb{N}_0$ and $n \in \mathbb{N}_0$, let $GW(\mu,n)$ denote the law of a Bienaymé–Galton–Watson tree with offspring distribution μ and conditioned to have n edges.

Following Addario-Berry's work [AB19], for $u > 0$ we aim at finding a measure μ^u such that the block tree $T_{\mathbf{M}_{n,u}}$ under $\mathbb{P}_{n,u}$ has law $GW(\mu^u, 2n)$. To that end, for any $y \in [0, \rho_{B_\circ}]$ we introduce the following probability distribution on $\mathbb{N}_0$:

$$\mu^{y,u}(2j) := \frac{b_j^\circ y^j u^{\mathbb{1}_{j\neq 0}}}{1 + u(B_\circ(y)-1)} \quad \text{and} \quad \mu^{y,u}(2j+1) := 0 \qquad \text{for all } j \in \mathbb{N}_0. \tag{2.7}$$

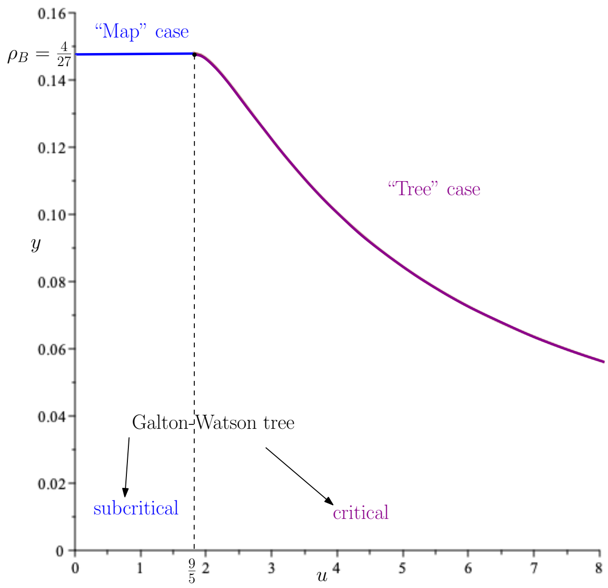

Figure 2.6: Plot of $y(u)$.

Moreover (see Remark 2.7 for a further discussion), we set

$$y(u) := \rho_\circ(u) M_\circ^2(\rho_\circ(u), u) \quad \text{and} \quad \mu^u := \mu^{y(u),u} \qquad \text{for any } u > 0, \tag{2.8}$$

where we recall that $\rho_\circ(u)$ is the radius of convergence of $z \mapsto M_\circ(z, u)$. On Fig. 2.6, the value of $y(u)$ is represented, using an explicit expression (see Remark 2.9). Notice that in view of (2.5), $y(u) \le \rho_{B_\circ}$ for all $u > 0$ and

$$1 + u(B_\circ(y(u)) - 1) = M_\circ(\rho(u), u). \tag{2.9}$$

Then, by (2.2), for all $u > 0$, we have:

$$\mu^u(\{2j\}) \sim \sqrt{\frac{3}{\pi}\frac{2}{27}\frac{u}{M_\circ(\rho(u), u)}} \left(\frac{27}{4} y(u)\right)^j j^{-5/2}, \quad \text{as } j \to \infty,$$

so that by setting

$$c(u) = \sqrt{\frac{3}{\pi}\frac{2}{27}\frac{u}{M_\circ(\rho(u), u)}}, \tag{2.10}$$

it holds that

$$\mu^u(\{2j\}) \sim c(u) \left(\frac{27}{4} y(u)\right)^j j^{-5/2}, \quad \text{as } j \to \infty. \tag{2.11}$$

The following proposition extends [AB19, Proposition 3.1] to our setting.

Proposition 2.6. *Let the random variable $\mathbf{M}$ follow the law $\mathbb{P}_u$, and denote by $(\mathbf{T}, (\mathbf{B}_v, v \in \mathbf{T}))$ the block tree of $\mathbf{M}$ where $(\mathbf{B}_v, v \in \mathbf{T})$ is its family $(\mathfrak{b}_v^{\mathbf{M}})_{v \in \mathbf{T}}$ of blocks. Then, for every $u > 0$, the law of the tree of blocks can be described as follows:*

- $\mathbf{T}$ *follows the law* $GW(\mu^u)$*;*
- *Conditionally given* $\mathbf{T} = \mathfrak{t}$*, the blocks* $(\mathbf{B}_v, v \in \mathfrak{t})$ *are independent random variables, and, for* $v \in \mathfrak{t}$*,* $\mathbf{B}_v$ *follows a uniform distribution on the set of blocks of size* $k_v(\mathfrak{t})/2$*, where* $k_v(\mathfrak{t})$ *is the number of children of* v *in* $\mathfrak{t}$*.*

For every $n \geq 1$*, the same statements hold under* $\mathbb{P}_{n,u}$*, only replacing* $GW(\mu^u)$ *with* $GW(\mu^u, 2n)$*.*

Proof. It suffices to prove the statement for $\mathbb{P}_u$ as, by Proposition 2.5, the block-tree of a map of size n has size $2n$.

Let $\mathfrak{t}$ be a tree where each vertex has an even number of children, and let $(\mathfrak{b}_v, v \in \mathfrak{t})$ be a family of 2-connected maps, with, for any $v \in \mathfrak{t}$, $2|\mathfrak{b}_v| = k_v(\mathfrak{t})$. Let $\mathfrak{m}$ be the map with block decomposition given by $(\mathfrak{t}, (\mathfrak{b}_v, v \in \mathfrak{t}))$.

Then, one has

$$\begin{aligned}
&\mathbb{P}_u\left(\mathbf{T} = \mathfrak{t}, \mathbf{B}_v = \mathfrak{b}_v \ \forall v \in \mathfrak{t}\right) = \mathbb{P}_u(\mathfrak{m}) \\
&= \frac{\rho(u)^{|\mathfrak{m}|} u^{b(\mathfrak{m})}}{M_\circ(\rho(u), u)} = \frac{\rho(u)^{\sum_{v\in\mathfrak{t}} k_v(\mathfrak{t})/2} u^{\sum_{v\in\mathfrak{t}} \mathbb{1}_{k_v(\mathfrak{t})\neq 0}}}{M_\circ(\rho(u), u)} \prod_{v\in\mathfrak{t}} \frac{b_{\frac{k_v(\mathfrak{t})}{2}}}{b_{\frac{k_v(\mathfrak{t})}{2}}} \\
&= \frac{1}{M_\circ(\rho(u), u)} \left(\frac{y(u)}{M_\circ^2(\rho(u), u)}\right)^{\sum_{v\in\mathfrak{t}} k_v(\mathfrak{t})/2} \prod_{v\in\mathfrak{t}} b_{\frac{k_v(\mathfrak{t})}{2}} u^{\mathbb{1}_{k_v(\mathfrak{t})\neq 0}} \times \prod_{v\in\mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}} \\
&= \frac{\prod_{v\in\mathfrak{t}} b_{\frac{k_v(\mathfrak{t})}{2}} y(u)^{k_v(\mathfrak{t})/2} u^{\mathbb{1}_{k_v(\mathfrak{t})\neq 0}}}{M_\circ(\rho(u), u)^{1+\sum_{v\in\mathfrak{t}} k_v(\mathfrak{t})}} \times \prod_{v\in\mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}} \\
&= \prod_{v\in\mathfrak{t}} \frac{b_{\frac{k_v(\mathfrak{t})}{2}} y(u)^{k_v(\mathfrak{t})/2} u^{\mathbb{1}_{k_v(\mathfrak{t})\neq 0}}}{M_\circ(\rho(u), u)} \times \prod_{v\in\mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}} \\
&= GW(\mu^u)(\mathfrak{t}) \times \prod_{v\in\mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}}.
\end{aligned}$$

This concludes the proof. □

It is well-known that the behaviour of μ-Bienaymé–Galton–Watson trees exhibits a phase transition depending on whether or not $\mathbb{E}[\mu] \leq 1$. Therefore, we compute:

$$\mathbb{E}\left[\mu^{y,u}\right] = \sum_{j\in\mathbb{N}_0} \frac{2j b_j^\circ y^j u^{\mathbb{1}_{j\neq 0}}}{1 + u(B_\circ(y) - 1)} = \frac{2uyB_\circ'(y)}{1 + u(B_\circ(y) - 1)}. \tag{2.12}$$

by (2.7). Then, it holds that

$$\mathbb{E}\left[\mu^{y,u}\right] = 1 \Leftrightarrow u = \frac{1}{2yB_\circ'(y) - B_\circ(y) + 1}. \tag{2.13}$$

The mapping $y \in (0, 4/27] \mapsto d(y) := 2yB_\circ'(y) - B_\circ(y) + 1$ is increasing. Indeed, for all

$y \in (0, 4/27]$,

$$d(y) = \sum_{n\geq 1} 2nb_n^\circ y^n - \sum_{n\geq 0} b_n^\circ y^n + 1 = \sum_{n\geq 1}(2n-1)b_n^\circ y^n.$$

Moreover, if follows from (2.3) that $d(0) = 0$ and $d(4/27) = 5/9$. So $1/d(y)$ maps bijectively $(0, 4/27]$ to $[9/5, +\infty)$. Therefore, if $u \in [9/5, +\infty)$, there exists a unique $y \in (0, 4/27]$ such that the law $\mu^{y,u}$, and, if $u \in (0, 9/5)$, such a value does not exist.

Using (2.9), it is shown in the proof of Theorem 2.8 that this value of y is exactly $y(u)$ defined in (2.8). Moreover, for fixed $u > 0$, the mapping $y \in [0, 4/27] \mapsto \mathbb{E}[\mu^{y,u}]$ is increasing [Jan12, Lemma 3.1] thus the maximal value is attained in ρ_B, which is the value of $y(u)$ for $u < 9/5$.

Remark 2.7. This highlights the reasons behind the choice of $y(u)$ in (2.8), which amounts to making the substitution of (2.5) *critical*. That way, when conditioning the trees to be of size $2n$, the conditioning is as little degenerate as possible. To achieve this, one wants the Bienaymé–Galton–Watson tree to be critical, and, if not possible, its expected size to be as high as possible. See [Jan12, §7] for a thorough discussion.

Indeed, when $u \geq 9/5$, $y(u)$ is the only $y \in [0, \rho_{B_\circ}]$ such that $\mathbb{E}[\mu^{y,u}] = 1$; when $u < 9/5$, there is no such y, and the value of $y(u)$ is the y that maximises $\mathbb{E}[\mu^{y,u}]$. These properties can also be used as a definition for $y(u)$.

Theorem 2.8. *Recall the definition of $c(u)$ given in (2.10). Then, depending on the value of u, the model $\mathbb{P}_u$ undergoes the following phase transition, driven by the properties of μ^u:*

Subcritical case. *For $u < u_C := 9/5$,*

$$E(u) := \mathbb{E}[\mu^u] = \frac{8u}{3(3+u)} < 1 \qquad \text{and} \qquad \mu^u(\{2j\}) \underset{j\to\infty}{\sim} c(u) j^{-5/2} \tag{2.14}$$

where $c(u) = \sqrt{\frac{3}{\pi}\frac{2u}{9(3+u)}}$. In this case, $y(u) = \rho_{B_\circ}$;

Critical case. *For $u = 9/5$,*

$$\mathbb{E}[\mu^{u_C}] = 1 \qquad \text{and} \qquad \mu^{u_C}(\{2j\}) \underset{j\to\infty}{\sim} \frac{1}{4\sqrt{3\pi}} j^{-5/2}.$$

It also holds that $y(u_C) = \rho_{B_\circ}$;

Supercritical case. *For $u > 9/5$,*

$$\mathbb{E}[\mu^u] = 1 \qquad \text{and} \qquad \mu^u(\{2j\}) \underset{j\to\infty}{\sim} c(u)\left(\frac{y(u)}{\rho_{B_\circ}}\right)^j j^{-5/2},$$

where $y(u) < \rho_{B_\circ}$ so that μ^u has exponential moments.

Notice that the case $u = 1$, which corresponds to uniform planar maps, as studied by Addario-Berry [AB19], falls in the subcritical regime.

Proof of Theorem 2.8. The only thing left is to explain how to obtain the values of $y(u)$, $\rho(u)$ and $M(\rho(u), u)$. For the sake of completeness, we recall an argument from [Bon16, §8.2.2]. We start from (2.5):

$$M_\circ(z,u) = uB_\circ(zM_\circ(z,u)^2) + 1 - u.$$

For a fixed $u > 0$, there are two possible sources of singularity:

1. The pair $(z_0 = \rho_\circ(u), m_0 = M_\circ(\rho_\circ(u), u))$ satisfies $\frac{\partial F}{\partial m}(z_0, m_0) = 0$ for $F : (z,m) \mapsto m - uB_\circ(zm^2) - 1 + u$, thus being a singularity by the contraposition of the implicit function theorem. In this case,

$$1 - 2\rho_\circ(u)M_\circ(\rho_\circ(u), u)uB_\circ'(\rho_\circ(u)M_\circ(\rho_\circ(u), u))^2 = 0,$$

so

$$2\rho_\circ(u)M_\circ(\rho_\circ(u), u)uB_\circ'(y(u)) = 1.$$

Then, by (2.9),

$$2y(u)B_\circ'(y(u)) - B_\circ(y(u)) + 1 = \frac{2\rho_\circ(u)M_\circ(\rho_\circ(u), u)^2}{2u\rho_\circ(u)M_\circ(\rho_\circ(u), u)} - \frac{M_\circ(\rho_\circ(u), u) + u - 1}{u} + 1 = \frac{1}{u}, \tag{2.15}$$

which is to say that $y(u) = \rho_\circ(u)M_\circ^2(\rho_\circ(u), u)$ satisfies (2.13). This is possible if and only if $u \geq 9/5$. Then, it follows that $\mathbb{E}[\mu^u] = 1$, and (2.11) gives the asymptotic behaviour of $\mu^u(2j)$.

2. A singularity of B is reached so $\rho_\circ(u)M_\circ^2(\rho_\circ(u), u) = \rho_{B_\circ} = 4/27$ *i.e.* $y(u) = 4/27$. Then, the value of $E(u)$ is obtained as an immediate consequence of Equations (2.3) and (2.12), and the asymptotic behaviour of $\mu^u(2j)$ comes from Equations (2.10) and (2.11). This happens if and only if $u \leq 9/5$.

Notice that at $u = u_C$, both types of singularity are reached. □

Remark 2.9. Using (2.13), we obtain an explicit expression for $y(u)$ in terms of u for $u \geq u_C$. By [Tut63], the series $B_\circ$ is algebraic and for all $y \in [0, 4/27]$,

$$B_\circ(y)^3 - B_\circ(y)^2 - 18yB_\circ(y) + 27y^2 + 16y = 0. \tag{2.16}$$

This gives an expression of $B_\circ'$ in terms of $B_\circ$, and polynomial elimination between this new equation and (2.13) allows to eliminate $B_\circ$. Initial conditions then give

$$u = \frac{1}{2yB_\circ'(y) - B_\circ(y) + 1} \Leftrightarrow y = \left(1 - \sqrt{1 - \frac{1}{u}}\right)\left(1 - \frac{1}{u}\right). \tag{2.17}$$

2.1.4 Extension to quadrangulations

In this section, we propose a first extension to another model: that of decomposing quadrangulations into simple blocks. Remember the definition of *quadrangulation* and *simple* from Definitions 1.15 and 1.16.

The class of all quadrangulations is denoted by $\mathcal{Q}$. Planar quadrangulations are *bipartite*, *i.e.* their vertices can be properly bicoloured in black and white. In the following, we always assume that they are endowed with the unique such colouring having a black root vertex. Observe that a quadrangulation has an even number of edges. Therefore, to avoid dealing with parity issues, we define the *size* of a quadrangulation as half its number of edges, or equivalently as its number of faces.

Definition 2.10. A *quadrangulation of the* 2*-gon* is a map where the root face has degree 2 and all other faces have degree 4.

A quadrangulation of the 2-gon with at least two faces can be identified with a quadrangulation (of the sphere) by simply gluing together both edges of the root face.

We start by describing how a quadrangulation can be decomposed into maximum simple quadrangular components, in the same way that a map can be decomposed into maximum 2-connected components. Next, we show that this decomposition is intimately linked to the decomposition of maps into 2-connected components and that the results previously obtained extend readily to the quadrangulation decomposition scheme.

To decompose a quadrangulation $\mathfrak{q}$, one can have a look at its 2-cycles, *i.e.* pairs $(e_1, e_2) \in \overrightarrow{E}(\mathfrak{q})^2$ of half-edges such that e_1, whose underlying edge is $f_1 \in E(\mathfrak{q})$, goes from $u \in V(\mathfrak{q})$ to $v \in V(\mathfrak{q})$ and e_2, whose underlying edge is $f_2 \neq f_1$, goes from v to u. Then, the notion of *interior* of a 2-cycle can be defined.

Definition 2.11. Let e_1e_2 be a 2-cycle of a quadrangulation $\mathfrak{q}$, its *interior* is the submap of $\mathfrak{q}$ between e_1 and e_2 (both included) which does not contain the root corner of $\mathfrak{q}$. A 2-cycle is *maximal* when it does not belong to the interior of another 2-cycle.

Let e_1e_2 be a maximal 2-cycle of a quadrangulation $\mathfrak{q}$, its *pendant subquadrangulation* is defined as its interior, which is turned into a quadrangulation of the 2-gon by rooting it at the corner incident to the unique black vertex of e_1e_2.

Let e be a half-edge of a quadrangulation $\mathfrak{q}$. If e is oriented from black to white and there exists a half-edge f such that ef is a maximal 2-cycle of $\mathfrak{q}$, then the *pendant subquadrangulation* of e is the pendant subquadrangulation of ef. Else, it is the edge map (which is also a quadrangulation of the 2-gon).

For $\mathfrak{q}$ a quadrangulation, its *simple core* $\mathfrak{q}_s$ — the simple block containing the root — is obtained by collapsing the interior of every maximal 2-cycle of $\mathfrak{q}$. Similarly as for maps, a *block tree* $T_{\mathfrak{q}}^{(q)}$ can be associated to a quadrangulation $\mathfrak{q}$, by recursively decomposing the pendant subquadrangulations at the simple core, see Fig. 2.7.

Simple blocks are recursively defined as the simple cores appearing in the underlying arborescent decomposition. Then, there is an exact parallel with the situation of maps and

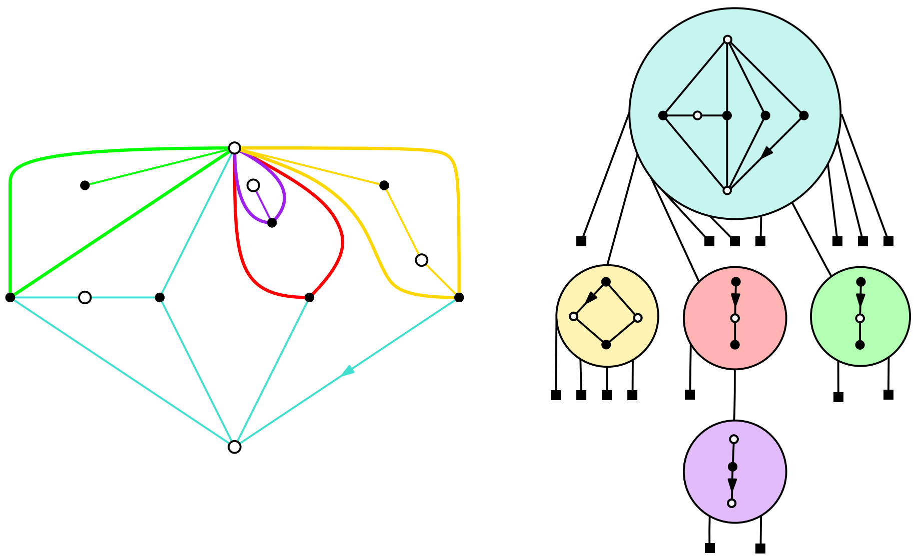

Figure 2.7: Quadrangulation associated to the map of Fig. 2.5 via Tutte's bijection, and its block tree.

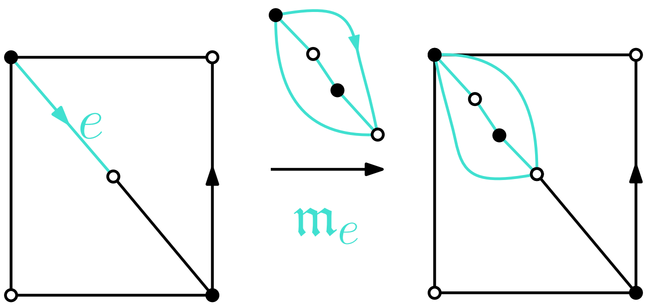

Figure 2.8: Reconstructing a quadrangulation from its simple core and the pendant subquadrangulations.

their 2-connected components.

Given a simple quadrangulation $\mathfrak{q}_s$ and a collection of $|E(\mathfrak{q}_s)| = 2|q_s|$ quadrangulations of the 2-gon $\{\mathfrak{m}_e, e \in E(\mathfrak{q}_s)\}$, it is possible to construct a quadrangulation: for each $\mathfrak{m}_e$ of root ρ_e replace e by $\mathfrak{m}_e$ such that ρ_e has the orientation of e. See Fig. 2.8 for an illustration. This transformation is invertible. Thus, a quadrangulation can be encoded as a simple quadrangulation where each edge is decorated by one quadrangulation of the 2-gon, *i.e.* each face is decorated by two quadrangulations of the 2-gon:

$$Q(z,u) + 1 = uS(z(Q(z,u)+1)^2) + 1 - u, \tag{2.18}$$

where Q is the generating series for quadrangulations (with a weight z for faces, and u for simple blocks) and S is the generating series for simple quadrangulations (with a weight z for faces, and by convention $S(0) = 1$). Note that this equation is isomorphic to (2.5).

This decomposition and the former one presented for general maps are in fact two sides of the same coin. Indeed, they can be related via Tutte's bijection as we now present: there exists an explicit bijective correspondence φ between quadrangulations of size n and (general)

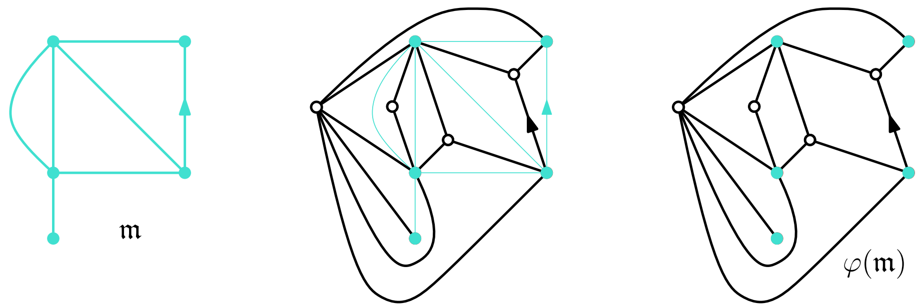

Figure 2.9: A map $\mathfrak{m}$ and its associated quadrangulation $\varphi(\mathfrak{m})$ via Tutte's bijection.

maps of size n. More precisely, for a map $\mathfrak{m}$ (rooted in ρ), its image by φ, called its *angular map*, can be constructed as follows, see Fig. 2.9.

1. Add a (white) vertex inside each face of $\mathfrak{m}$ and draw an edge from this new (white) vertex to each corner around the face (respecting the order of the corners);

2. The half-edge e created in the corner of ρ is now the root, oriented from black to white;

3. Remove the original edges.

Proposition 2.12. *[Tut63, Bro65] For $n \in \mathbb{N}_{>0}$, the function φ is a bijection between maps of size n and quadrangulations of size n. Moreover, it maps bijectively 2-connected maps of size n to simple quadrangulations of size n.*

The construction φ is due to Tutte [Tut63, §5] (he defines the notion of derived map, from which the angular map is extracted by deleting one of the 3 classes of vertices, as explained in [Bro65, §7]). The specialization to 2-connected maps is explained *e.g.* by Brown [Bro65]. In particular, it implies that $S(y) = B_\circ(y)$. Moreover, given Equations (2.5) and (2.18), this gives $M_\circ(z,u) = Q(z,u) + 1$.

Finally, when constructing the decomposition tree $T_{\mathfrak{q}}^{(q)}$ of a quadrangulation $\mathfrak{q}$, if the deterministic orders used for the half-edges of 2-connected maps and for the edges of simple quadrangulations are consistent via Tutte's bijection, then the decomposition trees of $\mathfrak{m}$ and of $\varphi(\mathfrak{m})$ are the same, and for each node v of the tree, the 2-connected map (resp. simple quadrangulation) at v are in correspondence by Tutte's bijection, *e.g.* the example of Fig. 2.5 is consistent with the example of Fig. 2.7 via Tutte's bijection. This can be rephrased as the following result.

Proposition 2.13. *For all $\mathfrak{m} \in \mathcal{M}^\circ$,*

$$T_{\varphi(\mathfrak{m})}^{(q)} = T_{\mathfrak{m}}$$

and, for all $v \in T_{\varphi(\mathfrak{m})}^{(q)}$,

$$\mathfrak{b}_v^{(q)} = \varphi(\mathfrak{b}_v).$$

Recall the model defined in (2.6) for general maps. We now define its analogue on quadrangulations. To that end, we set for all $u > 0$, $n \in \mathbb{N}_0$ and $\mathfrak{q} \in \mathcal{Q}$,

$$\mathbb{P}_u^{\text{quad}}(\mathfrak{q}) = \frac{\rho_\circ(u)^{|\mathfrak{q}|} u^{b(\mathfrak{q})}}{Q(\rho_\circ(u), u)} \quad \text{and} \quad \mathbb{P}_{n,u}^{\text{quad}}(\mathfrak{q}) = \frac{u^{b(\mathfrak{q})}}{[z^n]Q(z,u)} \mathbb{1}_{|\mathfrak{q}|=n}, \tag{2.19}$$

where, as introduced before, $b(\mathfrak{q})$ is the number of blocks of $\mathfrak{q}$ (Definition 2.2) and $Q(\rho_\circ(u), u) = M_\circ(\rho_\circ(u), u) - 1 < \infty$. By Proposition 2.13, one has:

Proposition 2.14. *For all $\mathfrak{q} \in \mathcal{Q}$ and $n \in \mathbb{N}_{>0}$,*

$$\mathbb{P}_{n,u}^{\text{quad}}(\mathfrak{q}) = \mathbb{P}_{n,u}\left(\varphi^{-1}(\mathfrak{q})\right) \quad \textit{and} \quad \mathbb{P}_u^{\text{quad}}(\mathfrak{q}) = \mathbb{P}_u\left(\varphi^{-1}(\mathfrak{q})\right),$$

so, denoting by $*$ the pushforward, for all $n \in \mathbb{N}_{>0}$,

$$\mathbb{P}_{n,u}^{\text{quad}} = \varphi_* \mathbb{P}_{n,u} \quad \textit{and} \quad \mathbb{P}_u^{\text{quad}} = \varphi_* \mathbb{P}_u.$$

In this way, if $\mathbf{M}$ follows $\mathbb{P}_u$ (resp. $\mathbb{P}_{n,u}$), by Proposition 2.14, $\mathbf{Q} = \varphi(\mathbf{M})$ has law $\mathbb{P}_u^{\text{quad}}$, (resp. $\mathbb{P}_{n,u}^{\text{quad}}$) and the block tree $\mathbf{T}$ associated to $\mathbf{M}$ is also associated to $\mathbf{Q}$ by Proposition 2.13). Therefore, all results of Section 2.1.3, including Proposition 2.6 and Theorem 2.8 apply to the weighted decomposition of quadrangulations into simple blocks as well: everything that has been said about the size of blocks (in the sense of maximum 2-connected components) can also be said about the maximum simple quadrangular components of quadrangulations; and likewise in everything that follows.

2.2 General framework for decomposition schemes

In the 60's, Tutte studied different families of maps and their relations, decomposing maps into 2-connected components and simple triangulations into irreducible components [Tut62b, Tut63]. In a seminal work [BFSS01], Banderier, Flajolet, Schaeffer and Soria have listed several decompositions *à la Tutte* and developed a uniform treatment via analytic combinatorics; focusing in particular on the size of the largest block, with the aim of achieving efficient random generation (see *e.g.* [Sch99] for the link between decompositions and random generation). In this section, we show that the block decomposition for maps presented in the previous sections can be extended to most of their models in a unified Lagrangean setting, by slightly rearranging some of the equations [BFSS01] (Section 2.2.1, see Section 7.1.1 for a discussion of the cases not covered here).

Here is formalism for the decompositions we consider. Planar maps (or quadrangulations, triangulations...) can be decomposed into loopless (or 2-connected, or simple, or 2-connected simple...) components, which are the so-called "blocks". We consider several models here (all relevant definitions can be found in Section 1.3.1):

1. *General maps decomposed into 2-connected blocks, or, equivalently, quadrangulations decomposed into simple blocks* (which is the main example used);

Scheme	maps, $\widehat{M}(\hat{z})$	blocks, $B(y)$	submaps, $H(\hat{z})$
1	all, $M_1(\hat{z}) = M_\circ(\hat{z}) - 1$	2-connected, $M_4(y) = B_\circ(y) - 1$	$\hat{z}(1 + \widehat{M}(\hat{z}))^2$
2	loopless, $M_2(\hat{z})$	simple, $M_3(y)$	$\hat{z}(1 + \widehat{M}(\hat{z}))$
3	2-connected $M_4(\hat{z}) - \hat{z}$	2-connected simple, $M_5(y)$	$\hat{z}(1 + \widehat{M}(\hat{z}))$
4	bipartite, $B_1(\hat{z})$	bipartite simple, $B_2(y)$	$\hat{z}(1 + \widehat{M}(\hat{z}))$
5	bipartite, $B_1(\hat{z})$	bipartite 2-connected, $B_4(y)$	$\hat{z}(1 + \widehat{M}(\hat{z}))^2$
6	bipartite 2-connected, $B_4(\hat{z})$	bipartite 2-connected simple $B_5(y)$	$\hat{z}(1 + \widehat{M}(\hat{z}))$
7	loopless triangulations, $T_1(\hat{z})$	simple triangulations, $y + yT_2(y)$	$\hat{z}(1 + \widehat{M}(\hat{z}))^3$
8	simple triangulations, $T_2(\hat{z})$	irreducible triangulations, $T_3(y)$	$\hat{z}(1 + \widehat{M}(\hat{z}))^2$

Table 2.1: Partial reproduction of [BFSS01, Table 3], which describes composition schemes of the form $\mathcal{M} = \mathcal{B} \circ \mathcal{H}$ except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times \mathcal{B} \circ \mathcal{H}$. The terminology and notation were slightly changed, but, for all i the series M_i, B_i and T_i are as defined in [BFSS01]. Moreover, for all i, $[\hat{z}^n]M_i(\hat{z})$ and $[\hat{z}^n]B_i(\hat{z})$ are the number of such maps with n edges, and $[\hat{z}^n]T_1(\hat{z})$ (resp. $[\hat{z}^n]T_2(\hat{z})$, $[\hat{z}^n]T_3(\hat{z})$) is the number of loopless (resp. simple, irreducible) triangulations with $n + 2$ (resp. $n + 3$) vertices.

2. Loopless maps decomposed into simple blocks;
3. 2-connected maps decomposed into 2-connected simple blocks;
4. Bipartite maps decomposed into bipartite simple blocks;
5. Bipartite maps decomposed into bipartite 2-connected blocks;
6. Bipartite 2-connected maps decomposed into bipartite 2-connected simple blocks;
7. Loopless triangulations decomposed into triangular simple blocks;
8. Simple triangulations decomposed into triangular irreducible blocks.

In a decomposition scheme, we let $\widehat{M}(\hat{z}) = \sum_{n\in\mathbb{N}_{>0}} m_n \hat{z}^n$ be the generating series of the class $\mathcal{M}$ of maps to be decomposed and similarly $B(y) = \sum_{n\in\mathbb{N}_{>0}} b_n y^n$ be the generating series of the class $\mathcal{B}$ of "blocks" into which the maps are decomposed. The coefficients m_n and b_n are the numbers of such maps of size n, where the size is as usual the number of edges except for

- Loopless triangulations where it is the number of vertices minus 2;
- Simple and irreducible triangulations where it is the number of vertices minus 3.

As for quadrangulations, this tweaking is made to avoid parity problems, see Table 2.1. In each case, the combinatorial properties of the models give an equation for the generating series:

$$\widehat{M}(\hat{z}) = B(H(\hat{z})) \qquad \text{or, for the last scheme,} \qquad \widehat{M}(\hat{z}) = (1 + \widehat{M}(\hat{z}))B(H(\hat{z}))$$

with H a series closely related to $\widehat{M}$: for scheme considered, there exists $d \in \mathbb{N}_{>0}$ such that it can be written

$$H(\hat{z}) = \hat{z}(1 + \widehat{M}(\hat{z}))^d. \tag{2.20}$$

The values of d can be found in Table 2.1.

Considering these equations, it is easy to add a weight u for blocks. Setting $\widehat{M}(\hat{z}, u) = \sum_{\mathfrak{m}\in\mathcal{M}} \hat{z}^{|\mathfrak{m}|} u^{b(\mathfrak{m})}$, where $b(\mathfrak{m})$ is the number of blocks of positive size in $\mathfrak{m}$, all the models we consider satisfy

$$\widehat{M}(\hat{z}, u) = uB(H(\hat{z}, u)), \tag{2.21}$$

or, for the last one,

$$\widehat{M}(\hat{z}, u) = (1 + \widehat{M}(\hat{z}, u)) \times uB(H(\hat{z}, u)); \tag{2.22}$$

where $H(\hat{z}, u) = \hat{z}(1 + \widehat{M}(\hat{z}, u))^d$.

For example, for the decomposition of general maps into 2-connected ones (which is the case studied in Section 2.1[3]), one has

$$\widehat{M}(\hat{z}, u) = uB(\hat{z}(1 + \widehat{M}(\hat{z}, u))^2).$$

For $u > 0$, denote by $\rho(u)$ the radius of convergence of $\hat{z} \mapsto M(\hat{z}, u)$. In view of the form of Equations (2.21) and (2.22) and in particular that they are non-linear in $\widehat{M}(\hat{z}, u)$, it holds that $\widehat{M}(\rho(u), u) < \infty$.

2.2.1 Block trees of a decomposition scheme

An underlying tree structure can be associated to each of the models we consider. As a first step, the decomposition equations are rewritten in the standard Lagrangean form $M(z) = z \times \Phi(M(z))$ for some function Φ. Taking the weight u into account, one wants a bivariate formal power series Φ with non-negative coefficients such that

$$M(z, u) = z \times \Phi(M(z, u), u).$$

Beware that Equations (2.21) and (2.22) are not of this form as the products by $\hat{z}$ are inside H.

Even if the size of a map is the number of edges (or other elements), blocks are not necessarily substituted on edges. It is natural to count the elements that are going to be substituted, and to do this, for any map $\mathfrak{m}$, we introduce a new size parameter $\mathrm{s}(\mathfrak{m})$, which is d times its (original) size plus one, *i.e.*,

$$\mathrm{s}(\mathfrak{m}) = d|\mathfrak{m}| + 1,$$

where d is defined in (2.20). Indeed, having $H(\hat{z}) = \hat{z}(1 + \widehat{M}(\hat{z}, u))^d$ means that each edge is transformed into an edge with d (potentially empty) elements $\mathfrak{m}$ attached. The additional element can be seen as dangling to the immediate left of the root, and serves as a beacon to

[3] Contrary to what was done in Section 2.1, here the vertex map is not counted in the class of maps nor in the class of 2-connected maps. This change makes the parallel with the other equations more obvious.

know how to attach the blocks when the substitution is done. Then, one can set

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{\mathrm{s}(\mathfrak{m})} u^{b(\mathfrak{m})}. \tag{2.23}$$

Then, it follows immediately that, for each decomposition scheme,

$$M(z,u) = z(1+\widehat{M}(z^d,u))$$

Therefore, each element counted by the size correspond to attaching one new structure. For the case of general maps decomposed into 2-connected blocks, one gets

$$M(z,u) = zM_\circ(z^2,u) = z + 2uz^3 + (8u^2+u)z^5 + (40u^3+12u^2+2u)z^7 + \ldots,$$

where $M_\circ$ is the series defined in (2.4). The parameter z counts the number of half-edges plus one, because, as explained in Section 2.1, the substitution corresponds to the pendant submap of each half-edge.

For all schemes except the last one, it now holds that, by (2.21),

$$M(z,u) = z(1+\widehat{M}(z^d,u)) = z\left(1+uB\left(z^d(1+\widehat{M}(z^d,u))^d\right)\right) = z(1+uB(M(z,u)^d)),$$

and for the last one, using (2.22),

$$\begin{aligned} M(z,u) &= z(1+\widehat{M}(z^d,u)) = z\left(1+u(1+\widehat{M}(z^d,u))B(z^d(1+\widehat{M}(z,u))^d)\right) \\ &= z + uM(z,u)B(M(z,u)^d). \end{aligned}$$

This gives the following result, and, from now on, we only consider $M(z,u)$.

Theorem 2.15. *For all decompositions models listed in Table 2.1, consider the generating series M defined from $\widehat{M}$ by*

$$M(z,u) = z(1+\widehat{M}(z^d,u)) \tag{2.24}$$

where d is defined in (2.20). Then, letting Φ be the generating function

$$\Phi(x,u) = 1 + uB(x^d) \qquad \text{or, for the last scheme,} \quad \Phi(x,u) = \frac{1}{1-uB(x^d)}, \tag{2.25}$$

it holds that

$$M(z,u) = z \times \Phi(M(z,u),u). \tag{2.26}$$

The radius of convergence $\rho_\Phi(u)$ of $x \mapsto \Phi(x,u)$ does not depend on u and is equal to $\rho_B^{1/d}$ for all schemes except the last, where it is equal to $\sup\{x \in [0,\rho_B^{1/d}] \mid uB(x^d) < 1\}$. In this last case, $\rho_\Phi(u) = \rho_B^{1/d}$ if $u \leq 1/B(\rho_B) = 32/5$. Furthermore, the series B is finite at its radius of convergence, therefore it is the case for the series Φ as well (except for the last

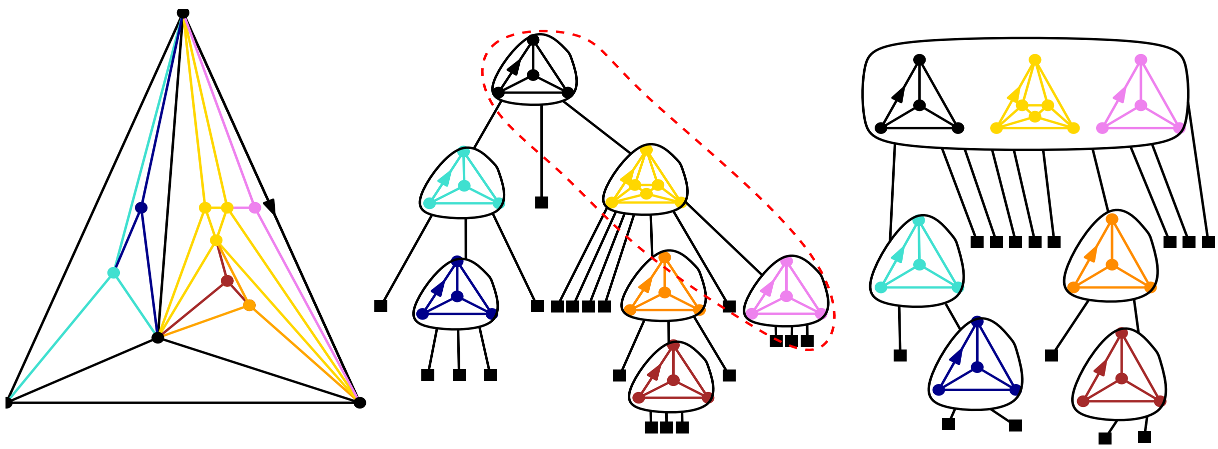

Figure 2.10: A simple triangulation, its tree of (irreducible) blocks where each node corresponds to a block, the adapted tree we consider here where blocks are grouped into sequences.

scheme when $u > 32/5$).

The Lagrangean form is intricately linked to trees (see *e.g.* [FS09, §I.5.1]). Thus, Theorem 2.15 sheds light to an underlying tree structure in each of the decomposition schemes. We built an *ad hoc* block tree for the decomposition of general maps into 2-connected blocks in Section 2.1.1, and, for each decomposition scheme, we can now systematically define the *block tree* $T_\mathfrak{m}$ of a map $\mathfrak{m}$ as the tree associated to the the decomposition of $\mathfrak{m}$. More precisely, each node of the tree is decorated by an object φ counted by Φ, with

$$\varphi \in \mathcal{E} + u\mathcal{B} \qquad \text{or, for the last scheme,} \quad \varphi \in \mathcal{E} + u\mathrm{Seq}_{\geq 1}(\mathcal{B})$$

where $\mathcal{E}$ denotes the combinatorial class constituted by one element of size 0 and the size of a sequence of objects is the sum of the sizes of its objects (remember that there are no objects of size 0 in $\mathcal{B}$). The first case is illustrated in Fig. 2.5, the second one in Fig. 2.10. The subtrees hanging at a node of $T_\mathfrak{m}$ (corresponding to some φ) are the trees of the components substituted into φ: for each edge (or the corresponding size parameter) of φ, the node has d children corresponding to the elements substituted. Then, one can write a generalised form of Proposition 2.5, where we extend the size notion to finite sequences of maps $\mathfrak{s} = (\mathfrak{m}_1, \dots, \mathfrak{m}_p)$ with

$$\mathrm{s}(\mathfrak{s}) = \sum_{i=1}^{p} \mathrm{s}(\mathfrak{m}_i).$$

Proposition 2.16. *For a decomposition scheme of Table 2.1 expressed as in Theorem 2.15, the block tree $T_\mathfrak{m}$ of a map $\mathfrak{m} \in \mathcal{M}$ satisfies the following properties:*

- *The tree $T_\mathfrak{m}$ has $\mathrm{s}(\mathfrak{m}) - 1 = d|\mathfrak{m}|$ edges and $\mathrm{s}(\mathfrak{m})$ vertices;*
- *An internal node v of $T_\mathfrak{m}$ corresponds to an object φ_v, which is either a block, or, in the last case, a sequence of blocks. The node v has $\mathrm{s}(\varphi_v) - 1 = d|\varphi_v|$ children;*

- *The map $\mathfrak{m}$ is entirely determined by $(T_{\mathfrak{m}}, (\varphi_v, v \in T_{\mathfrak{m}}))$ where φ_v is defined just above if v is an internal node; else, by convention, φ_v is the vertex map (for the last scheme, the empty list).*

In the last case, the number of blocks in the sequence corresponding to v is denoted by $\mathrm{el}(v)$. This definition is extended to all other cases where we set $\mathrm{el}(v) = 1$.

2.2.2 Probabilistic framework

As in Section 2.1, we introduce probability distributions to study the block trees and therefore the maps. For $u \in \mathbb{R}_{>0}$, $n \in \mathbb{N}_{>0}$ and $\mathfrak{m} \in \mathcal{M}$, we set

$$\mathbb{P}_u(\mathfrak{m}) = \frac{\rho(u)^{\mathrm{s}(\mathfrak{m})} u^{b(\mathfrak{m})}}{M(\rho(u), u)} \quad \text{and} \quad \mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}}{[z^n]M(z,u)} \mathbb{1}_{\mathrm{s}(\mathfrak{m})=n} \tag{2.27}$$

where $\rho(u)$ is the radius of convergence of $z \mapsto M(z,u)$. Recall the definition of Φ given in (2.25); then, for all $u > 0$ and $x \in [0, \rho_\Phi(u)]$, we introduce the following probability distribution: for all $j \in \mathbb{N}_0$,

$$\mu^{x,u}(j) = \frac{[X^j]\Phi(X,u)\, x^j}{\Phi(x,u)}; \tag{2.28}$$

recalling that $\Phi(\rho_\Phi(u), u) < \infty$. Then, for all decomposition schemes but the last, it holds that

$$\mu^{x,u}(d\,j) = \frac{\mathbb{1}_{j=0} + \mathbb{1}_{j\neq 0} u b_j x^{dj}}{1 + uB(x^d)}. \tag{2.29}$$

Since for all $j \in \mathbb{N}_0$ such that $d \nmid j$, it holds that $[X^j]\Phi(X,u) = 0$ then, for these values of j, it holds that $\mu^{x,u}(j) = 0$. Moreover (see Remark 2.7 for a discussion, which applies to all decomposition schemes), we set for all $u > 0$:

$$x(u) = M(\rho(u), u) \quad \text{and} \quad \mu^u := \mu^{x(u),u}.$$

By (2.26), it holds that

$$x(u) \leq \rho_\Phi(u) \quad \text{and} \quad \Phi(x(u), u) = \frac{M(\rho(u), u)}{\rho(u)} = \frac{x(u)}{\rho(u)}. \tag{2.30}$$

Now that the probabilistic context is set, one can state a generalisation of Proposition 2.6. Recall the definition of $\mathrm{el}(.)$ after Proposition 2.16.

Proposition 2.17. *Fix $u > 0$, let $\mathbf{M}$ be distributed according to $\mathbb{P}_u$, and denote by $(\mathbf{T}, (\varphi_v, v \in \mathbf{T}))$ the block tree of $\mathbf{M}$ where $(\varphi_v, v \in \mathbf{T})$ is its family of node decorations. Then, for every $u > 0$, the law of the tree of blocks can be described as follows:*

- *$\mathbf{T}$ follows the law $GW(\mu^u)$;*
- *Conditionally given $\mathbf{T} = \mathfrak{t}$, the decorations $(\varphi_v, v \in \mathfrak{t})$ are independent random variables, and, for $v \in \mathfrak{t}$, φ_v follows a uniform distribution on the set of sequences of*

$\mathrm{el}(v)$ *elements of total size* $k_v(\mathfrak{t})$*, where* $k_v(\mathfrak{t})$ *is the number of children of* v *in* $\mathfrak{t}$*.*

For every $n \geq 1$*, the same statements hold under* $\mathbb{P}_{n,u}$*, only replacing* $GW(\mu^u)$ *with* $GW(\mu^u, n-1)$*.*

Remark 2.18. In the case of simple triangulations decomposed into irreducible blocks, sampling the decorations of the tree amounts to drawing according to $\Gamma\Phi$ in the Boltzmann framework. As described in Section 2.2.3 and Algorithm 8, one can simply sample $\mathrm{el}(v)$ independently from all the other variables, according to a geometric law with parameter $uB(x(u)^d)$ (for all $u > 0$, if holds that $uB(x(u)^d) < 1$ by (2.33)).

Proof of Proposition 2.17. The proof is very similar to the proof of Proposition 2.6. Once again, it suffices to prove the statement for $\mathbb{P}_u$ as, by Proposition 2.16, the block-tree of a map of size n has size $n-1$.

Let $\mathfrak{t}$ be a tree, and let $(\varphi_v, v \in \mathfrak{t})$ be a family of (sequences of) blocks, with, for any $v \in \mathfrak{t}$, $|\varphi_v| = k_v(\mathfrak{t})$. Let $\mathfrak{m}$ be the map with block decomposition given by $(\mathfrak{t}, (\varphi_v, v \in \mathfrak{t}))$.

If $\mathbf{M}$ follows $\mathbb{P}_u$, denote by $(\mathbf{F}_v, v \in \mathbf{T})$ its (random) family of decorations. Then, one has

$$
\begin{aligned}
\mathbb{P}_u\left(\mathbf{T} = \mathfrak{t}, \mathbf{F}_v = \varphi_v \ \forall v \in \mathfrak{t}\right) &= \mathbb{P}_u(\mathfrak{m}) = \frac{\rho(u)^{|\mathfrak{m}|} u^{b(\mathfrak{m})}}{M(\rho(u), u)} \\
&= \frac{u^{b(\mathfrak{m})}}{x(u)} \left(\frac{x(u)}{\Phi(x(u), u)}\right)^{|\mathfrak{m}|} = \frac{u^{b(\mathfrak{m})} x(u)^{|\mathfrak{m}|-1}}{\Phi(x(u), u)^{|\mathfrak{m}|}} \\
&= \frac{u^{\sum_{v\in\mathfrak{t}} \mathrm{el}(v)} x(u)^{\sum_{v\in\mathfrak{t}} k_v(\mathfrak{t})}}{\Phi(x(u), u)^{|V(\mathfrak{t})|}} \prod_{v\in\mathfrak{t}} \frac{[X^{k_v(\mathfrak{t})}]\Phi(X, u)}{[X^{k_v(\mathfrak{t})}]\Phi(X, u)} \\
&= \prod_{v\in\mathfrak{t}} \frac{[X^{k_v(\mathfrak{t})}]\Phi(X, u) x(u)^{k_v(\mathfrak{t})}}{\Phi(x(u), u)} \times \prod_{v\in\mathfrak{t}} \frac{u^{\mathrm{el}(v)}}{[X^{k_v(\mathfrak{t})}]\Phi(X, u)} \\
&= GW(\mu^u)(\mathfrak{t}) \times \prod_{v\in\mathfrak{t}} \frac{1}{[X^{k_v(\mathfrak{t})} u^{\mathrm{el}(v)}]\Phi(X, u)}.
\end{aligned}
$$

This concludes the proof. □

This enables to put into light a phase transition on the tree structure, using the usual phase transition for Bienaymé–Galton–Watson trees [Nev86], which depends on the mean of the reproduction law. Following the same line of argument as for Theorem 2.8, one gets the following result.

Theorem 2.19. *For all models of Table 2.1, the law* $\mathbb{P}_u$ *undergoes the following phase transition, driven by the properties of* μ^u*, depending on the position of* u *with respect to*

$$
u_C = \frac{1}{d\rho_B B'(\rho_B) \mp B(\rho_B)}, \tag{2.31}
$$

where the $\mp$ *is a* $-$ *in all cases except the last, where it is a* $+$*. The explicit values of* u_C *can be found in Table 2.2.*

Scheme	u_C	$E(u)$	$1-E(1)$
1	$\frac{9}{5}$	$\frac{8u}{3(u+3)}$	$\frac{1}{3}$
2	$\frac{81}{17}$	$\frac{32u}{3(5u+27)}$	$\frac{2}{3}$
3	$\frac{135}{7}$	$\frac{32u}{5(5u+27)}$	$\frac{4}{5}$
4	$\frac{36}{11}$	$\frac{20u}{9(u+4)}$	$\frac{5}{9}$
5	$\frac{52}{27}$	$\frac{40u}{13(u+4)}$	$\frac{5}{13}$
6	$\frac{68}{3}$	$\frac{20u}{17(u+4)}$	$\frac{13}{17}$
7	$\frac{16}{7}$	$\frac{9u}{2(u+8)}$	$\frac{1}{2}$
8	$\frac{64}{37}$	$\frac{27u}{2(32-5u)}$	$\frac{1}{2}$

Table 2.2: Values of u_C, $E(u)$ and $1-E(1)$ when $u \leq u_C$ for all the decomposition schemes of Table 2.1.

Subcritical case. *For $u < u_C$, then there is no $x \in (0, \rho_\Phi(u)]$ such that $\mathbb{E}[\mu^{x,u}] = 1$,* i.e. *$\mu^{x,u}$ cannot be critical. In particular,*

$$E(u) := \mathbb{E}[\mu^u] = \frac{du\rho_B B'(\rho_B)}{1 \pm uB(\rho_B)} < 1 \tag{2.32}$$

where the $\pm$ is a $+$ in all cases except the last, where it is a $-$. The explicit expressions of $E(u)$ can be found in Table 2.2. In this case, $x(u) = M(\rho(u), u) = \rho_\Phi(u) = \rho_B^{1/d}$.

Critical case. *For $u = u_C$,*

$$\mathbb{E}[\mu^{u_C}] = 1.$$

In this case, $x(u_C) = M(\rho(u_C), u_C) = \rho_\Phi(u_C) = \rho_B^{1/d}$.

Supercritical case. *For $u > u_C$,*

$$\mathbb{E}[\mu^u] = 1.$$

In this case, $x(u)$ satisfies

$$dx(u)^d B'(x(u)^d) \mp B(x(u)^d) - \frac{1}{u} = 0, \tag{2.33}$$

where the $\mp$ is a $-$ in all cases except the last, where it is a $+$.

Proof. Using (2.28), one gets

$$\mathbb{E}[\mu^{x,u}] = \frac{x\Phi'(x,u)}{\Phi(x,u)}. \tag{2.34}$$

where the derivation of Φ is done with respect to the first variable. So, for $x \in [0, \rho_\Phi(u)]$,

$$\mathbb{E}[\mu^{x,u}] = 1 \iff x\Phi'(x,u) - \Phi(x,u) = 0. \tag{2.35}$$

In all cases but the last one,

$$x\Phi'(x,u) - \Phi(x,u) = dux^d B'(x^d) - uB(x^d) - 1$$

so

$$\mathbb{E}\left[\mu^{x,u}\right] = 1 \iff u = \frac{1}{dx^d B'(x^d) - B(x^d)}.$$

The function $f : x \in [0, \rho_\Phi(u)] \mapsto dx^d B'(x^d) - B(x^d) = \sum_{j\in\mathbb{N}_0}(dj-1)b_j x^{dj}$ is increasing in x. Moreover, one has $f(0) = 0$ and $f(\rho_\Phi(u)) = f(\rho_B^{1/d}) = d\rho_B B'(\rho_B) - B(\rho_B)$. So $1/f(x)$ maps bijectively $(0, \rho_\Phi(u)] = (0, \rho_B^{1/d}]$ to $[u_C, +\infty)$. Therefore, there exists $x \in (0, \rho_\Phi(u)]$ such that the law $\mu^{x,u}$ is critical if and only if $u \in [u_C, +\infty)$, and this x is unique. For the last case, one gets

$$\mathbb{E}\left[\mu^{x,u}\right] = 1 \iff u = \frac{1}{dx^d B'(x^d) + B(x^d)}$$

and the same conclusion applies[4].

Furthermore, by (2.26), it holds that for fixed $u > 0$, there are two possible sources of singularity for $M(z,u)$:

1. The pair $(z_0 = \rho(u), x_0 = M(\rho(u), u) = x(u))$ satisfies $\frac{\partial F}{\partial x}(z_0, x_0) = 0$ for $F : (z,x) \mapsto x - z\Phi(x,u)$, thus being a singularity by the contraposition of the implicit function theorem. In this case,

$$\Phi'(x(u), u) = \frac{1}{\rho(u)}, \tag{2.36}$$

   so

$$x(u)\Phi'(x(u),u) - \Phi(x(u),u) = \frac{x(u)}{\rho(u)} - \frac{x(u)}{\rho(u)} = 0;$$

   which is equivalent, by (2.35), to $\mathbb{E}\left[\mu^u\right] = 1$. By what precedes, this is possible if and only if $u \geq u_C$. Then, it follows that in this case $\mathbb{E}\left[\mu^u\right] = 1$.

2. A singularity of Φ is reached so $M(\rho(u), u) = \rho_\Phi(u)$ *i.e.* $x(u) = \rho_\Phi(u)$. Then, the value of $\mathbb{E}\left[\mu^u\right]$ is obtained as an immediate consequence of Equations (2.25) and (2.34). This happens if $u < u_C$.

At $u = u_C$, both types of singularity are in fact reached. All these discussions can be summed up as the following theorem. □

Notice that the case $u = 1$, which corresponds to the uniform law on $\mathcal{M}$, always falls in the subcritical regime.

Remark 2.20. As explained in Remark 2.7, it is possible to define $x(u)$ as the unique $x \in [0, \rho_B]$ such that $\mathbb{E}\left[\mu^{x,u}\right] = 1$ when such an x exists, and $x = \rho_B^{1/d}$ otherwise. Then, it would still hold that $x(u) = M(\rho(u), u)$. This choice is motivated by the need to obtain large (finite) structures without degenerate conditioning, as discussed by Janson [Jan12, §7].

[4]The fact that $\rho_\Phi(u)$ depends on u does not create problems because the singular point u_C arises before the singularity from the sequence.

Remark 2.21. It is possible to obtain more information on the values of $x(u)$ in the supercritical case than is indicated in the statement of Theorem 2.19. Indeed, all the block series B considered can be described by a Lagrangean specification ([BFSS01, Table 2]) and are therefore algebraic. By deriving the algebraic equation verified by $B(y)$, one obtains a new algebraic equation relating, for all $y \in [0, \rho_B]$, the values of $B'(y)$, $B(y)$ and y. Adding (2.33), this gives three algebraic equations for three unknowns. Polynomial elimination gives explicit polynomials P_1, P_2, P_3 such that, for all $u > u_C$,

$$P_1(x(u), u) = 0, \qquad P_2(B(x(u)^d), u) = 0 \qquad \text{and} \qquad P_3(B'(x(u)^d), u) = 0.$$

Here are the equations for the main example:

$$\begin{aligned} u^3 x(u)^4 - 2u^2(u-1)x(u)^2 + (u-1)^2 &= 0 \\ u^2 B(x(u)^2)^2 + 2u(4u-3)B(x(u)^2) - 8u + 9 &= 0 \\ (u-1)B'(x(u)^2)^2 - 4u &= 0. \end{aligned}$$

For other schemes, the equations can be found in Table 2.3 page 77. But one can do even better, by solving the system of equations (and using sign considerations to discriminate between the different possibilities): this yields the triplet $\left(x(u), B(x(u)^d), B'(x(u)^d)\right)$ as a function of u. However, expressions are generally neither short nor easy (or interesting) to manipulate. For our main example, one gets the following results:

$$\begin{aligned} x(u) &= \frac{\sqrt{(u-1)\sqrt{u} - (u-1)^{3/2}}}{u^{3/4}} \\ B(x(u)^2) &= \frac{3 - 4(u - \sqrt{u(u-1)})}{u} \\ B'(x(u)^2) &= \frac{2\sqrt{u}}{\sqrt{u-1}}. \end{aligned}$$

Notice that the value obtained for $x(u)$ is compatible with the fact that $y(u) = x(u)^2$ in that case, and $y(u)$ has be computed in Remark 2.9. Interestingly enough, solving gives similarly nice results for $x(u)$, $B(x(u)^d)$ and $B'(x(u)^d)$ in the last scheme as well, which one could assume to be more involved.

In addition to the expectation, the phase transition on the μ^u probability distribution can also be seen in the evolution of its asymptotic behaviour and its variance. Indeed, as is typical for families of maps, in each case the series B has asymptotic expansion of the form [BFSS01, §5]

$$B(y) = B(\rho_B) - \rho_B B'(\rho_B) Y + \gamma_B Y^{3/2} + O(Y^2). \tag{2.37}$$

where the value of $\gamma_B > 0$ can be found in [BFSS01, Table 2] and $Y = 1 - \frac{y}{\rho_B}$. As a consequence,

$$[y^n]B(y) = b_n \underset{n\to\infty}{\sim} \frac{3\gamma_B}{4\sqrt{\pi}} \rho_B^{-n} n^{-5/2}, \tag{2.38}$$

Therefore, using Equations (2.29) and (2.30), it is immediate that for all schemes except the last,

$$\mu^u(d\,j) \sim \frac{3\gamma_B}{4\sqrt{\pi}} \frac{u\rho(u)}{x(u)} \left(\frac{x(u)^d}{\rho_B}\right)^j j^{-5/2} \qquad \text{as} \quad j \to \infty,$$

with this expression simplifying in the subcritical and critical cases as $x(u) = \rho_B^{1/d}$. For the last decomposition scheme, the computation is not so straightforward as the expression of the coefficients of Φ in terms of those of B is not obvious; this case is treated later in Section 2.2.2.

Furthermore, one can also study the variance of μ^u. Indeed, using Equations (2.30) and (2.36), one gets

$$\sigma^2(\mu^u) = \sum_{j\in\mathbb{N}_0} j^2\mu^u(j) - \mathbb{E}\left[\mu^u\right]^2 = \frac{1}{\Phi(x(u),u)} \sum_{j\in\mathbb{N}_0} j^2[X^j]\Phi(X,u)x(u)^j - 1.$$

When $u \le u_C$, one can write $j^2[X^j]\Phi(X,u)x(u)^j \sim C(u)j^{-1/2}$ for $C(u) \neq 0$ as $j \to \infty$ so the sum diverges. When $u > u_C$, the sum converges, and, using (2.35), one can write

$$\begin{aligned}\sigma^2(\mu^u) &= \frac{1}{\Phi(x(u),u)}\left(x(u)^2 \sum_{j\in\mathbb{N}_0} j(j-1)[X^j]\Phi(X,u)x(u)^{j-2} + x(u)\sum_{j\in\mathbb{N}_0} j[X^j]\Phi(X,u)x(u)^{j-1}\right) - 1\\ &= \frac{x(u)^2\Phi''(x(u),u) + x(u)\Phi'(x(u),u)}{\Phi(x(u),u)} - 1 = \frac{x(u)^2\Phi''(x(u),u)}{\Phi(x(u),u)}.\end{aligned}$$

Therefore, the following holds.

Proposition 2.22. *For all models of Table 2.1, the law μ^u undergoes the following phase transition depending on the position of u with respect to u_C defined in (2.31). Define*

$$c(u) = \frac{3}{4\sqrt{\pi}}\gamma(u)\frac{\rho(u)}{x(u)} \tag{2.39}$$

where $\gamma(u) = u\gamma_B$ in all cases but the last, where $\gamma(u) = \frac{u\gamma_B}{(1-uB(\rho_B))^2}$. Then, the following holds.

Subcritical case. *For $u < u_C$,*

$$\mu^u(d\,j) \underset{j\to\infty}{\sim} c(u)j^{-5/2}, \qquad \mathbb{E}\left[\mu^u\right] = E(u) < 1 \qquad \textit{and} \qquad \mathbb{V}(\mu^u) = \infty.$$

Critical case. *For $u = u_C$,*

$$\mu^u(d\,j) \underset{j\to\infty}{\sim} c(u_C)j^{-5/2}, \qquad \mathbb{E}\left[\mu^{u_C}\right] = 1 \qquad \textit{and} \qquad \mathbb{V}(\mu^u) = \infty.$$

Supercritical case. *For $u > u_C$, μ^u has exponential moments and*

$$\mu^u(d\,j) \underset{j\to\infty}{\sim} c(u)j^{-5/2}\left(\frac{x(u)^d}{\rho_B}\right)^j, \qquad \mathbb{E}\left[\mu^u\right] = 1 \qquad \textit{and} \qquad \mathbb{V}(\mu^u) < \infty,$$

with

$$\sigma(u) := \mathbb{V}(\mu^u)^{1/2} = x(u)\sqrt{\frac{\Phi''(x(u),u)}{\Phi(x(u),u)}}; \tag{2.40}$$

except for the last decomposition scheme, where the asymptotic expansion for μ^u holds only for $u < \frac{32}{5}$ (see below).

Simple triangulations decomposed into irreducible blocks.

Recall that, by (2.30),

$$\mu^u(j) := \frac{[X^j]\Phi(X,u)\,x(u)^j}{\Phi(x(u),u)} = \frac{\rho(u)}{x(u)}[X^j]\Phi(X,u)\,x(u)^j.$$

To analyse the asymptotic behaviour of $\mu(j)$ when $j \to \infty$, we resort to singularity analysis of Φ: for fixed $u > 0$, determine the behaviour of $\Phi(x,u)$ when x is in a neighbourhood of its dominant singularities. In the last case, it holds, by (2.25), that

$$\Phi(x,u) = \frac{1}{1 - uB(x^2)}.$$

There are two possible sources of singularity for Φ : either $uB(x^2)$ reaches 1, or x^2 reaches a singularity of B.

We show that there is a phase transition in the asymptotic behaviour of $[X^j]\Phi(X,u)$. This phase transition is in some sense "independent" of the one for blocks. Here we take a fixed u, and study the singular behaviour of $\Phi(x,u)$ and deduce the asymptotic behaviour of $[X^j]\Phi(X,u)$. We show that that this behaviour changes from (2.42) to (2.43) at $u = v_C$ for

$$v_C := \frac{1}{B(\rho_B)} = \frac{32}{5}.$$

The first case is attained when x^2 is near ρ_B (which is the only singularity of B), *i.e.* x is near $\pm\rho_B^{1/2}$. In this case, the radius of convergence satisfies $\rho_\Phi(u) = \rho_B^{1/2}$. Using the known asymptotic of B in (2.37), one gets, for y in a neighbourhood of ρ_B and writing $Y = 1 - \frac{y}{\rho_B}$

$$\Phi(y^{1/2},u) = \frac{1}{1-uB(\rho_B)} - \frac{u\rho_B B'(\rho_B)}{(1-uB(\rho_B))^2}Y + \gamma(u)Y^{3/2} + O(Y^2), \tag{2.41}$$

where $\gamma(u) = \frac{u\gamma_B}{(1-uB(\rho_B))^2}$. Using the transfer rules around the dominant singularity ρ_B, one gets

$$[y^n]\Phi(y^{1/2},u) \underset{n\to\infty}{\sim} \frac{3\gamma(u)}{4\sqrt{\pi}}\rho_B^{-n}n^{-5/2}$$

(from the definition of Φ it is obvious that for all $n \in \mathbb{N}_0$, $[x^n]\Phi(x,u) = 0$ if $2 \nmid n$), which can be rewritten

$$[x^{2n}]\Phi(x,u) \underset{n\to\infty}{\sim} \frac{3\gamma(u)}{4\sqrt{\pi}}\rho_B^{-n}n^{-5/2}.$$

Therefore, in this case,

$$\mu^u(2j) \underset{j\to\infty}{\sim} \frac{3}{4\sqrt{\pi}} \frac{u\gamma_B}{(1-uB(\rho_B))^2} \frac{\rho(u)}{x(u)} \left(\frac{x(u)^2}{\rho_B}\right)^j j^{-5/2}. \tag{2.42}$$

This result has been obtained in a very broad setting by Stufler [Stu20c, Lemma 3.2], using probabilistic arguments (such as those in Section 3.2). The latter source of singularity is attained when $uB(x^2)$ reaches 1. If $u < \frac{1}{B(\rho_B)} = v_C$, this does not happen as a singularity for B (case above) is reached before $uB(x^2)$ reaches 1. Let us now consider the case $u > v_C > u_C$, which is included in the supercritical case. Then, by (2.33),

$$uB(x(u)^2) = 1 - 2x(u)^2 B'(x(u)^2)$$

so $uB(x(u)^2) < 1$. But since $B(\rho_B)$ is finite, for all fixed $u > 0$, there exists $x \in (x(u), \rho_B^{1/2}]$ such that $uB(x^2) = 1$. Then, such x is the radius of convergence $\rho_\Phi(u)$ and it is the only dominant singularity of Φ. This case is what Flajolet and Sedgewick call *the supercritical sequence schema* and describe in [FS09, §V.2]. Since the function Φ is periodic, the computation requires a bit of care.

When $u \geq v_C$, it holds that $\rho_\Phi(u) \leq \rho_B^{1/2}$, so, when x is in a neighbourhood of $\rho_\Phi(u)$, it holds that, writing $X = \rho_{\Phi(u)} - x$,

$$\Phi(x,u) = \frac{1}{1-uB(x^2)} = \frac{1}{1-u\left(B(\rho_\Phi(u)^2) - 2B'(\rho_\Phi(u)^2)X + o(X)\right)} \sim \frac{1}{2uB'(\rho_\Phi(u)^2)X}.$$

Thus, Φ has two *poles* of order 1: $\pm\rho_\Phi(u)$ so, by [FS09, Theorem IV.10], when $n \to \infty$,

$$[x^n]\Phi(x,u) = \frac{\rho_\Phi(u)^{-n}}{2uB'(\rho_\Phi(u)^2)} + \frac{(-1)^n \rho_\Phi(u)^{-n}}{2uB'(\rho_\Phi(u)^2)} + O(R^n)$$

with $|R| < \rho_\Phi(u)$, so

$$[x^{2n}]\Phi(x,u) \underset{n\to\infty}{\sim} \frac{\rho_\Phi(u)^{-n}}{uB'(\rho_\Phi(u)^2)}.$$

(Remember that for all $n \in \mathbb{N}_0$, $[x^n]\Phi(x,u) = 0$ if $2 \nmid n$.) Therefore, in this case,

$$\mu^u(2j) \underset{j\to\infty}{\sim} \frac{\rho(u)}{x(u)uB'(\rho_\Phi(u)^2)} \left(\frac{x(u)}{\rho_\Phi(u)}\right)^{2j}, \tag{2.43}$$

so μ^u still has exponential moments since $x(u) < \rho_\Phi(u)$.

In the end, the supercritical case of Proposition 2.22 can be extended as follows for the last decomposition scheme, the last case remaining to cover being $u \geq v_C$.

Proposition 2.23. *Consider the decomposition of simple triangulations into irreducible blocks. Let $u \geq v_C > u_C$. Then, μ^u has exponential moments and*

$$\mu^u(dj) \underset{j\to\infty}{\sim} \frac{\rho(u)}{uB'(\rho_\Phi(u)^d)x(u)} \left(\frac{x(u)}{\rho_\Phi(u)}\right)^{dj}$$

and $x(u) < \rho_\Phi(u)$. Moreover, it still holds that

$$\mathbb{E}[\mu^u] = 1 \qquad \text{and} \qquad \sigma(u) := \mathbb{V}(\mu^u)^{1/2} = x(u)\sqrt{\frac{\Phi''(x(u),u)}{\Phi(x(u),u)}} < \infty.$$

2.2.3 Restatement as Boltzmann samplers

To sample and represent maps according to $\mathbb{P}_{n,u}$ for large n and different values of $u > 0$, we implement a *Boltzmann generator* for maps with a Boltzmann weight u on their blocks (see Section 1.2.4). This framework is what one could call a "combinatorial counterpart" to the probabilistic framework just introduced[5]. In this subsection, we simply sample maps according to $\mathbb{P}_u$; we condition the size of the output in Section 3.3. More precisely, we specify Boltzmann generators to our case and show their equivalence with the Bienaymé–Galton–Watson framework introduced earlier.

One can obtain a Boltzmann generator $\Gamma M(z,u)$ for $\mathcal{M}$ with weight u on the blocks using the Lagrangean expression of $M(z,u)$ (2.26). Assuming the existence of a Boltzmann generator $\Gamma\Phi$ drawing an element from the class described by the series $\Phi(.,u)$ of (2.25), one can write the following for $\Gamma M(z,u)$:

Algorithm 6 $\Gamma M(z,u)$ for $z \leq \rho(u)$

```
x = M(z, u)
φ = ΓΦ(x, u)  // see Algorithms 7 and 8 below
n = s(φ)
for i = 1 to n do
    m_i = ΓM(z, u)
    Attach m_i to the i-th element of φ
end for
return φ
```

Algorithm 6 produces an object whose internal structure is a Bienaymé–Galton–Watson tree: its root φ has n children, where n is the size of a random object obtained by $\Gamma\Phi(M(z,u),u)$, and for each child the same process applies. More precisely, one can consider the tree obtained by considering, at each step of the algorithm, φ as a node and the φ_i as its n children. The reproduction law of this Bienaymé–Galton–Watson tree is given by the size distribution of the output of $\Gamma\Phi(M(z,u),u)$. The generator $\Gamma\Phi$ is a Boltzmann generator, therefore for all $j \in \mathbb{N}_0$,

$$\mathbb{P}\left(\mathrm{s}(\Gamma\Phi(x,u)) = j\right) = \frac{[X^j]\Phi(X,u)x^j}{\Phi(x,u)} = \mu^{x,u}(j)$$

by (2.28); which shows that the underlying tree structure has distribution $GW(\mu^u)$ for $z = \rho(u)$.

Moreover, if the tree structure is fixed, *i.e.*, conditioning the sizes of the various elements obtained, the decorations for the nodes are drawn according to the Boltzmann generator $\Gamma\Phi$

[5]One might have already suspected as much, given the similarities between Equations (1.5) and (2.34).

with fixed size, so they are distributed uniformly among all elements of the corresponding size. Together with Proposition 2.17, this shows that tree of blocks of a map drawn according to $\Gamma M(\rho(u), u)$ has the same distribution as the tree of blocks of a map $\mathbf{M}$ drawn according to $\mathbb{P}_u$ (defined in (2.27)). By Proposition 2.16, the tree of blocks determines uniquely the map, so *maps drawn according to* $\Gamma M(\rho(u), u)$ *have the distribution* $\mathbb{P}_u$. This shows that the Boltzmann point of view and the Bienaymé–Galton–Watson one are two sides of the same coin.

Now, we explain the construction of the Boltzmann sampler $\Gamma\Phi$. Using the definition of Φ in (2.25), the generator $\Gamma\Phi$ can be written in function of the Boltzmann generator for the blocks ΓB. Depending on the decomposition schemes, the following Boltzmann generators can be written:

Algorithm 7 $\Gamma\Phi(x, u)$
when $\Phi(x, u) = 1 + uB(x^d)$

if $\mathrm{Bern}\left(\frac{1}{1+uB(x^d)}\right)$ **then**
  return the trivial block
else
  return $\Gamma B(x^d, u)$
end if

Algorithm 8 $\Gamma\Phi(x, u)$
when $\Phi(x, u) = 1/(1 - uB(x^d))$

$k = \mathrm{Geom}\left(uB(x^d)\right)$
$\mathfrak{b}_1, \ldots, \mathfrak{b}_k = (\Gamma B(x^d, u))_{1 \le i \le k}$ // *k independent calls*
for $i = k$ **to** 2 **do**
  Attach $\mathfrak{b}_i$ in the last element of $\mathfrak{b}_{i-1}$
end for
return $\mathfrak{b}_1$ // *see Fig. 2.10 for an illustration of the gluing process*

So the question of a Boltzmann generator $\Gamma\Phi$ comes back to that of a Boltzmann generator ΓB for blocks. For many families of maps, there exist linear-time uniform generators for fixed size objects, using bijections with trees which are much easier to generate [Fus07, §4.2]. Fusy implemented a fixed-size uniform generator for simple quadrangulations (and simple triangulations), which is why we focus on generating quadrangulations weighted by their simple blocks, *i.e.* the quadrangular rewriting of our main example, detailed in Section 2.1.4. However, the following study can be done for all decomposition schemes, provided a (Boltzmann) generator for the blocks is available.

Fusy's generator is an exact size generator, *i.e.*, it takes a size as input and returns a simple quadrangulation of the requested size. The difficulty then lies in drawing the size to set as the input to the algorithm so that it can be used as a Boltzmann generator for $\mathcal{B}$, and therefore a Boltzmann generator for $\Gamma\Phi$ and ultimately for ΓM. One must sample a size S such that, for $k \in \mathbb{N}_{>0}$,

$$\mathbb{P}\left(S = k\right) = \frac{u b_k y^k}{1 + uB(y)} = \mu^{y^{1/d}, u}(k),$$

for some $y \in (0, \rho_B]$. In practice, since the goal is that ΓM samples according to the law $\mathbb{P}_u$, we choose $y = x(u)^d$. One can write the Boltzmann generator ΓB as follows:

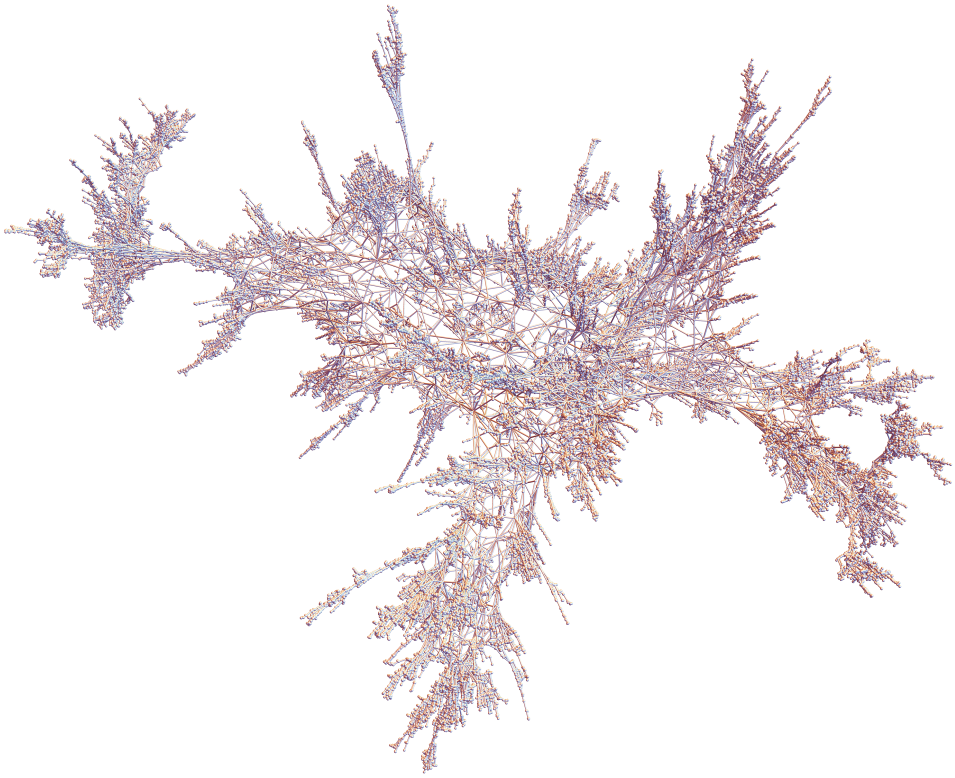

Figure 2.11: Quadrangulation drawn according to the subcritical model $\mathbb{P}_{n,1}$ of size around $55\ 000$.

Algorithm 9 $\Gamma B(y, u)$

Sample k according to $\mu^{y^{1/d},u}$

return a block of size k using a uniform fixed-size generator

---

The sampling operation can *e.g.* be done using the coefficients b_k to sample, but, when the tail is heavy, there are also faster options. This is developed in Section 3.3, in which we give complexity considerations for the generators.

In the end, Fusy's generator and size generation considerations allow us to implement the ΓM generator, which draws block-weighted maps according to $\mathbb{P}_u$. Figures 2.11 to 2.15 are drawings of large maps generated according to this law for different values of u[6]. Although one should not over-interpret the results drawn by the generator, it is clear that the maps drawn change considerably as a function of u: the maps drawn for $u < u_C$ are "bushier" than those for $u > u_C$, which are more tree-like.

[6]In this thesis, drawings of large maps and large trees of Figs. 4 to 6 and 2.11 to 2.15 use the *spring electrical embedding*, which is a technique to arrange graph vertices such that edges are roughly the same length and crossings are minimized. In this method, each vertex acts like a charged particle and each edge acts like a spring so that, at equilibrium, connected nodes are uniformly spaced due to spring forces, while unconnected nodes are pushed apart due to electrical repulsion.

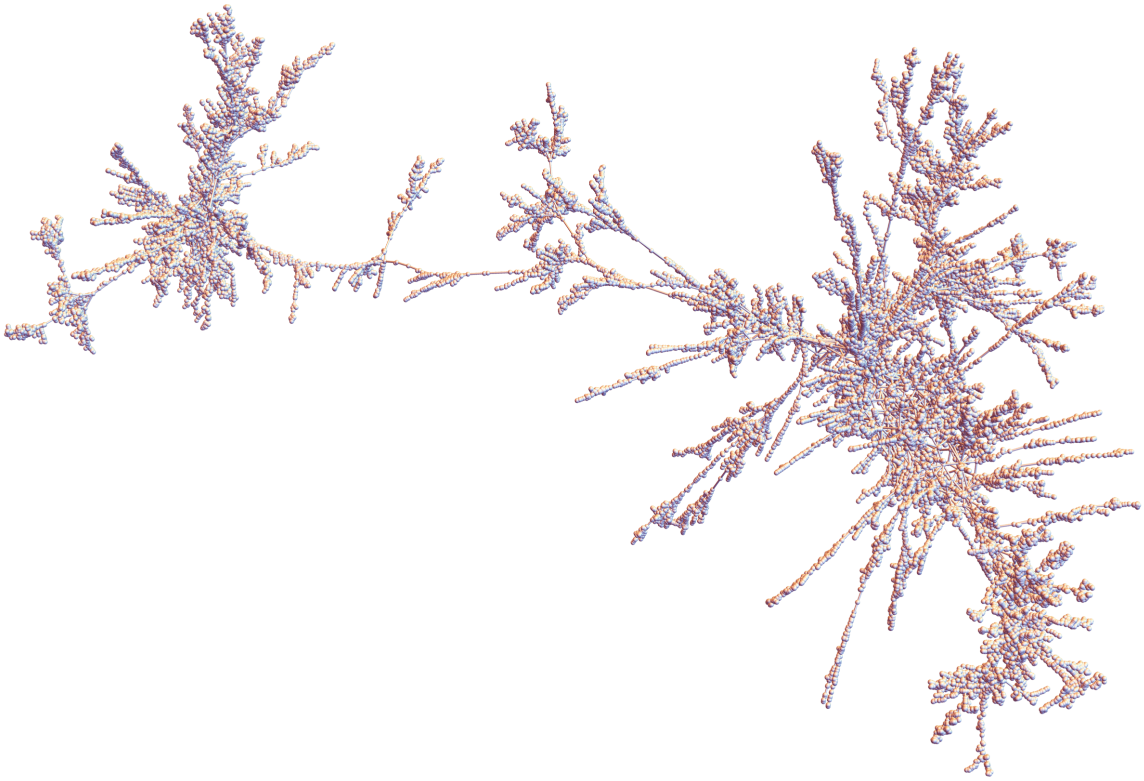

Figure 2.12: Quadrangulation drawn according to the subcritical model $\mathbb{P}_{n,8/5}$ of size around $55\ 000$.

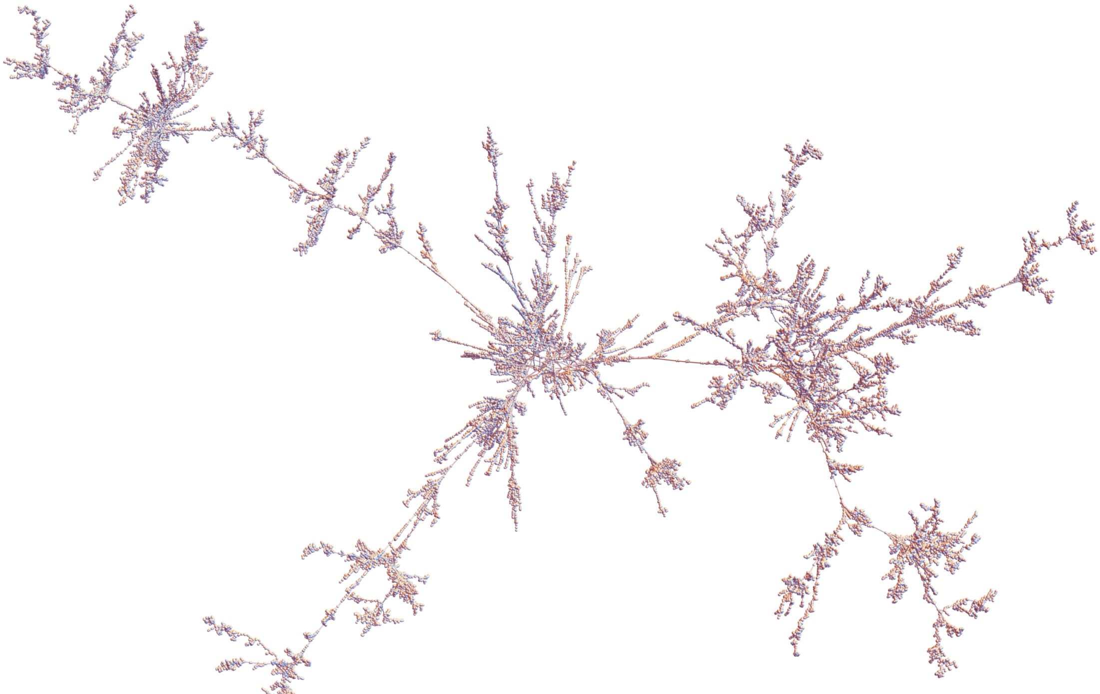

Figure 2.13: Quadrangulation drawn according to the critical model $\mathbb{P}_{n,9/5}$ of size around $80\ 000$.

Figure 2.14: Quadrangulation drawn according to the supercritical model $\mathbb{P}_{n,5/2}$ of size around 75 000.

Figure 2.15: Quadrangulation drawn according to the supercritical model $\mathbb{P}_{n,5}$ of size around 50 000.

Scheme	Algebraic equations for $x(u)$, $B(x(u)^d)$ and $B'(x(u)^d)$
1	$u^3x^4-2u^2(u-1)x^2+(u-1)^2=0;$ $u^2B^2+2u(4u-3)B-8u+9=0;$ $(u-1)B'^2-4u=0.$
2	$(u-1)^2x^4+4(5u-1)(u-1)x^3+2(u^2+28u+3)x^2-4(u-1)x-u+1=0;$ $u^3B^4+4u^3B^3+2u^2(u+2)B^2+4uB-u^2+u+1=0;$ $u^4B'^4-4u^3(9u-1)B'^3+2u^2(35u^2+90u+3)B'^2-4u(9u-1)(u-1)^2B'$ $+(u-1)^4=0.$
3	$u^6x^4+6u^4(9u-2)(u-1)x^3+24u^2(11u+2)(u-1)^2x^2$ $+8(5u^3+27u^2-24u+8)(u-1)^2x-27(u-1)^5=0;$ $B^4u^3-2u^2(u-3)B^3-6u(u-2)B^2-2(u^2+u-4)B+u^2-5u+5=0;$ $27u(u-1)^2B'^4-8(2u-1)(5u^2-5u+8)B'^3+6u(13u^2-13u-8)B'^2$ $-12u^2(2u-1)B'-u^3=0.$
4	$u(3u-4)^2x^4-2u(u-1)(3u-16)x^3-(u-1)(2u^2+15u+1)x^2-2(u-1)^2x$ $+(u-1)^2=0;$ $B^4u^2+2u(3u-1)B^3+(6u^2-u+1)B^2+4B-3u+4=0;$ $B'^4u^3-2u^2(10u+1)B'^3+u(10u^2+108u+1)B'^2$ $+2u(18u^2+69u-76)B'-(3u-4)^3=0.$
5	$u^3(3u-4)^2x^8-u^2(u-1)(7u^2-47u+36)x^6$ $-(32u^3-55u^2+54u-27)(u-1)^2x^4-16u^2(u-1)^3x^2+16(u-1)^4=0;$ $B^4u^2+2u(5u-3)B^3+(19u^2-22u+9)B^2+2(3u^2-7u+6)B-6u+7=0;$ $64u^2(u-1)^2B'^4-48u^2(u-1)^2B'^3-3(u-1)(41u^3+19u^2-21u+9)B'^2$ $+2u^2(67u-4)(u-1)B'-u(3u-4)^3=0.$
6	$243u^5(3u-4)^2x^4+54u^3(189u^3-594u^2+609u-200)x^3$ $+(3645u^6-1701u^5-26730u^4+56403u^3-47178u^2+18750u-3125)x^2$ $+54(126u^2-123u+41)(u-1)^3x-729(u-1)^6=0;$ $9B^4u^2+30B^3u+5(3u+5)B^2+26B-3u+10=0;$ $729u^2(u-1)^4B'^4-54u(54u^3-108u^2+117u-41)(u-1)^2B'^3$ $+(4374u^6-17496u^5+32562u^4-39528u^3+32697u^2-15750u+3125)B'^2$ $-54u^2(u-1)(54u^3-162u^2+207u-100)B'+27u^3(3u-4)^3=0.$
7	$(u-1)^4x^9+2u(31u-6)(u-1)^2x^6+u^2(128u^2+449u+48)x^3-64u^3=0;$ $64B^3u^3+3u^2(23u+16)B^2+6u(3u^2+2u+2)B-9u^2-3u+1=0;$ $4096B'^3u^6-u^4(4393u^2+10464u+768)B'^2+2u^2(149u-24)(u-1)^3B'$ $-(u-1)^6=0.$

Scheme	Algebraic equations for $x(u)$, $B(x(u)^d)$ and $B'(x(u)^d)$
8	$(u-1)^3x^6+3(3u-4)(u-1)^2x^4+3(u-1)(49u^2-64u+16)x^2-(7u-8)^2=0;$ $u^2(7u-8)B^3-3u^2(6u-7)B^2+3u(4u^2-4u-1)B-4u^2+6u-1=0;$ $u^3(7u-8)^3B'^3-3u^2(121u^2-184u+64)(u-1)^2B'^2+3u(7u-8)(u-1)^4B'$ $-(u-1)^6=0.$

Table 2.3: Algebraic relations relating $x(u)$ and u, $B(x(u)^d)$ with u and $B'(x(u)^d)$ with u. For the sake of legibility, in this table, $x(u)$ is written x, $B(x(u)^d)$ written B and $B'(x(u)^d)$ written B'.

Chapter 3

Enumerative phase transition

We start by shedding light on the phase transition for the enumeration: at a fixed size n, we focus on the sum of the weights $u^{b(\mathfrak{m})}$ for all maps $\mathfrak{m}$ of size n. This reveals a phase transition in the asymptotic behaviour of the enumeration. There exists a critical value u_C such that the polynomial correction for $u < u_C$ (subcritical case) is the same as for planar maps, whereas when $u > u_C$ (supercritical case) it is the same as for plane trees. Moreover, at $u = u_C$, a new asymptotic behaviour appears with a polynomial correction in $n^{-5/3}$, which was predicted by Flajolet and Sedgewick [FS09, VI.18] and showed for the case of general maps decomposed into 2-connected blocks by Bonzom [Bon16].

These enumeration results can be obtained using both analytic combinatorics (Section 3.1) and a probabilistic approach (Section 3.2). In Section 3.3, they are used to analyse the complexity of the block-weighted Boltzmann generator introduced in Section 2.2.3.

3.1 Enumeration by analytic combinatorics

Recall that $M(z,u)$ denotes generically the generating series of maps with the control of their number of blocks (variable u) for any decomposition scheme considered in Section 2.2: as stated in (2.23),

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{\mathrm{s}(\mathfrak{m})} u^{b(\mathfrak{m})};$$

and, for fixed $u > 0$, $\rho(u)$ denotes the radius of convergence of $z \mapsto M(z,u)$. We employ the classical method of singularity analysis, described in Section 1.2.3: we obtain the singular expansion of $M(z,u)$ at its dominant singularities $\omega_j \rho(u)$, where the ω_j are the d-th roots of unity and then apply the transfer lemma [FO90] which gives immediately an asymptotic expansion of $[z^n]M(z,u)$.

3.1.1 Radius of convergence

We start by comments on $\rho(u)$. Equation (2.30) gives that, for all $u > 0$,

$$\rho(u) = \frac{x(u)}{\Phi(x(u),u)}, \qquad (3.1)$$

Scheme	$\rho(u)$	$M(\rho(u),u)$	$\rho(1)$	$\rho(u_C)$	$\hat{\rho}(1)$
1	$\frac{2\sqrt{3}}{3(u+3)}$	$\frac{2\sqrt{3}}{9}$	$\frac{\sqrt{3}}{6}$	$\frac{5\sqrt{3}}{36}$	$\frac{1}{12}$
2	$\frac{27}{8(5u+27)}$	$\frac{1}{8}$	$\frac{27}{256}$	$\frac{17}{256}$	$\frac{27}{256}$
3	$\frac{128}{27(5u+27)}$	$\frac{128}{729}$	$\frac{4}{27}$	$\frac{28}{729}$	$\frac{4}{27}$
4	$\frac{5}{8(u+4)}$	$\frac{5}{32}$	$\frac{1}{8}$	$\frac{11}{128}$	$\frac{1}{8}$
5	$\frac{5\sqrt{2}}{4(u+4)}$	$\frac{5\sqrt{2}}{16}$	$\frac{\sqrt{2}}{4}$	$\frac{27\sqrt{2}}{128}$	$\frac{1}{8}$
6	$\frac{125}{128(u+4)}$	$\frac{125}{512}$	$\frac{25}{128}$	$\frac{75}{2048}$	$\frac{25}{128}$
7	$\frac{3\cdot 2^{\frac{1}{3}}}{u+8}$	$\frac{3\cdot 2^{\frac{1}{3}}}{8}$	$\frac{2^{1/3}}{3}$	$\frac{7\cdot 2^{1/3}}{24}$	$\frac{2}{27}$
8	$\frac{\sqrt{3}(32-5u)}{144}$	$\frac{2\sqrt{3}}{9}$	$\frac{3\sqrt{3}}{16}$	$\frac{6\sqrt{3}}{37}$	$\frac{27}{256}$

Table 3.1: Values of $\rho(u)$ and $x(u) = M(\rho(u), u)$ when $u \leq u_C$, with the special cases $\rho(1)$, $\rho(u_C)$ and $\hat{\rho}(1) = \rho(1)^d$.

and, by Theorem 2.19, when $u \leq u_C$, $x(u) = \rho_B^{1/d}$ so the following holds.

Proposition 3.1. *For $0 < u \leq u_C$, it holds that the radius of convergence of $M(z, u)$ satisfies*

$$\rho(u) = \frac{\rho_B^{1/d}}{\Phi\left(\rho_B^{1/d}, u\right)} = \begin{cases} \frac{\rho_B^{1/d}}{1+uB(\rho_B)} & \text{for all schemes but the last} \\ \rho_B^{1/d}(1 - uB(\rho_B)) & \text{for the last scheme} \end{cases}.$$

The explicit values can be found in Table 3.1.

One can check that, by computing $\rho(1)$ for all values of Table 3.1, we retrieve the (already known) radius of convergence of $M(z, 1)$ (see *e.g.* [BFSS01, Table 2]). The difference between the values in this table and those in [Sal23, Table 2] is due to the fact that in Theorem 2.15 the series M is what we call $\widehat{M}$ here (see Theorem 2.15), which satisfies that its radius of convergence is

$$\widehat{\rho}(u) = \rho(u)^d \tag{3.2}$$

and its value at its radius of convergence is

$$\widehat{M}(\widehat{\rho}(u), u) = \frac{M(\rho(u), u)}{\rho(u)} - 1.$$

Notice that the function ρ has a finite limit when $u \to 0$ (our study is restricted to $u > 0$):

$$\lim_{\substack{u \to 0 \\ u > 0}} \rho(u) = \rho_B^{1/d}.$$

As explained in Remark 2.21, in the supercritical case, one can also obtain $x(u)$ and $B(x(u)^d)$ (and therefore $\Phi(x(u), u)$) as explicit functions of u. However, as these expressions are very cumbersome, one can find it easier to take the equations relating $x(u)$, $B(x(u)^d)$

Scheme	Algebraic equation for $\rho(u)$ when $u > u_C$
1	$256\rho^4 u^3 - 32\rho^2 u^2 + 1$
2	$(u-1)^4\rho^4 - 4(9u-1)(u-1)^2\rho^3 + (70u^2+180u+6)\rho^2 + (-36u+4)\rho + 1$
3	$u^6\rho^4 + 12u^4(2u-1)\rho^3 - 6u^2(13u^2-13u-8)\rho^2 + 8\left(2u-1\right)\left(5u^2-5u+8\right)\rho - 27\left(u-1\right)^2$
4	$u\left(3u-4\right)^3\rho^4 - 2u\left(18u^2+69u-76\right)\rho^3 + (-10u^2-108u-1)\rho^2 + 2(10u+1)\rho - 1$
5	$16u^3\left(3u-4\right)^4\rho^8 - 4u^2\left(81u^4-2066u^3+2129u^2-376u+144\right)\rho^6 + (-1553u^4 + 3440u^3 + 62u^2 - 72u + 27)\rho^4 - 16u\left(19u+3\right)\rho^2 + 16$
6	$27u^5\left(3u-4\right)^3\rho^4 - 54u^3\left(u-1\right)\left(54u^3-162u^2+207u-100\right)\rho^3 + (4374u^6 - 17496u^5 + 32562u^4 - 39528u^3 + 32697u^2 - 15750u + 3125)\rho^2 + -54\left(54u^3-108u^2+117u-41\right)\left(u-1\right)^2\rho + 729\left(u-1\right)^4$
7	$19683\left(u-1\right)^{10}\rho^9 - 1458u\left(2u^3-13u^2+3520u-384\right)\left(u-1\right)^5\rho^6 + 27u^2\left(27833u^4-183232u^3+4776448u^2+4947968u+196608\right)\rho^3 - 16777216u^3$
8	$\rho^6 - 12(u-1)\rho^4 + 48\left(9u^2-9u+1\right)\left(u-1\right)^2\rho^2 - 64u^3 + 192u^2 - 192u + 64$

Table 3.2: Algebraic equations for $\rho \equiv \rho(u)$ when $u > u_C$.

Scheme	$\rho(u)$	$\rho(u)^{-d} \underset{u\to\infty}{\sim}$
1	$\frac{\sqrt{\sqrt{u}+\sqrt{u-1}}}{4u^{3/4}}$	$8u$
2	$\frac{(9u-1)\sqrt{u-1}-4\sqrt{(-5\sqrt{u}+9u^{3/2})\sqrt{u-1}-9u^2+10u-1}(u-1)^{1/4}u^{1/4}+8\sqrt{u}-8u^{3/2}}{(u-1)^{5/2}}$	u
8	$2(u-1)^{1/6}\sqrt{(u-1)^{2/3}+3u^{1/3}(u-1)^{4/3}+3u^{2/3}-u^{5/3}}$	$\frac{27}{4}u$

Table 3.3: Radius of convergence $\rho(u)$ when $u > u_C$.

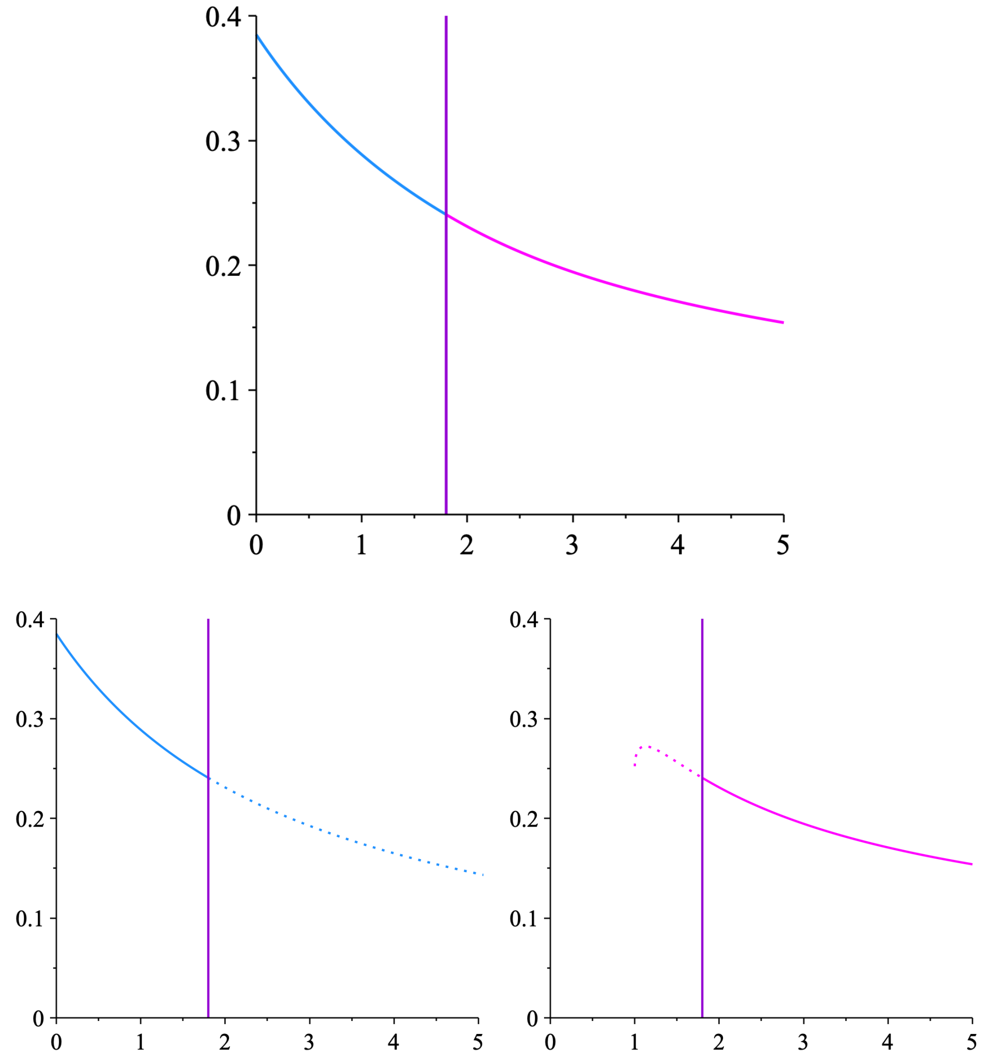

Figure 3.1: (Top) Radius of convergence $\rho(u)$ of $M(z,u)$ of the decomposition scheme 1 (general maps into 2-connected components, which is our main example in this manuscript) when $u \leq u_C$ (in blue) and when $u \geq u_C$ (in pink). The purple line corresponds to the value $u = u_C$. The lower figures show the continuation of the subcritical branch for $u > u_C$ and the supercritical branch for $u < u_C$.

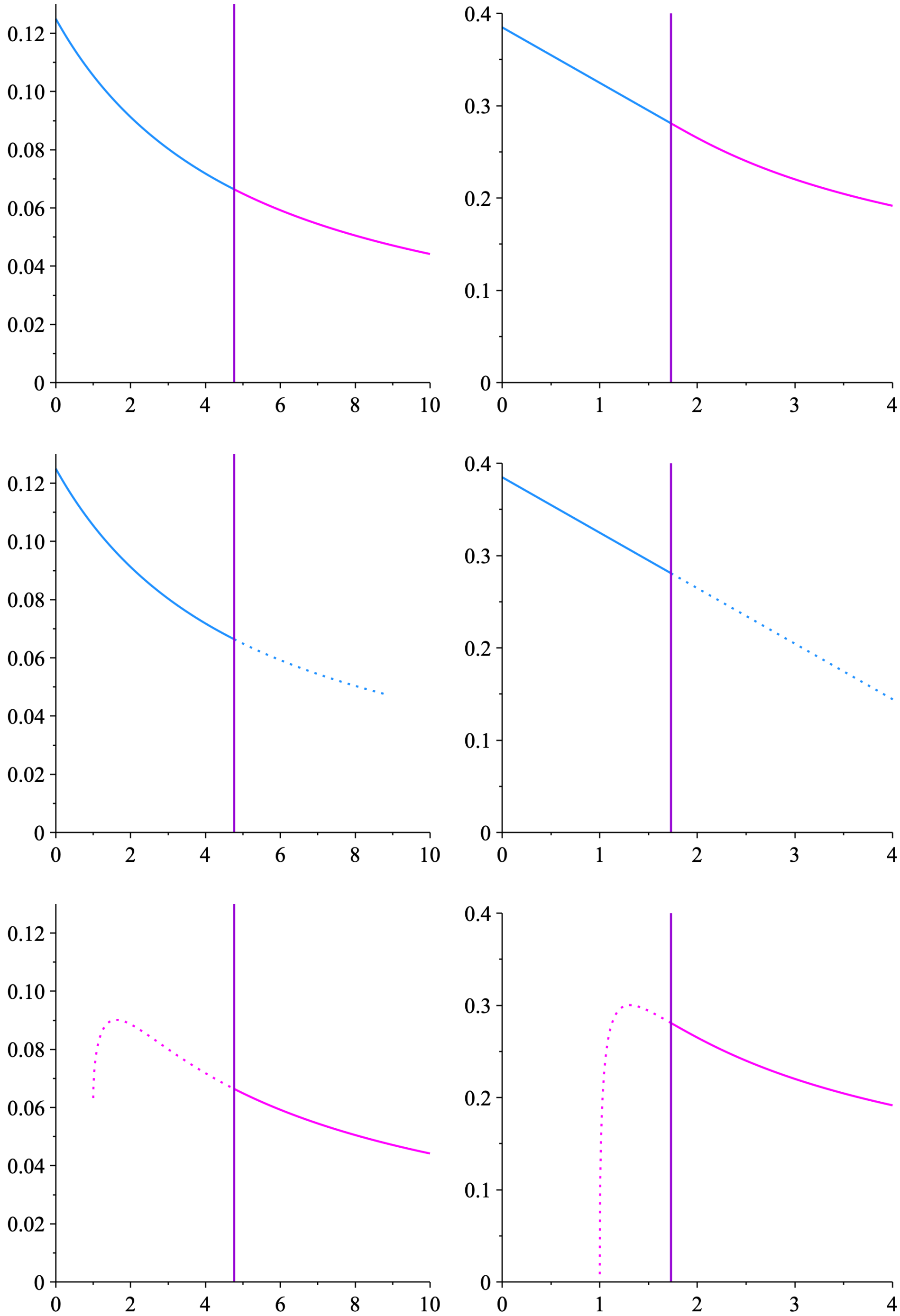

Figure 3.2: (Top) Radius of convergence $\rho(u)$ of $M(z,u)$ of the decomposition scheme 2 (on the left) and of decomposition scheme 8 (on the right) when $u \leq u_C$ (in blue) and when $u \geq u_C$ (in pink). The purple line corresponds to the value $u = u_C$. The lower figures show the continuation of the subcritical branch for $u > u_C$ (middle) and the supercritical branch for $u < u_C$ (bottom).

Scheme	u_C	c
1	$\frac{9}{5}$	$\frac{1}{\sqrt{3}}$
2	$\frac{81}{17}$	$\frac{3^{4/3}}{32}$
3	$\frac{135}{7}$	$\frac{160\times 3^{1/3}}{729}$
4	$\frac{36}{11}$	$\frac{9\times 2^{1/3}}{64}$
5	$\frac{52}{27}$	$\frac{13\times 65^{1/3}}{2^{9/2}}$
6	$\frac{68}{3}$	$\frac{425\times 2^{1/3}}{1024}$
7	$\frac{16}{7}$	$\frac{3\times 2^{1/3}\times 3^{2/3}}{16}$
8	$\frac{64}{37}$	$\frac{2^{2/3}\times 3^{5/6}}{9}$

Table 3.4: Values for the critical case.

and $B'(x(u)^d)$ and add to them the equation (3.1) relating $\rho(u)$ with $x(u)$ and $B(x(u)^d)$. Polynomial elimination manipulations (introduced in Section 1.2.2) give an explicit algebraic equation linking $\rho(u)$ and u, written in Table 3.2. And they are even algebraic equations[1] in $\rho(u)^d$, of degree at most 4. In particular, one can even obtain a closed formula with radicals for $\rho(u)$ in the supercritical case, see Table 3.3 for the cases of schemes 1, 2 and 8. One can then show that, for these schemes, $\rho(u)$ is in fact $\mathcal{C}^2$: Figs. 3.1 and 3.2 show that $\rho(u)$ looks indeed very regular around u_C.

3.1.2 Enumerative results

Now that we characterised $\rho(u)$ as the root of a polynomial equation, we can study the critical behaviour of $M(z,u)$ around $\rho(u)$, from which we derive the following singular expansion theorem. Note that the algebraicity of B (or M) is not necessary to obtain this result. In the case of general maps decomposed into 2-connected components, these asymptotics have already been obtained by Bonzom, who derived the singularity analysis solving a polynomial system obtained from the Lagrangean parametrisation of $B(y)$ [Bon16, §8.2.2], whereas the method here does not rely on the algebraicity of $B(y)$ but stems from the analysis of the non-polynomial equation $M(z,u) = z\Phi(M(z,u),u)$ (and thus can be used for the model of Chapter 6 where the series are not algebraic).

Theorem 3.2. *The series $M(z,u)$ displays the following asymptotic behaviours when z is in a Δ-domain neighbourhood of $\rho(u)$, where we let $Z = 1 - \frac{z}{\rho(u)}$.*

Subcritical regime. *When $u < u_C$, there exist $r(u), s(u) > 0$ such that*

$$M(z,u) = \rho_B^{1/d} - r(u)Z + s(u)Z^{3/2} + \Theta\left(Z^2\right),$$

[1]It is not surprising as these computations can also be done for the original $M(z,u)$, whose radius of convergence is $\rho(u)^d$.

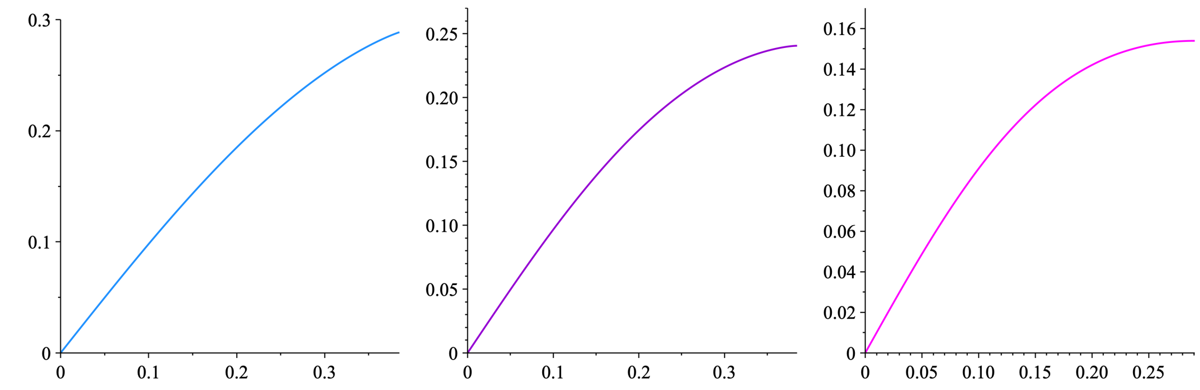

Figure 3.3: Plot of $\Psi(M)$ around its singularity M_0 for the main example: when $u < u_C$ (left), $\Psi'(M_0) \neq 0$ so there is a non-horizontal tangent at M_0, when $u = u_C$ (centre) and $u > u_C$ (right), $\Psi'(M_0) = 0$ so the tangent is horizontal at M_0. In the supercritical case, the curve is a (half-)parabola.

Critical regime. *There exists a constant c satisfying*

$$M(z, u_C) = \rho_B^{1/d} - cZ^{2/3} + \Theta\left(Z\right).$$

Supercritical regime. *When $u > u_C$, there exists $s(u) > 0$ such that*

$$M(z, u) = M(\rho(u), u) - s(u)Z^{1/2} + \Theta\left(Z\right).$$

The constants are made explicit in Lemma 3.3, Proposition 3.4, and Table 3.4.

As previously claimed, the polynomial corrections in each case correspond to the general understanding of the three phases: for $u < u_C$, one has a "general map phase", with the correction $n^{-5/2}$ typical for maps, for $u > u_C$ one has an "arborescent phase", with the correction $n^{-3/2}$ of plane trees. Moreover, at $u = u_C$, a new asymptotic behaviour emerges in between, with a polynomial correction in $n^{-5/3}$.

The rest of this subsection is devoted to proving Theorem 3.2 and deducing the asymptotic behaviour of $[z^n]M(z, u)$. Consider the functional inverse $\Psi(M)$ of $M(z)$ which satisfies

$$\Psi(M) = \frac{M}{\Phi(M)} \tag{3.3}$$

as

$$M = z\Phi(M) \quad \text{so} \quad z = \frac{M}{\Phi(M)} = \Psi(M).$$

The behaviour of $\Psi(M)$ around its singularity M_0 gives the behaviour of M around its singularity. Three distinct situations arise, depending on the position of u with respect to u_C; they are illustrated on Fig. 3.3. For instance, if the singular behaviour of $\Psi(M)$ is quadratic, such as in the supercritical case, we write

$$z - z_0 = \Psi(M) - \Psi(M_0) \sim -c(M - M_0)^2$$

and all that remains is to invert to obtain the behaviour of M as a function of z:

$$M - M_0 \sim -C\sqrt{z_0 - z}.$$

This is now discussed formally:

Proof of Theorem 3.2. The expansions of the $M(z)$ and $B(y)$ series around their singular points are known [BFSS01, Table 2] (therefore, so is that of $\Phi(x,u)$), so one can use (2.26) to obtain the asymptotic expansion of $M(z,u)$.

In the terminology of Meir and Moon [MM78], Equation (2.26) indicates that the class under consideration is a *simple variety of trees* and it is known that, in this case, the asymptotic development of $M(z,u)$ depends on whether the derivative $\Psi'(x,u)$ cancels out in $(0, \rho_\Phi(u))$ [FS09, §VII.3], where $\Psi(x,u)$ is the functional inverse of $M(z,u)$ and satisfies

$$\Psi(x,u) = \frac{x}{\Phi(x,u)}. \tag{3.4}$$

However, using (2.34), it holds that:

$$\Psi'(x,u) = 0 \iff \Phi(x,u) - t\Phi'(x,u) = 0 \iff \mathbb{E}\left[\mu^{x,u}\right] = 1. \tag{3.5}$$

Theorem 2.19 and the preceding discussion show that this equation has a solution if and only if $u \geq u_C$, and that this solution is in $(0, \rho_\Phi(u))$ if and only if $u > u_C$. In this case, $x(u)$ is a singularity by the contraposition of the implicit function theorem.

Supercritical case. The latter case is straightforward as it falls into the *smooth-inverse function schema* [FS09, §VII.3, Def VII.3], so, by [FS09, §VII.3, Th VII.2], the series $M(z,u)$ has a dominant singularity of square-root type at $\rho(u)$ and the asymptotic development is as announced with the following expression for $s(u)$. Given the definition of Φ in (2.25), Φ is twice differentiable on the interior of its disk of convergence. Therefore, the latter expression does make sense because it always holds that $x(u) < \rho_B^{1/d}$ in the supercritical case.

Lemma 3.3. *In the supercritical case, it holds that*

$$s(u) = \sqrt{\frac{2\Phi(x(u),u)}{\Phi''(x(u),u)}}.$$

Subcritical and critical cases. In these cases, there is no solution to (3.5) on $[0, \rho_\Phi(u))$. When x is in a neighbourhood $x(u) = \rho_\Phi(u) = \rho_B^{1/d}$, x^d is in a neighbourhood of ρ_B so one can make use of the expansion of $B(y)$ around ρ_B. Remember (2.37):

$$B(y) = B(\rho_B) - \rho_B B'(\rho_B)Y + \gamma_B Y^{3/2} + O(Y^2).$$

As a consequence, for all schemes but the last, denoting $Y(x) = 1 - \frac{x^d}{\rho_B}$,

$$\Phi(x,u) = 1 + uB(\rho_B) - u\rho_B B'(\rho_B)Y(x) + u\gamma_B Y(x)^{3/2} + O(Y(x)^2);$$

and, in the last case, as was obtained in (2.41),

$$\Phi(x,u) = \frac{1}{1-uB(\rho_B)} - \frac{u\rho_B B'(\rho_B)}{(1-uB(\rho_B))^2}Y(x) + \frac{u\gamma_B}{(1-uB(\rho_B))^2}Y(x)^{3/2} + O(Y(x)^2).$$

Inverting and bootstrapping then give the singular expansion of $M(z,u)$. We now explain how one can derive the singular expansion of $M(z,u)$ from a singular expansion of $\Phi(x,u)$ of the form

$$\Phi(x,u) = \alpha(u) + \beta(u)Y(x) + \gamma(u)Y(x)^{3/2} + O(Y(x)^2),$$

where $\alpha(u) = \Phi(x(u),u)$, and $\beta(u)$ and $\gamma(u)$ are explicit rational fractions in u and $Y(x) = 1 - \frac{x^d}{x(u)^d}$. Using (3.4) and writing $x = x(u)(1-Y(x))^{1/d}$, this can be rephrased as, when x is in a neighbourhood of $x(u)$,

$$\Psi(x,u) = \frac{x}{\Phi(x,u)} = \rho(u)\left(1 - \frac{\alpha(u)+d\beta(u)}{d\alpha(u)}Y(x) - \frac{\gamma(u)}{\alpha(u)}Y(x)^{3/2} + O(Y(x)^2)\right) \tag{3.6}$$

since $x(u)/\alpha(u) = \rho(u)$. Then, when z is in a neighbourhood of $\rho(u)$, by continuity, $M(z,u) \to M(\rho(u),u) = x(u)$), so one can write

$$\frac{\Psi(M(z,u),u)}{\rho(u)} = 1 - \frac{\alpha(u)+d\beta(u)}{d\alpha(u)}\widehat{Y}(z) - \frac{\gamma(u)}{\alpha(u)}\widehat{Y}(z)^{3/2} + O(\widehat{Y}(z)^2),$$

with $\widehat{Y}(z) = Y(M(z,u)) = 1 - \frac{M(z,u)^d}{x(u)^d}$. Since Ψ is the compositional inverse of $M(z,u)$, it holds that $\Psi(M(z,u),u) = z$ so

$$Z := 1 - \frac{z}{\rho(u)} = \frac{\alpha(u)+d\beta(u)}{d\alpha(u)}\widehat{Y}(z) + \frac{\gamma(u)}{\alpha(u)}\widehat{Y}(z)^{3/2} + O(\widehat{Y}(z)^2). \tag{3.7}$$

Subcritical case It is now possible to bootstrap this result. By definition of $\widehat{Y}(z)$, it holds that $M(z,u) = x(u)(1-\widehat{Y}(z))^{1/d}$, so understanding the behaviour of $Y(z)$ is sufficient to conclude. It holds that, when $u < u_C$,

$$Z \sim \frac{\alpha(u)+d\beta(u)}{d\alpha(u)}\widehat{Y}(z) \qquad \textit{i.e.} \qquad \widehat{Y}(z) \sim \delta(u)Z$$

for an explicit $\delta(u)$. Therefore,

$$\widehat{Y}(z)^{3/2} = \delta(u)^{3/2}Z^{3/2} + o(Z^{3/2}) \quad \text{and} \quad O(\widehat{Y}(z)^2) = O(Z^2)$$

so, multiplying each side of (3.7) by $\delta(u)$, it holds that

$$\delta(u)Z = \widehat{Y}(z) + \delta(u)\frac{\gamma(u)}{\alpha(u)}\delta(u)^{3/2}Z^{3/2} + O(Z^2),$$

which gives the singular expansion of $\widehat{Y}(z)$ in function of Z.

Finally,

$$M(z,u) = x(u)(1-\widehat{Y}(z))^{1/d} = x(u) - \frac{x(u)\delta(u)}{d}Z + \frac{x(u)\gamma(u)\delta(u)^{5/2}}{d\alpha(u)}Z^{3/2} + O(Z^2)).$$

Furthermore, for all schemes, it holds that

$$\frac{1}{\delta(u)} = \frac{\alpha(u)+d\beta(u)}{d\alpha(u)} = \frac{1-E(u)}{d}$$

so

$$r(u) = \frac{x(u)}{1-E(u)} = \frac{\rho_B^{1/d}}{1-E(u)}.$$

Moreover, it holds that

$$\frac{\gamma(u)}{\alpha(u)} = \frac{u\gamma_B}{1 \pm uB(\rho_B)}$$

where the $\pm$ is a $+$ for all schemes except the last, where it is a $-$; so

$$s(u) = \frac{u\gamma_B\rho_B^{1/d}d^{3/2}}{(1-E(u))^{5/2}(1 \pm uB(\rho_B))}.$$

Making $1-E(u)$ appear helps in understanding why this result is only valid in the subcritical case: in the critical case, $1-E(u)=0$.

Critical case In this case, (3.7) can be rewritten as follows:

$$Z := 1 - \frac{z}{\rho(u_C)} = \frac{\gamma(u_C)}{\alpha(u_C)}\widehat{Y}(z)^{3/2} + O(\widehat{Y}(z)^2), \tag{3.8}$$

so it now holds that

$$Z \sim \frac{\gamma(u_C)}{\alpha(u_C)}\widehat{Y}(z)^{3/2} \qquad \textit{i.e.} \qquad \widehat{Y}(z) \sim \left(\frac{\gamma(u_C)}{\alpha(u_C)}\right)^{-2/3} Z^{2/3} \quad \text{so} \quad O(\widehat{Y}(z)^2) = O(Z^{4/3}).$$

Therefore, (3.8) gives

$$\left(\frac{\gamma(u_C)}{\alpha(u_C)}\right)^{2/3} \widehat{Y}(z) = \left(Z + O(Z^{4/3})\right)^{2/3} = Z^{2/3} + O(Z)$$

which finally gives, using $M(z,u) = x(u)(1-\widehat{Y}(z))^{1/d}$,

$$M(z,u_C) = x(u_C) - \frac{x(u_C)}{d}\left(\frac{\alpha(u_C)}{\gamma(u_C)}\right)^{2/3} Z^{2/3} + O(Z).$$

Thus, one gets, using (2.31),

$$c = \frac{\rho_B^{1/d}}{d}\left(\frac{1 \pm u_C B(\rho_B))}{u_C\gamma_B}\right)^{2/3} = \frac{\rho_B^{1/d}}{d\gamma_B^{2/3}}(d\rho_B B'(\rho_B))^{2/3} = \frac{\rho_B^{1/d+2/3}B'(\rho_B)^{2/3}}{d^{1/3}\gamma_B^{2/3}}.$$

□

In particular, the proof of Theorem 3.2 gives that

Proposition 3.4. *In the subcritical case, using notation of Theorem 3.2 it holds that*

$$r(u) = \frac{\rho_B^{1/d}}{1 - E(u)}$$

where $E(u)$ is defined in (2.32); and, using γ_B defined in (2.37), one has

$$s(u) = \frac{u\gamma_B \rho_B^{1/d} d^{3/2}}{(1 - E(u))^{5/2}(1 \pm uB(\rho_B))} \tag{3.9}$$

where the $\pm$ is a $+$ for all schemes except the last, where it is a $-$. In the critical case, the constant c of Theorem 3.2 satisfies

$$c = \frac{\rho_B^{1/d+2/3} B'(\rho_B)^{2/3}}{d^{1/3}\gamma_B^{2/3}}.$$

The explicit values for each case are written in Table 3.4.

Contrary to what happens in the simplest case, for every fixed $u > 0$, the radius of convergence $\rho(u)$ is not the only *dominant* singularity of the series $z \mapsto M(z, u)$ if $d \neq 1$, *i.e.* the series can have singularities distinct from $\rho(u)$ on the closed disk $\{z \mid |z| \leq \rho(u)\}$. This is due to the Lagrangean equation (2.26) being periodic, in the sense that Φ is periodic. Taking that into account, the singular developments allows to conclude with the transfer theorem to get the asymptotic expansion of $[z^n]M(z, u)$, which was stated with less details in [Sal23, Theorem 1].

Proposition 3.5. *Using the notation of Theorem 3.2, it holds that, when $n \to \infty$ with $n \equiv 1[d]$,*

Subcritical regime.

$$[z^n]M(z, u) \sim d\,\frac{3s(u)}{4\sqrt{\pi}} n^{-5/2} \rho(u)^{-n};$$

Critical regime.

$$[z^n]M(z, u_C) \sim d\,\frac{c\Gamma(2/3)}{\sqrt{3}\pi} n^{-5/3} \rho(u_C)^{-n};$$

Supercritical regime.

$$[z^n]M(z, u) \sim d\,\frac{s(u)}{2\sqrt{\pi}} n^{-3/2} \rho(u)^{-n}.$$

Remark 3.6. One can use the results of Proposition 3.5 to immediately deduce the analogous formulae for $[z^n]\widehat{M}(z,u)$. By (2.24), for all $n \in \mathbb{N}_{>0}$,

$$[z^n]\widehat{M}(z,u) = [z^{dn+1}]M(z,u).$$

Moreover, as expressed in (3.2), $\widehat{\rho}(u) = \rho(u)^d$, so one gets that, as $n \to \infty$,

$$[z^n]\widehat{M}(z,u) \sim \begin{cases} \frac{3s(u)}{4\sqrt{\pi}d^{3/2}\rho(u)} n^{-5/2}\widehat{\rho}(u)^{-n} & \text{if} \quad u < u_C \\ \frac{c\Gamma(2/3)}{\sqrt{3}\pi d^{2/3}\rho(u_C)} n^{-5/3}\ \widehat{\rho}(u_C)^{-n} & \text{if} \quad u = u_C \\ \frac{s(u)}{2\sqrt{\pi}d^{1/2}\rho(u)} n^{-3/2}\widehat{\rho}(u)^{-n} & \text{if} \quad u > u_C \end{cases}.$$

Proof of Proposition 3.5. The series $M(z,u)$ has $\rho(u)$ as its smallest singularity on the positive real line. To find its dominant singularities, its suffices to look at the circle $\{z \in \mathbb{C} \mid |z| = \rho(u)\}$.

The series $M(z,u)$ is periodic of period d and singular in $\rho(u)$, so it is singular in all the $\omega\rho(u)$, where the ω span the set $\mathbb{U}_d$ of the d-th roots of unity.

There are two sources of singularity for $M(z,u)$: either a singularity of Φ is reached; or the value of the series satisfies

$$\frac{M(z,u)\Phi'(M(z,u),u)}{\Phi(M(z,u),u)} = 1.$$

If $z \in \{z \in \mathbb{C} \mid |z| = \rho(u)\} \setminus \{\omega\rho(u) \mid \omega \in \mathbb{U}_d\}$, then by the Daffodil Lemma, it holds that

$$|M(z,u)| < M(\rho(u),u) = x(u).$$

Therefore, it holds that $M(z,u)$ does not reach a singularity of Φ, nor does $\frac{M(z,u)\Phi'(M(z,u),u)}{\Phi(M(z,u),u)}$ reach 1. Therefore, z cannot be a singularity of $M(z,u)$.

This concludes the search of the dominant singularities of $M(z,u)$, which are the elements of $\{\omega\rho(u) \mid \omega \in \mathbb{U}_d\}$. Since the coefficients $[z^n]M(z,u)$ are non-zero only for $n \equiv 1[d]$, it holds that, for $\omega \in \mathbb{U}_d$,

$$M(\omega z,u) = \omega M(z,u),$$

it holds immediately that for z in a neighbourhood of $\omega\rho(u)$, if $u < u_C$:

$$M(z,u) = \omega\rho_B^{1/d} - \omega r(u)Z + \omega s(u)Z^{3/2} + \Theta\left(Z^2\right),$$

and, if $u = u_C$,

$$M(z,u_C) = \omega\rho_B^{1/d} - \omega c Z^{2/3} + \Theta\left(Z\right).$$

Therefore, the transfer theorem gives, in the subcritical case

$$[z^n]M(z,u) \underset{n\to\infty}{\sim} \sum_{\omega\in\mathbb{U}_d} \omega\frac{3s(u)}{4\sqrt{\pi}} n^{-5/2}(\omega\rho(u))^{-n};$$

and in the critical case

$$[z^n]M(z,u_C) \underset{n\to\infty}{\sim} \sum_{\omega\in\mathbb{U}_d} \omega c\frac{\Gamma(2/3)}{\sqrt{3}\pi} n^{-5/3}(\omega\rho(u_C))^{-n}.$$

Notice that, for any $n \in \mathbb{N}_0$,

$$\sum_{\omega \in \mathbb{U}_d} \omega^{-n+1} = \begin{cases} 0 & \text{if} \quad d \nmid n-1 \\ d & \text{if} \quad d \mid n-1 \end{cases} \quad ;$$

which allows to conclude.

Finally, for the supercritical case, the periodic version of the smooth-inverse function schema gives immediately the result [FS09, §VI.7, Note VI.17 and §VII.3, Th VII.2]. □

The Lagrange inversion formula, although a potential method, is not straightforward here as it necessitates prior knowledge of the asymptotics of the expansion of $\Phi(x,u)^n$, which is *a priori* unknown and would for example require a saddle point method.

3.1.3 Bijective comments

Finally, we end this section with bijective arguments to recover the asymptotic behaviours of $\rho(u)$ when $u \to \infty$ for the cases of Table 3.3.

In all generality, estimating $[z^n]M(z,u)$ by combinatorial considerations is not easy, but it is a lot easier when $u \to \infty$. This allows us to obtain in another way the equivalent of the radius of convergence $\rho(u)$ when $u \to \infty$ (and therefore allows to check the previous computations). When $u \to \infty$, the maps are in a tree-like phase: they tend to behave like trees, in which each edge is a block. So, when $u \to \infty$, maps of size dn have around n blocks and therefore the asymptotic behaviour of the coefficient $[z^{dn}]M(z,u)$ is dominated by the contribution of $[z^{dn}u^n]M(z,u)$ when $u \to \infty$. This heuristic is true computationally as long as $[z^{dn}u^n]M(z,u) \neq 0$.

On the one hand, for the decomposition of general maps into 2-connected components, the computation from the expression of $\rho(u)$ gives, when $u \to \infty$,

$$\rho(u)^{-d} = \rho(u)^{-2} \sim 8u.$$

On the other hand, the class of maps with n edges and n 2-connected blocks can be bijectively characterised. This class is larger that the class of plane trees as loops are authorized. In fact, there are 2 types of size-1 2-connected components: edges $\mathfrak{e}$ (with two distinct extremities) and loops $\mathfrak{l}$. Such a map $\mathfrak{m}$ has block-tree $T_{\mathfrak{m}}$ whose internal node have as decoration $\mathfrak{e}$ or $\mathfrak{l}$, and have $\mathrm{s}(\mathfrak{e}) = \mathrm{s}(\mathfrak{l}) = 2$ children. Moreover, a binary tree with $2n$ edges has n internal nodes, and there are $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ such trees, with Cat_n being the n-th Catalan number. Since the n nodes have 2 possible decorations, the total number of possibilities for such block trees is $2^n \mathrm{Cat}_n$. Since block trees are in bijection with their corresponding maps, this concludes the enumeration.

Proposition 3.7. *For $n \in \mathbb{N}_{>0}$, the number of maps with n edges and n 2-connected*

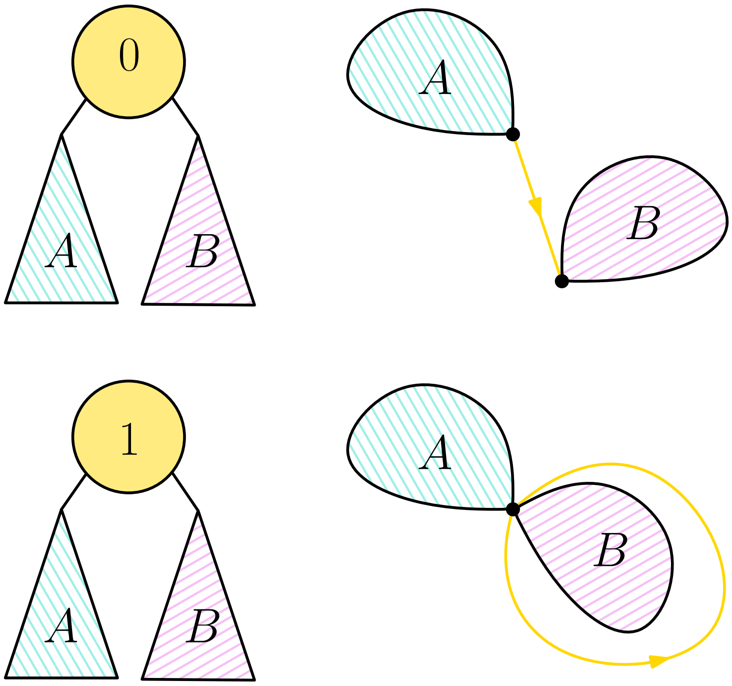

Figure 3.4: Bijection between rooted plane binary trees with n internal nodes where each internal node has a label in $\{0, 1\}$ (on the left) and maps of size n with and n blocks (on the right).

blocks is

$$[z^{dn}u^n]M(z,u) = 2^n \mathrm{Cat}_n = \frac{2^n}{n+1}\binom{2n}{n} \underset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi}} 8^n n^{-3/2}.$$

There is no need to go through the block tree to prove this result as one can write explicitly a bijection between binary trees with internal nodes labelled 0 or 1 and maps of size n with n blocks: this is the object of Fig. 3.4.

Thus, one gets that, when $u \to \infty$,

$$[z^{2n}]M(z,u) \approx u^n[z^{dn}u^n]M(z,u) \bowtie 8^n u^n = (8u)^n \approx \rho(u)^{-dn},$$

where $(a_n) \bowtie R^n$, read (a_n) *is of exponential order of* R^n, means that $\limsup |a_n|^{1/n} = R$. This allows to recover the asymptotic behaviour of $\rho(u)$ when $u \to \infty$.

The case of simple triangulations decomposed into irreducible components is similar, but this time the bijection is with *ternary trees* and there is only one block of size 1, so these trees need no decoration. More precisely, simple triangulations of size n are in bijection with ternary trees with n internal nodes, as shown by Fig. 3.5. Therefore, it holds that:

Proposition 3.8. *For $n \in \mathbb{N}_{>0}$, the number of simple triangulations with $n+3$ vertices and n irreducible blocks is*

$$[z^{dn}u^n]M(z,u) = \frac{1}{2n+1}\binom{3n}{n} \underset{n\to\infty}{\sim} \frac{\sqrt{3}}{4\sqrt{\pi}}\left(\frac{27}{4}\right)^n n^{-3/2}.$$

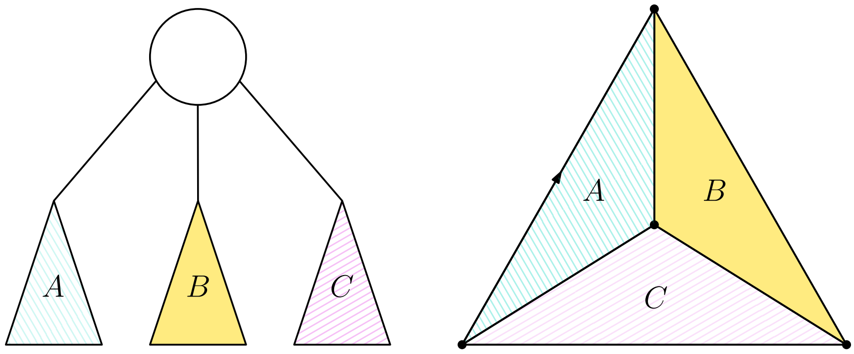

Figure 3.5: Bijection between rooted plane ternary trees with n internal nodes (on the left) and simple triangulations of size n with and n blocks (on the right).

Thus, one gets that, when $u \to \infty$

$$[z^n]M(z,u) \approx u^n[z^nu^n]M(z,u) \bowtie \left(\frac{27}{4}u\right)^n \approx \rho(u)^{-dn};$$

while computation from the closed form of $\rho(u)$ in the supercritical case gives

$$\rho(u)^{-d} \underset{u\to\infty}{\sim} \frac{27}{4}u.$$

Finally, the case of loop-free maps decomposed into simple blocks is unfortunately not very interesting: the only loop-free map of size $dn = n$ with n blocks is the map composed of two vertices connected by n parallel edges. Therefore,

$$[z^nu^n]M(z,u) = 1.$$

However, it still gives the asymptotic behaviour of $\rho(u)$ when $u \to \infty$ as

$$[z^n]M(z,u) \approx u^n[z^nu^n]M(z,u) \bowtie u^n \approx \rho(u)^{-dn};$$

which is indeed what the closed form of $\rho(u)$ gives.

3.2 Enumeration by probabilistic approach

Enumerative aspects can also be obtained using the probabilistic method, but for the subcritical case this method requires an estimate of the size of the largest blocks. Write $m_{n,u} = [z^n]M(z,u)$.

By Proposition 2.17, the probability to draw a map of size n according to $\mathbb{P}_u$ can be expressed in two distinct ways

- Using the definition of $\mathbb{P}_u$:

$$\sum_{\mathfrak{m}\in\mathcal{M}_n} \mathbb{P}_u(\mathfrak{m}) = \sum_{\mathfrak{m}\in\mathcal{M}_n} \frac{u^{b(\mathfrak{m})}\rho(u)^n}{M(\rho(u),u)} = \frac{m_{n,u}\rho(u)^n}{M(\rho(u),u)};$$

- As the probability that a μ^u-Bienaymé–Galton–Watson tree has size $n-1$: $\mathbb{P}\left(|GW(\mu^u)| = n-1\right)$.

Therefore,

$$m_{n,u} = M(\rho(u), u)\rho(u)^{-n}\mathbb{P}\left(|GW(\mu^u)| = n-1\right).$$

Depending on the position of u relative to u_C, the quantity $\mathbb{P}\left(|GW(\mu^u)| = n-1\right)$ does not have the same asymptotic behaviour. Let S_n denote the random variable equal to the sum of the sum of n *i.i.d.* variables ξ_i of law $\widehat{\mu^u}(k) := \mu^u(k+1)$ supported on $\{-1\} \cup \mathbb{N}_0$. The probability of obtaining a μ^u-Bienaymé–Galton–Watson tree of size $n-1$ is the probability that the random walk with increments ξ_i always remains (strictly) above -1 except after the n-th step where it reaches -1. Using the Cycle Lemma [DM47], one can remove the conditioning and get, for $n \equiv 1[d]$:

$$\mathbb{P}\left(|GW(\mu^u)| = n-1\right) = \frac{1}{n}\mathbb{P}\left(S_n = -1\right).$$

We conclude the proof by case-by-case analysis using local limit theorems for S_n, as follows.

Subcritical case. In this case, the asymptotic properties of μ^u described in Proposition 2.22 show that there is a form of *condensation*: one block condenses a macroscopic portion of the mass, and all the others behave like *i.i.d.* variables. The probability of obtaining this size is asymptotically the same as that of obtaining this linear jump, which is around $(1-E(u))n$ as $n \to \infty$ [JS11]. This jump can happen in any of the n possible positions. This case can be shown to be dominant, which is done in Chapter 4 in the context of the study of the largest block. Therefore, by Proposition 2.22,

$$\mathbb{P}\left(S_n = -1\right) \underset{n\to\infty}{\sim} n\, c(u) \left(\frac{(1-E(u))n}{d}\right)^{-5/2} = \frac{d^{5/2}c(u)}{(1-E(u))^{5/2}} n^{-3/2}$$

so

$$m_{n,u} \underset{n\to\infty}{\sim} \frac{d^{5/2}x(u)c(u)}{(1-E(u))^{5/2}} n^{-5/2}\rho(u)^{-n}$$

which indeed what is stated in Proposition 3.5 as, for all schemes, it holds that

$$d\,\frac{3s(u)}{4\sqrt{\pi}} = \frac{d^{5/2}x(u)c(u)}{(1-E(u))^{5/2}}$$

by Proposition 3.1 and Equations (2.39) and (3.9).

Critical case. In this case, S_n behaves like a $3/2$-stable random walk, which already been studied in detail by Janson [Jan12, Example 19.27], which gives the following local limit result:

$$\mathbb{P}\left(S_n = -1\right) \underset{n\to\infty}{\sim} (c(u_C)\Gamma(-3/2))^{-2/3}|\Gamma(-2/3)|^{-1}n^{-2/3} = \frac{2^{2/3}3^{1/6}\Gamma(2/3)}{4\pi^{4/3}c(u_C)^{2/3}} n^{-2/3},$$

so

$$m_{n,u_C} \underset{n\to\infty}{\sim} \frac{2^{2/3}3^{1/6}\Gamma(2/3)}{4\pi^{4/3}c(u_C)^{2/3}} M(\rho(u_C), u_C) n^{-5/3}\rho(u_C)^{-n}.$$

This is the same result than in Proposition 3.5 since $x(u_C) = \rho_B^{1/d}$ and, by Equations (2.31) and (2.39), for all schemes it holds that

$$c(u_C) = \frac{3\gamma_B}{4\sqrt{\pi} d B'(\rho_B)\rho_B}. \tag{3.10}$$

Supercritical case. In this case, by Proposition 2.22, μ^u has finite variance $\sigma(u)^2$ so S_n falls in the scope of a local limit theorem [GK54, p. 233] and, as $n \to \infty$,

$$\mathbb{P}(S_n = -1) \underset{n\to\infty}{\sim} \frac{d}{\sqrt{2\pi n}\sigma(u)},$$

so

$$m_{n,u} \sim d\frac{M(\rho(u),u)}{\sqrt{2\pi}\sigma(u)}\rho(u)^{-n}n^{-3/2}.$$

which is the result of Proposition 3.5, since, by (2.40),

$$\frac{M(\rho(u),u)}{\sqrt{2\pi}\sigma(u)} = \frac{x(u)}{\sqrt{2\pi}x(u)\sqrt{\frac{\Phi''(x(u),u)}{\Phi(x(u),u)}}} = \frac{1}{2\sqrt{\pi}}\sqrt{\frac{2\Phi(x(u),u)}{\Phi''(x(u),u)}} = \frac{s(u)}{2\sqrt{\pi}}.$$

Notice that this holds as well for the last case with $u \geq v_C$ by Proposition 2.23.

3.3 Application to complexity analysis of sampling

Now that we have studied the behaviour of $[z^n]M(z,u)$, one can answer questions along the lines of "what is the probability that a map drawn by the Boltzmann generator of Section 2.2.3 is of size n". Indeed, maps drawn according to $\mathbb{P}_u$ have size n with probability

$$\frac{[z^n]M(z,u)\cdot\rho(u)^n}{M(\rho(u),u)} = \begin{cases} \Theta(n^{-5/2}) & \text{if} \quad u < u_C \\ \Theta(n^{-5/3}) & \text{if} \quad u = u_C \\ \Theta(n^{-3/2}) & \text{if} \quad u > u_C \end{cases} \tag{3.11}$$

by (1.4) and Proposition 3.5. Section 2.2.3 presents the Boltzmann generator $\Gamma M(\rho(u),u)$ for drawing quadrangulations according to $\mathbb{P}_u$, *i.e.* with a weight u on simple blocks. The generator uses Algorithms 6, 7 and 9.

We propose an approximate-size random sampler drawing under $\mathbb{P}_u$ conditioned on the size being in a target interval $[\![n, 2n]\!]$, and analyse its complexity.

Here, we consider the *real-arithmetic model*, *i.e.* we consider that the addition, subtraction, multiplication and division of two reals have a unit cost. In particular, in this model, calculations between reals are carried out in an exact way and one does not raise questions about the representation of the reals by a machine or the precision of the calculations. We also assume that sampling a "uniform" real number in $[0,1]$ has a unit cost. As explained in the end of Section 2.2.3, this study is based on a linear-time exact-size sampler for the blocks, which can be found in Fusy's work for the case of simple quadrangulations [Fus07, §4.2].

More precisely, the Boltzmann framework allows approximate-size and exact-size sampling through *rejection*, *i.e.* drawing a map without conditioning its size; then, if the result is not of the requested size, starting again until getting a suitable map. In this section, we do not try to sample a map of size exactly n, but relax the constraint by asking to obtain a map of size between n and $2n$, where n is a value chosen by the user. We show here how to obtain a linear complexity for approximate-size sampling in the supercritical case, which is not trivial because of the substitution operation. We also discuss the complexity for the other cases.

As detailed in the original article [DFLS03], another classical method from the Boltzmann toolbox to fasten rejection is *truncation*: instead of building the whole object and then checking if the size is acceptable, one slightly modifies the generator to keep track of the current size and rejects as soon as it exceeds the bound k (which is an additional input parameter). When the number of atoms generated is greater than k, the algorithm terminates immediately and returns an error symbol $\perp$. This algorithm with a ceiling is called $\Gamma M^{\leq k}$ and avoids generating very large objects and then rejecting them.

The generator can therefore be written as follows.

Algorithm 10 Sampling of a map of size in $[\![n, 2n]\!]$

repeat
  $\mathfrak{m} = \Gamma M^{\leq 2n}(\rho(u), u)$ // *call Algorithm 6*
until $|\mathfrak{m}| \geq n$ (**and** $|\mathfrak{m}| \leq 2n$)
return $\mathfrak{m}$

Algorithm 10 takes as input an integer n and returns a map $\mathfrak{m}$, which, conditionally on its size $s = |\mathfrak{m}| \in [\![n, 2n]\!]$, is sampled according to $\mathbb{P}_{s,u}$.

The map can be rejected if the sum of the sizes of the blocks drawn is either less than n or greater than $2n$ (using early rejection in the latter case). For each size, the choice of the block of the size in question has no influence on the rejection. One can therefore avoid useless operations by first finding a sequence of block sizes that creates an admissible size for the map, and only then drawing each of the blocks of the requested sizes. This way, sampling blocks (as opposed to sampling *block sizes*) only happens once. The cost of drawing blocks is, for each block, linear with the size of the block. The sum of the block sizes is between n and $2n$, so the total complexity of these steps is $\Theta(n)$. It remains to determine the cost of finding a sequence of block sizes which has the right final sum (and which respects the desired Boltzmann law).

As explained in Section 2.2.3, the block size follows the law μ^u. The sequence of block sizes is the sequence of degrees of the block tree, and as such it must verify that the sequence of degrees minus one is a Łukasiewicz path. Informally, this means that one cannot create new nodes when all the existing nodes in the tree already have all their children. In other words, the final size is obtained when all the existing nodes are filled, and the goal is for this to happen when the total size is between n and $2n$. Once again, as soon as the sum exceeds $2n$, one can stop prematurely and start again.

There are therefore two elements to take into account in order to understand the cost of this operation: the cost of drawing a random variable according to the μ^u distribution, and the number of draws required to obtain an admissible sequence. It is classical that when using rejection, the complexity of the algorithm then depends on the number of unsuccessful attempts made before drawing an object of the desired size, and on the cost of a single attempt. Formally, they are related by this classical result for rejection generators:

Lemma 3.9 (Folklore). *Let $\mathfrak{A}$ be an algorithm returning a random output, and let p be the probability that the output of $\mathfrak{A}$ is in a target set $\mathcal{S}$. Consider the algorithm $\mathfrak{A}'$ that runs algorithm $\mathfrak{A}$ until the output is in $\mathcal{S}$. Then expected cost of $\mathfrak{A}'$ satisfies*

$$\mathbb{E}\left[\mathrm{cost}(\mathfrak{A}')\right] = \frac{\mathbb{E}\left[\mathrm{cost}(\mathfrak{A})\right]}{p}.$$

Sampling according to μ^u has a different cost depending on the value of u.

Complexity analysis for the supercritical case. In the supercritical case, as is shown in Section 4.1.3, the sampled values are asymptotically almost surely bounded by $K(u)\ln(n)$ for an explicit (simple) function $K(u)$. One can therefore pre-compute the values of the coefficients $\mu^u(k)$ for $k \in [\![0, K(u)\ln(n)]\!]$ which gives the distribution function of μ^u, then draw a real number between $[0,1]$, and determine the value of k corresponding to the value. The $\Theta(\ln(n))$ values are pre-computed once and for all, and then one can make successive comparisons to find the corresponding k. To find k, $k+1$ operations are required (one could even reduce this cost by using a dichotomy). Each sequence of values according to μ^u therefore has a total cost limited by $O(n)$ because the sum never (except with exponentially small probability) exceeds $2n-1+K(u)\ln(n)$ (size of an output which does not make the algorithm stop and size of the last number sampled).

In the supercritical case, the probability that the procedure gives a map of size k is $\Theta(k^{-3/2})$ and the cost of drawing a corresponding sequence is $\Theta(k)$. So the expected cost of a sample is

$$\sum_{k=1}^{2n} k^{-3/2}k + \sum_{k>2n} O(n)k^{-3/2} = O(\sqrt{n}), \tag{3.12}$$

and the sample is rejected with probability $\Theta(n^{-1/2})$, so by Lemma 3.9, the cost of this phase is also $\Theta(n)$.

Theorem 3.10. *When $u > u_C$, the cost in the real-arithmetic model of the generator $\Gamma M(\rho(u), u)$ with rejection to obtain an object with size between n and $2n$ is $\Theta(n)$ as $n \to \infty$.*

Remark 3.11. In the supercritical case, one could obtain linear complexity for a fixed-size generator, using Devroye's method [Dev12]. Devroye showed that it is possible to have linear global complexity for sampling Bienaymé–Galton–Watson trees whose reproduction laws have finite variance. The method proceeds by drawing (sequentially) for all d the number n_d of nodes of degree d, each following a binomial distribution. This makes it possible to construct the *profile* of the tree: the multiset of the degrees of the internal nodes. This is repeated until the multiset of the degrees with the fact that a tree has one edge less than vertices. This multiset is then randomly shuffled and the resulting sequence is transformed into a walk describing a tree by the cycle lemma. One of the difficulties lie in drawing variables according to a binomial distribution with $O(1)$ cost.

This line has been studied in detail by Panagiotou, Ramzews and Stufler [PRS23] in the general framework of so-called *subcritical classes* that can be bijectively encoded by *enriched trees*, a class of decorated planar trees; to which our supercritical case belongs. They present sampling procedures for uniformly generating objects of these classes with a given size n in an expected time $O(n)$ (assuming cost $O(1)$ to draw a binomial random variable), using a combination of the Devroye tree sampler and Boltzmann sampling techniques via decorated Bienaymé–Galton–Watson trees.

Sampling according to μ^u in the subcritical and critical cases. In the subcritical and critical cases, the coefficients exhibit exponential growth, potentially necessitating the storage of up to $O(n^{2/3})$ (critical case) or even $O(n)$ (subcritical case) coefficients. As a consequence, computing tables of coefficients is too costly in these cases. Therefore, we use bijective considerations to find a direct method of generating numbers according to the law μ^u. Inspiration is drawn from a similar method used by Fusy [Fus09]. For $k \in \mathbb{N}_{>0}$, define the number t_k of ternary trees with k internal nodes (*i.e.*, $3k$ edges):

$$t_k = \frac{1}{2k+1}\binom{3k}{k},$$

so it holds that, by (2.2),

$$b^{\circ}_{k+1} = \frac{2}{k+1} t_k.$$

Therefore, the generating series satisfies

$$T(y) = \sum_{k=1}^{\infty} t_k y^k = \frac{B'_{\circ}(y)}{2} = \frac{B'(y)}{2}$$

and has radius of convergence $\rho_{B_\circ}$. Ternary trees with k internal nodes are in bijection with Dyck paths with k jumps of size $+2$ and $2k+1$ jumps of size -1, ending in -1. In practice, one draws a Dyck path where $+2$-steps have probability $1/3$ and -1-steps have probability $2/3$ and stop as soon as -1 is reached. The number of increasing steps in this path follows the Boltzmann law ν with, for $k \in \mathbb{N}_{>0}$,

$$\nu(k) = \frac{t_k \rho^k_{B_\circ}}{T(\rho_{B_\circ})}.$$

To convert from ν to μ^u, notice that, for $k \in \mathbb{N}_{>0}$, using that $x(u) = \rho_{B_\circ}^{1/d}$ in these cases:

$$\frac{\nu(k)}{\mu^u(k+1)} = \frac{\frac{t_k \rho_{B_\circ}^k}{T(\rho_{B_\circ})}}{\frac{u b_{k+1}^\circ \rho_{B_\circ}^{k+1}}{1+uB(\rho_{B_\circ})}} = \frac{t_k}{b_{k+1}^\circ} \frac{2(1+uB(\rho_{B_\circ}))}{u\rho_{B_\circ} B'(\rho_{B_\circ})} = \frac{E(u)}{4}(k+1) =: \frac{k+1}{C(u)}.$$

Thus, if the Dyck path procedure returns k, this result is rejected with probability $1-C(u)/(k+1)$, otherwise it is retained and 1 is added to obtain the μ^u distribution. This can be further improved by anticipated rejection. Indeed, as explained in [Fus09, Lemma 12],

$$\frac{C(u)}{k+1} = \frac{C(u)}{1} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{k}{k+1},$$

so one can proceed by rejecting at each stage (new up step) with probability $1 - k/(k+1)$, where k is the number of up steps in the Dyck path drawn so far. One cannot speed up the computation by interrupting as soon as the variable k exceeds $2n$ because one does not yet know whether the value being computed is going to be rejected. All these operations are summarised in Algorithm 11.

Algorithm 11 Sampler according to μ^u for $u \le u_C$

```
1:  if Bern(μ^u(0)) then
2:      return 0
3:  else // draw k > 0
4:      currentHeight ← 0
5:      k ← 0
6:      while currentHeight ≥ 0 do
7:          if Bern(1/3) then // add up step
8:              currentHeight ← currentHeight + 2
9:              if k > 0 and Bern(1 − k/(k+1)) then
10:                 go to line 3
11:             end if
12:             k ← k + 1
13:         else // add down step
14:             currentHeight ← currentHeight − 1
15:         end if
16:     end while
17: end if
18: return k + 1
```

With the same arguments developed by Fusy [Fus09], the expected cost of sampling according to μ using this anticipated rejection trick is $O(1)$. Notice that without anticipated rejection, one samples according to ν which has infinite expectation.

We then expect that the expected cost of one trial of Algorithm 10 should be $\Theta(n^{1/3})$ in the critical case, and $O(1)$ in the subcritical case. Given (3.11) and Lemma 3.9, this would yield expected complexity $\Theta(n)$ for Algorithm 10 in the critical case and $\Theta(n^{3/2})$ in the subcritical case.

Chapter 4

Phase transition for block sizes

The previous chapter highlighted a phase transition for the properties of the generating series $M(z, u)$ and the consequences for the form of the asymptotic enumeration. We now look at the properties of the maps drawn according to $\mathbb{P}_{n,u}$ when $n \to \infty$, namely the size of the largest blocks.

As we explained earlier in Section 2.2.1, the block sizes of a map are completely determined by the degrees of its (undecorated) block tree. Therefore, we follow the probabilistic approach introduced by Addario-Berry [AB19] and use results on random trees to obtain results on maps drawn according to $\mathbb{P}_{n,u}$. Recall that, by Proposition 2.17 and Theorem 2.19, the block tree of a map drawn according to $\mathbb{P}_{n,u}$ has the law of a μ^u-Bienaymé–Galton–Watson tree, which is subcritical if $u < u_C$ and critical otherwise. Therefore, we rely on Janson's survey [Jan12], in which there is an extensive study of the largest degree of Bienaymé–Galton–Watson trees.

On the one hand; in all cases but the last, we show that if $u < u_C$, a condensation phenomenon occurs and the largest block is of size $\Theta(n)$ (Section 4.1.1); when $u > u_C$, the largest block is of size $\Theta(\ln(n))$ (Section 4.1.3); for $u = u_C$, the largest block is of size $\Theta(n^{2/3})$ (Section 4.1.2), with explicit limit laws (after renormalising).

On the other hand, simple triangulations decomposed into irreducible blocks present a challenge, as the vertices of the block trees are not decorated with a single block (or none) but with a sequence of blocks. Hence, the size of the blocks cannot be immediately read from the degrees in the block tree. However, an extreme condensation phenomenon occurs, concentrating mass in only one element of the sequence (as in [Gou98, Theorem 1]), resulting in a similar behaviour (Section 4.2).

Notation. For $\mathfrak{m}$ a map, denote by $\mathrm{s}(\mathrm{LB}_1(\mathfrak{m})) \geq \cdots \geq \mathrm{s}(\mathrm{LB}_{b(\mathfrak{m})}(\mathfrak{m}))$ the sizes of its blocks in decreasing order. By convention, set $\mathrm{s}(\mathrm{LB}_k(\mathfrak{m})) = 1$ if $k > b(\mathfrak{m})$. Then, it holds that

$$\mathrm{s}(\mathfrak{m}) = \sum_{k=1}^{b(\mathfrak{m})} (\mathrm{s}(\mathrm{LB}_k(\mathfrak{m})) - 1) + 1.$$

In the following, the random variable $\mathbf{M}_{n,u}$ is distributed according to $\mathbb{P}_{n,u}$, and set $\mathbf{T}_{n,u} = T_{\mathbf{M}_{n,u}}$. To compare the results here with those of [FS24], be careful that $\mathbf{M}_{n,u}$ of the article would be written $\mathbf{M}_{2n+1,u}$ here, and that the sizes of the largest blocks here use the function $\mathrm{s}(.)$ and not the number of edges. However, the change is straightforward: if $\widehat{\mathbf{M}_{n,u}}$ is the $\mathbf{M}_{n,u}$ of the article, then one can simply write, for $k \in \mathbb{N}_{>0}$,

$$|\mathrm{LB}_k(\widehat{\mathbf{M}_{n,u}})| = |\mathrm{LB}_k(\mathbf{M}_{dn+1,u})| = \frac{\mathrm{s}(\mathrm{LB}_k(\mathbf{M}_{dn+1,u})) - 1}{d}.$$

4.1 All models but the last

All results stated here for the block sizes are for the decomposition schemes 1 to 7 (as numbered in Table 2.1).

4.1.1 Subcritical case

When $u < u_C$, $\mathbf{T}_{n,u}$ is distributed as a subcritical Bienaymé–Galton–Watson tree. Subcritical Bienaymé–Galton–Watson trees exhibit a *condensation* phenomenon: Jonsson and Stefánsson showed that exactly one of the nodes has a degree linear in the size [JS11]. Janson's survey gives a more general statement for the condensation phenomenon, and in particular shows the following result. Recall that denoting by d_{TV} the total variation distance, we write $X_n \overset{(d)}{\approx} Y_n$ if $d_{TV}(X_n, Y_n) \to 0$ as $n \to \infty$.

Proposition 4.1. *[Jan12, Theorem 19.34] Let μ be a probability distribution on $\mathbb{N}_0$ such that $\mu(0) > 0$, $\mathbb{E}[\mu] < 1$ and there exists c satisfying $\mu(k) \sim_{k\to\infty} ck^{-5/2}$. Let $D_{n,1} \geq D_{n,2} \geq \cdots \geq D_{n,n}$ be the ranked list of the number of children of a μ-Bienaymé–Galton–Watson tree conditioned to have n edges. Then, letting $\xi_1, \ldots, \xi_{n-1}$ be a family of $n-1$ independent random variables of law μ and $\left(\xi_1^{(n-1)}, \ldots, \xi_{n-1}^{(n-1)}\right)$ their decreasing reordering, it holds that:*

$$(D_{n,1}, \ldots, D_{n,n}) \overset{(d)}{\approx} \left(n - \sum_{i=1}^{n-1} \xi_i, \xi_1^{(n-1)}, \ldots, \xi_{n-1}^{(n-1)}\right). \tag{4.1}$$

When $u < u_C$, the distribution μ^u satisfies the hypotheses of Proposition 4.1, which gives the following result. This is a rephrasing of the results for trees of [Jan12], to which we add the proof of the joint convergence. For the case of general maps decomposed into 2-connected maps, the case $u = 1$ was proved by Addario-Berry [AB19, Theorem 3.3].

Theorem 4.2. *Let $u \in (0, u_C)$. Recall that $E(u)$ and $c(u)$ are defined in Equations (2.31) and (2.32). Then, as $n \to \infty$ with $n \equiv 1[d]$*

$$\mathrm{s}(\mathrm{LB}_1(\mathbf{M}_{n,u})) = (1 - E(u))n + O_{\mathbb{P}}(n^{2/3}) \quad \textit{and} \quad \mathrm{s}(\mathrm{LB}_2(\mathbf{M}_{n,u})) = O_{\mathbb{P}}(n^{2/3}).$$

Moreover, the following joint convergence holds:

$$\frac{1}{d(c(u)n)^{2/3}}\left((1-E(u))n-(\mathrm{s}(\mathrm{LB}_1(\mathbf{M}_{n,u}))-1),(\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u}))-1,j\geq 2)\right)$$
$$\xrightarrow[n\to\infty]{(d)}\left(L_1,\left(\Delta L_{(j-1)},j\geq 2\right)\right) \tag{4.2}$$

where $(L_t)_{t\in[0,1]}$ is a stable process of parameter $3/2$ such that

$$\mathbb{E}\left[e^{-sL_1}\right]=e^{\Gamma(-3/2)s^{3/2}}$$

and $\Delta L_{(1)}\geq\Delta L_{(2)}\geq\dots$ is the ranked sequence of its jumps.

When $u\to 0$, we have $1-E(u)\to 1$: as expected, if the map has only one block, its size is n.

As noticed by Addario-Berry who treated the case $u=1$[AB19] for the case of general maps decomposed into 2-connected maps, the probabilistic approach has the advantage that we obtain a joint limit law for the largest block and the subsequent ones. Banderier, Flajolet, Schaeffer and Soria used analytic techniques to obtain a local limit for the size of the largest block in the case $u=1$ (and their technique could be extended to any $u<u_C$); but their technique does not give information for the subsequent blocks. We retrieve the results of of [BFSS01, Table 4]: indeed, our $1-E(1)$, computed in Table 2.2, corresponds to their α_0. Moreover, the probabilistic approach can also yield local limit theorems for the size of the largest block, as discussed by Stufler [Stu20b].

Remark 4.3. If $(L_t)_{t\in[0,1]}$ is a stable process of parameter $3/2$ satisfying $\mathbb{E}\left[e^{-sL_1}\right]=e^{\Gamma(-3/2)s^{3/2}}$ for s such that $Re(s)\geq 0$; then, the following is well-known (see [Ber96, Theorem 1] and its proof):

$$\left(\sum_{j:\Delta L_{(j)}\geq\varepsilon}\Delta L_{(j)}-\frac{2}{\sqrt{\varepsilon}}\right)\xrightarrow[\varepsilon\to 0]{(d)}L_1.$$

Proof of Theorem 4.2. Theorem 2.8 shows that the hypotheses of Proposition 4.1 are satisfied in the subcritical case.

Let $(\xi_i)_{i\geq 1}$ be a family of *i.i.d.* random variables of law μ^u and let $\left(\xi_1^{(n)},\dots,\xi_n^{(n)}\right)$ be the decreasing reordering of its first n variables (take the convention $\xi_i^{(n)}=0$ if $i>n$). Let us consider the following cumulative process:

$$L_t^{(n)}=\frac{\sum_{i=1}^{\lceil nt\rceil}\xi_i-ntE(u)}{C(u)n^{2/3}}\quad\text{for } t\in[0,1],\quad\text{where}\quad C(u)=dc(u)^{2/3}.$$

It is standard [Fel71, Theorem XVII.5.2] [JS87, Chapter VII, Corollary 3.6] that there exists a Lévy process $(L_t)_{t\in[0,1]}$ with Lévy measure $\pi(dx)=x^{-5/2}dx\mathbb{1}_{\{x>0\}}$ so that for s such that $Re(s)\geq 0$,

$$\mathbb{E}\left[e^{-sL_1}\right]=e^{\Gamma(-3/2)s^{3/2}},$$

and such that the following convergence holds in the so-called *Skorokhod topology*

$$\left(L_t^{(n)}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{(d)} (L_t)_{t\in[0,1]}. \tag{4.3}$$

By definition of the process $L_t^{(n)}$, $\frac{\xi_i}{C(u)(dn)^{2/3}}$ is its i-th jump. In particular, denoting by ΔP_t the jump of the process (P_t) at time t (which may equal 0),

$$\frac{\xi_1^{(n)}}{C(u)n^{2/3}} = \sup_{0\le t\le 1} \Delta L_t^{(n)}.$$

By [JS87, Chapter VI, Proposition 2.4], (4.3) gives

$$\frac{\xi_1^{(n)}}{C(u)n^{2/3}} \xrightarrow[n\to\infty]{(d)} \sup_{0\le t\le 1} \Delta L_t := \Delta L_{(1)}.$$

By construction of a Lévy process, $(\Delta L_{(j)})_{j\ge 1}$ has same law as the decreasing rearrangement of the atoms of a Poisson random measure with intensity π on $\mathbb{R}^+$ (see *e.g.* [Ber96, Theorem 1]). By denoting $t_1^{(n)}$ the time at which the jump $\xi_1^{(n)}$ of the process $L_t^{(n)}$ is realised, one has:

$$\frac{\xi_2^{(n)}}{C(u)n^{2/3}} = \sup_{0\le t\le 1} \Delta\left(L_t^{(n)} - \frac{\xi_1^{(n)}}{C(u)(dn)^{2/3}}\mathbb{1}_{t\ge t_1^{(n)}}\right)_t.$$

So, applying again [JS87, Chapter VI, Proposition 2.4], one gets, denoting by t_1 the time of the largest jump of (L_1):

$$\frac{\xi_2^{(n)}}{C(u)(dn)^{2/3}} \xrightarrow[n\to\infty]{(d)} \sup_{0\le t\le 1} \Delta\left(L_t - \Delta L_{(1)}\mathbb{1}_{t\ge t_1}\right)_t = \Delta L_{(2)}.$$

It is again possible to iterate by subtracting the largest jump: for all $k\ge 1$,

$$\frac{1}{C(u)n^{2/3}}\left(\xi_1^{(n)},\dots,\xi_k^{(n)}\right) \xrightarrow[n\to\infty]{(d)} \left(\Delta L_{(1)},\dots,\Delta L_{(k)}\right). \tag{4.4}$$

However, by Proposition 4.1, one has (recall that a map $\mathfrak{m}$ such that $\mathrm{s}(\mathfrak{m}) = n \equiv 1[d]$ has n components, some of which might be empty):

$$(\mathrm{s}(\mathrm{LB}_1(\mathbf{M}_{n,u})) - 1,\dots,\mathrm{s}(\mathrm{LB}_n(\mathbf{M}_{n,u})) - 1) \overset{(d)}{\approx} \left(n - \sum_{i=1}^{n-1}\xi_i, \xi_1^{(n-1)},\dots,\xi_{n-1}^{(n-1)}\right).$$

Therefore, for all $k\ge 2$ fixed

$$\left(\frac{(1-E(u))n - (\mathrm{s}(\mathrm{LB}_1(\mathbf{M}_{n,u}) - 1))}{C(u)n^{2/3}}, \frac{\mathrm{s}(\mathrm{LB}_2(\mathbf{M}_{n,u})) - 1}{C(u)n^{2/3}},\dots,\frac{\mathrm{s}(\mathrm{LB}_k(\mathbf{M}_{n,u})) - 1}{C(u)n^{2/3}}\right)$$
$$\overset{(d)}{\approx} \left(\frac{\sum_{i=1}^{n-1}\xi_i - E(u)n}{C(u)n^{2/3}}, \frac{\xi_1^{(n-1)}}{C(u)n^{2/3}},\dots,\frac{\xi_k^{(n-1)}}{C(u)n^{2/3}}\right) \xrightarrow[n\to\infty]{(d)} \left(L_1, \Delta L_{(1)},\dots,\Delta L_{(k)}\right).$$

This allows to conclude since k is arbitrary. □

4.1.2 Critical case

The critical case corresponds to $u = u_C$. Then, by Theorem 2.19 and Proposition 2.22, $\mathbf{T}_{n,u_C}$ is distributed as a critical Bienaymé–Galton–Watson tree with power law tail in $c(u_C)j^{-\alpha-1}$, with $\alpha = 3/2 \in (1,2)$ conditioned to have n vertices. This case is directly treated in Janson's survey [Jan12, Example 19.27 and Remark 19.28].

Theorem 4.4. *As $n \to \infty$ with $n \equiv 1[d]$, the following convergence holds:*

$$\left(\frac{\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u_c})) - 1}{dn^{2/3}}, j \geq 1\right) \xrightarrow{(d)} \left(E_{(j)}, j \geq 1\right), \tag{4.5}$$

where the $\left(E_{(j)}\right)$ are the ordered atoms of a Point Process E on $[0,\infty]$, satisfying that the random variable $E_{a,b} = \#\left(E \cap [a,b]\right)$ has a probability generating function convergent for all $z \in \mathbb{C}$ with

$$\mathbb{E}\left[z^{E_{a,b}}\right] = \frac{1}{2\pi g(0)} \int_{-\infty}^{\infty} \exp\left(c\Gamma(-3/2)(-it)^{3/2} + (z-1)c\int_a^b x^{-5/2} e^{itx} dx\right) dt,$$

where $c = c(u_C) = \frac{3\gamma_B}{4\sqrt{\pi} dB'(\rho_B)\rho_B}$ by (3.10) and

$$g : x \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt + c\Gamma(-3/2)(-it)^{3/2}} dt.$$

The intensity measure π of E satisfies, for $x > 0$,

$$\pi(dx) = cx^{-5/2} \frac{g(-x)}{g(0)} dx,$$

and, for all $j \geq 1$,

$$E_{(j)} > 0 \qquad \text{almost surely.}$$

Remark 4.5. By Theorem 2.8 and [Duq03, Proposition 4.3], one has the convergence of the (appropriately) rescaled Łukasiewicz path of $\mathbf{T}_{n,u_C}$ towards a $3/2$-stable excursion. Therefore, using [JS87, Chapter VI, Proposition 2.4] and following the same line of arguments as in the proof of Theorem 4.2, one gets that the $E_{(j)}$ are distributed like the reordered jumps of a $3/2$-stable excursion (multiplied by a constant factor). (Duquesne's result is stated under an aperiodicity hypothesis for the reproduction law, which can be omitted; see the discussion in the forthcoming proof of Proposition 5.10.)

4.1.3 Supercritical case

The supercritical case corresponds to $u \in (u_C, +\infty)$ and $x(u) \in (0, \rho_B)$. Recall that in this case $\mathbf{T}_{n,u}$ is distributed as a critical Bienaymé–Galton–Watson tree with finite exponential moments by Theorem 2.19 and Proposition 2.22.

Properties of the maximum degree of critical Bienaymé–Galton–Watson trees have been extensively studied by Janson [Jan12], building on work by Meir and Moon [MM90]. For the case where the offspring distribution admits finite exponential moments, Janson shows the following result.

Proposition 4.6. *[Jan12, Theorem 19.16] Let μ be a probability distribution on $\mathbb{N}_0$ such that $\mu(0) > 0$, and $\mu(k+1)/\mu(k)$ converges to a finite limit as $k \to \infty$. Let $D_{n,1} \geq D_{n,2} \geq \cdots \geq D_{n,n}$ be the ranked list of the number of children of a μ-Bienaymé–Galton–Watson tree conditioned to have n edges. Denote by ρ the radius of convergence of $\Phi : t \mapsto \sum_{k \in \mathbb{N}_0} \mu(k) t^k$, and $\nu = \lim_{x \to \rho^-} x \frac{\Phi'(x)}{\Phi(x)}$. Suppose $\nu > 1$. Then, denoting $k(n) = \max\{k \in \mathbb{N}_0 \mid \mu(k) \geq 1/n\}$, for all $j \geq 1$,*

$$D_{n,j} = k(n) + O_{\mathbb{P}}(1).$$

In our case, the asymptotic of $k(n)$ can be computed thanks to results about the Lambert W function, which is the compositional inverse of $x \in \mathbb{R} \mapsto xe^x \in [-e^{-1}, +\infty)$. This gives the following theorem.

Theorem 4.7. *Let $u > u_C$. For all fixed $j \geq 1$, it holds as $n \to \infty$ with $n \equiv 1[d]$ that*

$$\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u})) = \frac{\ln(n)}{\ln\left(\frac{\rho_B}{x(u)^d}\right)} - \frac{5\ln(\ln(n))}{2\ln\left(\frac{\rho_B}{x(u)^d}\right)} + O_{\mathbb{P}}(1). \tag{4.6}$$

Proof. The probability $\mu^u(\{dk\})$ is decreasing with k. So, by Proposition 2.22, for n large enough, to study $k(n)$ it is sufficient to study for which k it holds that

$$c(u)\left(\frac{x(u)^d}{\rho_B}\right)^k k^{-5/2}\,(1+o(1)) \geq \frac{1}{n}.$$

For sake of compactness, set $w(u) = \rho_B x(u)^{-d}$. Note that $w(u) > 1$ since $u > u_C$. Consequently, the previous inequality is equivalent to

$$w(u)^k k^{5/2} \leq c(u) n \,(1+o(1))\,.$$

Notice that this is equivalent to

$$\frac{2}{5}\ln(w(u))k \cdot e^{\frac{2}{5}\ln(w(u))k} \leq \frac{2}{5}\ln(w(u))(nc(u))^{2/5}\,(1+o(1))\,.$$

Therefore, $k(n)$ is the largest integer such that:

$$\frac{2}{5}\ln(w(u))k(n) \leq W\left(\frac{2}{5}\ln(w(u))(nc(u))^{2/5}\,(1+o(1))\right)$$

where W denotes the Lambert W function. It is known that W satisfies, for $x \to +\infty$,

$$W(x) = \ln(x) - \ln(\ln(x)) + o(1),$$

which concludes the proof. □

4.2 Simple triangulations decomposed into irreducible blocks

In this case, as expressed by Proposition 2.16, the nodes of the block tree are not blocks but sequences of blocks. What has been said about the degrees of the block tree in the previous section still applies to the block tree of this decomposition scheme, but the translation into results for the block sizes is not as immediate. Recall from (2.25) that the sequence is described by the following generating series

$$\Phi(x, u) = \frac{1}{1 - uB(x^2)}.$$

Subcritical sequences were studied by Gourdon who showed that in that case, "the size of the largest component is nearly the size of the whole structure" [Gou98]. There is an extreme condensation phenomenon: contrary to what happens for blocks of maps in the subcritical case (Section 4.1.1) where some non-maximal blocks are still relatively large, here the largest block is of size $n - O(1)$ where n is the total size of sequence. Gourdon's results are not sufficient for our case, so we use Stufler's work on Gibbs partitions [Stu20c, Stu24].

Let $u < v_C$ (recall from Section 2.2.2 that $v_C = 32/5$ is the point where the sequence ceases to be subcritical and the second source of singularity for Φ arises). Let $\mathbf{s}_{m,u}$ be distributed as a sequence of blocks of total size m, where each block has weight u. Then, [Stu24, Theorem 3.11] shows that the number of elements (*i.e.*, blocks) of $\mathbf{s}_{m,u}$ satisfies

$$\mathrm{el}(\mathbf{s}_{m,u}) \xrightarrow[m\to\infty]{(d)} \mathrm{El},$$

where El is a random variable satisfying

$$\mathbb{E}\left[t^{\mathrm{El}}\right] = \frac{t(1 - uB(\rho_B))^2}{(1 - utB(\rho_B))^2} = t\left(\frac{32 - 5u}{32 - 5tu}\right)^2.$$

In particular, this gives

$$\mathbb{E}\left[\mathrm{El}\right] = 1 + \frac{2}{\frac{32}{5u} - 1},$$

which gives the intuition that, as u increases, the number of elements in the sequence tends to increase as well. When $u \to 0$, triangulations drawn according to $\mathbb{P}_{n,u}$ are irreducible, so their tree is composed of one internal node with one block, which is consistent with the fact that $\mathbb{E}\left[\mathrm{El}\right] \xrightarrow[u\to 0]{} 1$. When u reaches v_C, the sequence is no longer "subcritical" and other behaviours emerge. Stufler's extensive work [Stu24] can be used to study the behaviour of this case.

Moreover, Stufler shows that the size of the largest component of $\mathbf{s}_{m,u}$ behaves as follows

$$m - \mathrm{s}(\mathrm{LB}_1(\mathbf{s}_{m,u})) \xrightarrow[m\to\infty]{(d)} \sum_{i=1}^{\mathrm{El}-1} \xi_i \tag{4.7}$$

where the $(\xi_i)_i$ are *i.i.d.* random variables such that

$$\mathbb{E}\left[t^{\xi_i}\right] = \frac{B(t\rho_B)}{B(\rho_B)},$$

i.e., for all $k \in \mathbb{N}_{>0}$,

$$\mathbb{P}\left(\xi_i = k\right) = \frac{b_k \rho_B^k}{B(\rho_B)} \underset{k\to\infty}{\sim} \frac{3\gamma_B}{4\sqrt{\pi}} k^{-5/2}$$

by (2.38). It is interesting to notice that, in the right-hand member of (4.7), only El depends on u. One can compute that

$$\mathbb{E}\left[\sum_{i=1}^{\mathrm{El}-1} \xi_i\right] = \frac{\rho_B B'(\rho_B)}{B(\rho_B)}(\mathbb{E}\left[\mathrm{El}\right] - 1) = \frac{27u}{2(32-5u)} = E(u),$$

where $E(u)$ is defined in (2.32) and satisfies $E(u) < 1$ if $u < u_C$.

The node of greatest degree in $\mathbf{T}_{n,u}$ corresponds to a block sequence of total size equal to its degree. By the above discussion, the largest block in the sequence concentrates all the mass except $O_{\mathbb{P}}(1)$. Thus, the joint law of largest node degrees is asymptotically the same as the joint law of largest blocks in the three regimes, which gives the following result.

Theorem 4.8. *Let $u > 0$. The block-weighted simple triangulation $\mathbf{M}_{n,u}$ exhibits the following behaviours when $n \to \infty$ with $n \equiv 1[d]$.*

Subcritical case. *If $u < u_C$, (4.2) still holds:*

$$\frac{1}{2(c(u)n)^{2/3}}\left((1-E(u))n - (\mathrm{s}(\mathrm{LB}_1(\mathbf{M}_{n,u})) - 1), (\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u})) - 1, j \geq 2)\right)$$
$$\xrightarrow[n\to\infty]{(d)} \left(L_1, \left(\Delta L_{(j-1)}, j \geq 2\right)\right)$$

where $(L_t)_{t\in[0,1]}$ is a stable process of parameter $3/2$ such that $\mathbb{E}\left[e^{-sL_1}\right] = e^{\Gamma(-3/2)s^{3/2}}$ and $\Delta L_{(1)} \geq \Delta L_{(2)} \geq \dots$ is the ranked sequence of its jumps.

Critical case. *If $u = u_C$, (4.5) still holds:*

$$\left(\frac{\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u_C})) - 1}{2n^{2/3}}, j \geq 1\right) \xrightarrow[n\to\infty]{(d)} \left(E_{(j)}, j \geq 1\right),$$

where the $E_{(j)}$ are as defined in Theorem 4.4.

Supercritical case. *If $u_C < u < v_C$, (4.6) still holds:*

$$\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u})) = \frac{\ln(n)}{\ln\left(\frac{\rho_B}{x(u)^2}\right)} - \frac{5\ln(\ln(n))}{2\ln\left(\frac{\rho_B}{x(u)^2}\right)} + O_{\mathbb{P}}(1).$$

In the subcritical case, there is almost surely one vertex of the block-tree with a linear degree, and almost surely one block of linear size. Therefore, the largest block is almost surely in the sequence decorating the largest vertex. In the critical case, the largest nodes are almost surely of degree $\Theta(n^{2/3})$, so the $O_{\mathbb{P}}(1)$ of the mass which does not go towards the largest block of the sequence do not change the order of sizes of the blocks: almost surely, the biggest node corresponds to the biggest block. On the contrary, in the supercritical case, the size of the nodes differ by only $O_{\mathbb{P}}(1)$, so there is no guarantee the biggest node corresponds to the biggest block.

Chapter 5

Phase transition for scaling limits

We have highlighted phase transition phenomena for maps drawn according to $\mathbb{P}_{n,u}$ from multiple perspectives: the probability law (Chapter 2), enumerative properties (Chapter 3), and block sizes (Chapter 4). Notably, we can also observe a phase transition in the geometric properties of the maps. A central objective of the study of our main model was to explore these geometric properties, proposing a model that interpolates between the *Brownian sphere* $\mathcal{S}$ (also called Brownian map, see Fig. 4) and the *Brownian Continuum Random tree* $\mathcal{T}^{(2)}$ (see Fig. 5), but where, unlike previous such models, the transition does not appear through the boundary; which is why we use instead a parameter tuning the density of separating elements (the Boltzmann weight u on blocks).

To that end, we use Gromov–Hausdorff–Prokhorov's topology which allows to make sense of the convergence of a sequence of maps to a certain limit, considering them as (isometry classes of) compact metric spaces; we show that, after an appropriate rescaling, the sequence $(\mathbf{M}_{n,u})_{n\geq 1}$ of maps drawn according to $\mathbb{P}_{n,u}$ converges in distribution towards some random metric space (called the *scaling limit* and formally defined in Section 5.1.1).

Contrary to what was done in previous chapters, here we do not give a unified proof of convergence for all models, but concentrate on our main example of general maps decomposed into 2-connected blocks and its reformulation in the form of quadrangulations decomposed into simple blocks (Section 5.1). Then, in a final section, we discuss how these results can be extended to the other decomposition schemes considered (Section 5.2). Section 5.1 is the verbatim of Section 5 of a joint article with William Fleurat [FS24][1].

The scaling limit of a tree-decomposed model like ours depends on the geometries of the blocks and of the underlying decomposition tree. In our setting, one of the behaviours always ends up dominating. Indeed, in the subcritical case, the geometry of the largest block dominates, whereas, in the critical and supercritical cases, the geometry of maps is driven by the geometry of the underlying tree. But this is not always the case: Sénizergues, Stefánsson and Stufler study situations where both geometries play a role in the scaling limit, and define the *decorated α-stable trees* which are the corresponding scaling limits [SSS23]. Our results for the scaling limits in the critical and supercritical cases (which are, up to a constant factor, respectively $\mathcal{T}^{(3/2)}$ after rescaling by $n^{1/3}$ and $\mathcal{T}^{(2)}$ after rescaling by $n^{1/2}$) are exactly what

[1] Except for the first paragraph and up to notation changes.

they conjecture in [SSS23, Remark 1.1].

More precisely, in Section 5.1.2, we give a unified proof of the convergence towards the Brownian tree $\mathcal{T}^{(2)}$, after renormalising distances by $n^{1/2}$ in the supercritical case $u > u_C$, and towards the *stable tree* $\mathcal{T}^{(3/2)}$ *of parameter* $3/2$ (see Fig. 6), after renormalising distances by $n^{1/3}$, in the critical case $u = u_C$ (Theorem 5.4). For $u > u_C$, we retrieve a previous result by Stufler for more general weighted models [Stu20a]. All these results hold for both maps and their 2-connected components, and quadrangulations and their simple components. Finally, when $u < u_C$, we show in Theorem 5.22 of Section 5.1.3 that quadrangulations converge towards the Brownian sphere when renormalising distances by $n^{1/4}$. We rely crucially on the convergence of uniform *simple* quadrangulations with the same normalisation, which is proven by Addario-Berry and Albenque [ABA17], and recalled in Proposition 5.23 below. A similar convergence result for uniform 2-connected maps would be needed in order to prove a version of Theorem 5.22 for maps, see the discussion after the statement of Proposition 5.23. Such a convergence is expected to hold and hinted at for instance by Lehéricy's results [Leh22], which show a uniform map of size n and its quadrangulation via Tutte's bijection are asymptotically isometric in the scale $n^{1/4}$.

It can be interesting to compare the approximations of the scaling limits (Figs. 4 to 6) with those obtained by the random generator for large sizes (Figs. 2.11 to 2.15).

Sénizergues, Stefánsson and Stufler build on a model introduced by Archer, which, contrary to this work, develops the *local limit* point of view [Arc20, Chapter 6]. In particular, Archer shows that the fractal dimension of the local limit for the critical and supercritical cases are respectively 3 and 2[2]. Both cases correspond to what Archer calls the "tree regime", where the local geometry of the tree is preponderant in the limit. Both articles consider only *critical* offspring distributions for the trees, which does not hold in our subcritical regime.

Notation. Recall the definitions in Equations (2.6) and (2.19) of the probability distributions $\mathbb{P}_{n,u}$ and $\mathbb{P}_u$ on maps and $\mathbb{P}_{n,u}^{\mathrm{quad}}$ and $\mathbb{P}_u^{\mathrm{quad}}$ on quadrangulations[3]. When $\mathbf{M}$ is a random variable on the space of maps, let $\mathbf{Q} = \varphi(\mathbf{M})$ be its image by Tutte's bijection. Let $\mathbf{T}$ denote the block tree associated to $\mathbf{M}$ (and also to $\mathbf{Q}$ by Proposition 2.13). In this way, when $\mathbf{M}$ has law $\mathbb{P}_u^{\mathrm{map}}$ (resp. $\mathbb{P}_{n,u}^{\mathrm{map}}$), then, by Proposition 2.14, $\mathbf{Q}$ has law $\mathbb{P}_u^{\mathrm{quad}}$, (resp. $\mathbb{P}_{n,u}^{\mathrm{quad}}$). Since we restrict to these particular cases, we use $y(u)$ as defined in (2.8) at the beginning of Chapter 2.

For v a vertex of $\mathbf{T}$, let $\mathfrak{b}_v^{\mathbf{M}}$ (resp. $\mathfrak{b}_v^{\mathbf{Q}}$) denote the 2-connected block of $\mathbf{M}$ (resp. simple block of $\mathbf{Q}$) represented by v in $\mathbf{T}$. By Proposition 2.13, it holds that $\mathfrak{b}_v^{\mathbf{Q}} = \varphi(\mathfrak{b}_v^{\mathbf{M}})$ for all $v \in \mathbf{T}$, where φ is Tutte's bijection.

These random variables will be studied under probability measures $\mathbb{P}_u$ and $(\mathbb{P}_{n,u})_{n\geq 1}$, which were introduced in Section 2.1.4. We write accordingly $\mathbb{E}_u[\dots]$ and $\mathbb{E}_{n,u}[\dots]$ the expectations with respect to these probability measures. In this chapter, unless mentioned otherwise or if it is clear from context, other random variables shall be viewed as defined on some probability space (Ω, P), and the according expectations will be written as $E[\dots]$. In particular we will

[2]This uses that the diameter for uniform blocks is $\Theta(n^{1/4})$, which is known for simple quadrangulations but only conjectured for 2-connected maps.

[3]For the sake of simplicity, we do not use the $\mathbb{P}_u$ and $\mathbb{P}_{n,u}$ of the generalised framework (2.27).

use the following random variables defined on (Ω, P):

- For each $u \geq 0$, the triplet $(\mathbf{T}_{n,u}, \mathbf{M}_{n,u}, \mathbf{Q}_{n,u})$ is $(\mathbf{T}, \mathbf{M}, \mathbf{Q})$ under the law $\mathbb{P}_{n,u}$.
- For each $k \geq 1$, the pair $(B_k^{\mathrm{map}}, B_k^{\mathrm{quad}})$ consists of a 2-connected map B_k^{map} with k edges sampled uniformly, together with $B_k^{\mathrm{quad}} = \varphi(B_k^{\mathrm{map}})$ its image by Tutte's bijection. By Proposition 2.12, the latter is a simple quadrangulation with k faces sampled uniformly at random.

5.1 Scaling limits of block-weighted general maps and block-weighted quadrangulations

The goal of the present section is to expand on this phase transition by considering metric properties of the models in each phase — both for the model on general maps and the one on quadrangulations — in the sense of taking *scaling limits*, see Section 5.1.1 for definitions.

Because Tutte's bijection commutes with the block decomposition of both models under consideration, as stated in Proposition 2.13, the combinatorial picture of Chapter 4 is the same for both models. However, obtaining global metric properties under either model requires a good understanding of the metric behaviour of the underlying blocks. As of now, the required results exist only for simple quadrangulations. Consequently, our scaling limit results are complete only for the quadrangulation model.

In Section 5.1.1, we introduce the relevant formalism to state our scaling limit results, as well as a deviation estimate for the diameters of blocks, which will be useful for all values of u.

In Section 5.1.2, we prove Theorem 5.4, which identifies scaling limits simultaneously when $u > u_C$ and $u = u_C$. For both models, there is convergence after suitable rescaling to a random continuous tree, namely a *Brownian tree* when $u > u_C$ and a $3/2$-*stable tree* when $u = u_C$. This convergence holds in the Gromov-Hausdorff-Prokhorov (GHP) sense – between measured metric spaces – when maps and quadrangulations are equipped with the uniform measure on their vertices.

Finally in Section 5.1.3, we prove Theorem 5.22 which deals with the GHP scaling limit when $u < u_C$. In this phase, the one-big-block identified in Section 4.1.1 converges after rescaling to a scalar multiple of the *Brownian sphere*, and the contribution of all other blocks is negligible. This result is proved only for the quadrangulation model since it relies crucially on the scaling limit result for uniform simple quadrangulations obtained in [ABA17]. No such result is available yet for uniform 2-connected general maps, although one expects that it should hold.

5.1.1 Preliminaries

The Gromov-Hausdorff and Gromov-Hausdorff-Prokhorov topologies

Originating from the ideas of Gromov, the following notions of metric geometry have become widely used in probability theory to state scaling limit results. We refer the interested reader to [BBI01] for general background on metric geometry and [Mie07, Section 6] for an exposition of the main properties of the Gromov-Hausdorff and Gromov-Hausdorff-Prokhorov topologies, and especially their definition *via* correspondences and couplings that we use here.

Define a *correspondence* between two sets X and Y as a subset C of $X \times Y$ such that for all $x \in X$, there exists $y \in Y$ such that $(x,y) \in C$, and *vice versa*. The set of correspondences between X and Y is denoted as $\mathrm{Corr}(X,Y)$. If (X,d_X) and (Y,d_Y) are compact metric spaces and $C \in \mathrm{Corr}(X,Y)$ is a correspondence, one may define its *distortion*:

$$\mathrm{dis}(C; d_X, d_Y) = \sup\Big\{|d_X(x,\widetilde{x}) - d_Y(y,\widetilde{y})| \colon (x,y) \in C,\ (\widetilde{x},\widetilde{y}) \in C\Big\}.$$

This allows to define the *Gromov-Hausdorff* distance between (isometry classes of) compact metric spaces

$$d_{\mathrm{GH}}\big((X,d_X),(Y,d_Y)\big) = \frac{1}{2}\inf\Big\{\mathrm{dis}(C; d_X, d_Y) \colon C \in \mathrm{Corr}(X,Y)\Big\}.$$

One can modify this notion of distance in order to get a distance between compact measured metric spaces. For measured spaces (X,ν_X) and (Y,ν_Y) such that ν_X and ν_Y are *probability* measures, let us denote by $\mathrm{Coupl}(\nu_X,\nu_Y)$ the set of couplings between ν_X and ν_Y, *i.e.* the set of measures γ on $X \times Y$ with respective marginals ν_X and ν_Y. Then the *Gromov-Hausdorff-Prokhorov* distance is defined as

$$\begin{aligned} &d_{\mathrm{GHP}}\big((X,d_X,\nu_X),(Y,d_Y,\nu_Y)\big) \\ &= \inf\Big\{\max\Big(\tfrac{1}{2}\,\mathrm{dis}(C; d_X, d_Y), \gamma\big((X\times Y)\setminus C\big)\Big) \colon C \in \mathrm{Corr}(X,Y),\ \gamma \in \mathrm{Coupl}(\nu_X,\nu_Y)\Big\}. \end{aligned}$$

When (X,d_X) and (Y,d_Y) are the same metric space, one can bound this distance by the *Prokhorov distance* between the measures ν_X and ν_Y. This distance is defined for ν_1 and ν_2 two Borel measures on the same metric space (X,d) by

$$d_{\mathrm{P}}^{(X,d)}(\nu_1,\nu_2) = \inf\Big\{\varepsilon > 0 \colon \nu_1(A) \leq \nu_2(A^\varepsilon) + \varepsilon \text{ and } \nu_2(A) \leq \nu_1(A^\varepsilon) + \varepsilon, \forall A \in \mathcal{B}(X)\Big\}, \tag{5.1}$$

where A^ε is the set of points $x \in X$ such that $d(x,A) < \varepsilon$. The bound mentioned above then corresponds to the inequality

$$d_{\mathrm{GHP}}\big((X,d,\nu_1),(X,d,\nu_2)\big) \leq d_{\mathrm{P}}^{(X,d)}(\nu_1,\nu_2), \tag{5.2}$$

which is a consequence of Strassen's Theorem, see [Dud02, Section 11.6].

Finally we will use the following fact, the proof of which is left to the reader. For $a \in [0,1)$ and Borel probability measures μ, ν and ν' on some metric space (X,d), it holds that

$$d_{\mathrm{P}}^{(X,d)}\big(a\mu + (1-a)\nu, a\mu + (1-a)\nu'\big) \leq d_{\mathrm{P}}^{(X,d)}(\nu,\nu'). \tag{5.3}$$

Formulation of the GHP-scaling limit problem

Let us begin by setting the notation for the measured metric spaces that one can canonically associate to the combinatorial objects under consideration. We associate to a tree (resp. map or quadrangulation) the following measured metric spaces:

- For $\mathfrak{t}$ a *tree*, denote by $V(\mathfrak{t})$ the set of its vertices, by $d_{\mathfrak{t}}$ the distance that the graph distance induces on $V(\mathfrak{t})$, by $\nu_{\mathfrak{t}}$ the uniform probability measure on $V(\mathfrak{t})$ and by $\underline{\mathfrak{t}}$ the measured metric space $\underline{\mathfrak{t}} = (V(\mathfrak{t}), d_{\mathfrak{t}}, \nu_t)$. For $v \in \mathfrak{t}$, the number of children of v is denoted by $k_v(\mathfrak{t})$.

- For $\mathfrak{m}$ a *map*, recall that $V(\mathfrak{m})$ is its vertex set, and denote by $d_{\mathfrak{m}}$ the graph distance on $V(\mathfrak{m})$, by $\nu_{\mathfrak{m}}$ the uniform probability measure on $V(\mathfrak{m})$ and by $\underline{\mathfrak{m}}$ the measured metric space $\underline{\mathfrak{m}} = (V(\mathfrak{m}), d_{\mathfrak{m}}, \nu_m)$.

- For $\mathfrak{q}$ a *quadrangulation*, denote by $V(\mathfrak{q})$ its vertex set, by $d_{\mathfrak{q}}$ the graph distance on $V(\mathfrak{q})$, by $\nu_{\mathfrak{q}}$ the uniform probability measure on $V(\mathfrak{q})$ and by $\underline{\mathfrak{q}}$ the measured metric space $\underline{\mathfrak{q}} = (V(\mathfrak{q}), d_{\mathfrak{q}}, \nu_q)$.

The problem of finding a GHP-scaling limit consists in finding a suitable rescaling of a sequence of random compact measured metric spaces so that it admits a non-trivial limit in distribution for the GHP-topology. Let us introduce a convenient notation for the rescaling operation on a measured metric space. For $\underline{X} = (X, d, \nu)$ a measured metric space and $\lambda > 0$, we denote by $\lambda \cdot \underline{X}$ the measured metric space $(X, \lambda d, \nu)$.

A useful deviation estimate

We shall now prove a deviation estimate for the diameters of the blocks of $\mathbf{M}$ and $\mathbf{Q}$. It will prove useful for all values of $u > 0$. We recall the definition of *stretched-exponential* quantities, as this notion provides a concise way to deal with the probabilities of exceptional events.

Definition 5.1. A sequence (p_n) of real numbers is said to be *stretched-exponential* as $n \to \infty$ if there exist constants $\gamma, C, c > 0$ such that

$$|p_n| \leq C \exp(-cn^{\gamma}).$$

As is evident from the definition, if $(p_n)_n$ and $(q_n)_n$ are stretched-exponential sequences, then so are the sequences $(p_n + q_n)_n$, $(p_n q_n)_n$, $(n^{\alpha} p_n)_n$ and $(n^{\alpha} \sup_{k \geq n^{\beta}} p_k)_n$ with arbitrary $\alpha, \beta > 0$.

The input we shall rely on to derive our estimate is a deviation estimate for the diameter of **one** block, in both the case of 2-connected blocks of maps and simple blocks of quadrangulations.

Proposition 5.2. *For any $\varepsilon > 0$, the probabilities*

$$P\left(\operatorname{diam}(B_k^{\mathrm{map}}) \geq k^{1/4+\varepsilon}\right) \qquad \textit{and} \qquad P\left(\operatorname{diam}(B_k^{\mathrm{quad}}) \geq k^{1/4+\varepsilon}\right)$$

are stretched-exponential as $k \to \infty$.

Proof. The estimate for uniform 2-connected maps $(B_k^{\mathrm{map}})_{k\geq 0}$ is obtained from [CFGN15, Theorem 3.7, specialized to $x = 1$]. To obtain the estimate for uniform simple blocks of quadrangulations $(B_k^{\mathrm{quad}})_{k\geq 0}$, one easily checks that for any path of length $l \geq 0$ in a map $\mathfrak{m}$, there exists a path with same endpoints and length at most $2l$ in $\varphi(\mathfrak{m})$, its image by Tutte's bijection. Therefore for every map $\mathfrak{m}$ one has $\mathrm{diam}(\varphi(\mathfrak{m})) \leq 2\,\mathrm{diam}(\mathfrak{m})$. In particular $\mathrm{diam}(B_k^{\mathrm{quad}}) \leq 2\,\mathrm{diam}(B_k^{\mathrm{map}})$, and the conclusion follows from the estimate for $(B_k^{\mathrm{map}})_{k\geq 0}$ □

This deviation estimate for the diameter of **one** block allows to control the deviations of the diameter of **every** block of $\mathbf{M}_{n,u}$ and $\mathbf{Q}_{n,u}$, in the sense of the following corollary.

Corollary 5.3. *For all $u > 0$ and all $\delta > 0$, the probabilities*

$$P\left(\exists v \in \mathbf{T}_{n,u},\ \mathrm{diam}(\mathfrak{b}_v^{\mathbf{M}_{n,u}}) \geq \max\left(n^{1/6}, k_v(\mathbf{T}_{n,u})^{(1+\delta)/4}\right)\right), \quad n \geq 1, \tag{5.4}$$
$$P\left(\exists v \in \mathbf{T}_{n,u},\ \mathrm{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) \geq \max\left(n^{1/6}, k_v(\mathbf{T}_{n,u})^{(1+\delta)/4}\right)\right), \quad n \geq 1, \tag{5.5}$$

are stretched-exponential as $n \to \infty$.

Proof. Let $\mathfrak{b}$ be either a 2-connected map, or a simple quadrangulation. Then $\mathrm{diam}(\mathfrak{b})$ is bounded by its number of edges, which is $|\mathfrak{b}|$ if $\mathfrak{b}$ is a map, and $2|\mathfrak{b}|$ if it is a quadrangulation. In particular, recalling that the outdegrees in the block-tree are twice the sizes of the respective blocks, we get for all $u > 0$ and $n \geq 1$, that

$$\forall v \in \mathbf{T}_{n,u},\ \left[\mathrm{diam}(\mathfrak{b}_v^{\mathbf{M}_{n,u}}) \leq k_v(\mathbf{T}_{n,u})/2\right] \text{ and } \left[\mathrm{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) \leq 2 \cdot k_v(\mathbf{T}_{n,u})/2\right].$$

Denote by $\mathrm{A}(\mathbf{M}_{n,u})$ the "bad" subset of $\mathbf{T}_{n,u}$ made of the vertices v such that both $k_v(\mathbf{T}_{n,u})/2 \geq n^{1/6}$ and $\mathrm{diam}(\mathfrak{b}_v^{\mathbf{M}_{n,u}}) \geq k_v(\mathbf{T}_{n,u})^{(1+\delta)/4}$. By the above trivial bound on diameters, to show that the probabilities (5.4) are stretched-exponential as $n \to \infty$, it suffices to see that the probability of the event $\{\mathrm{A}(\mathbf{M}_{n,u}) \neq \emptyset\}$ is stretched-exponential as $n \to \infty$.

By Proposition 2.6, conditionally on $\mathbf{T}_{n,u}$, each block $\mathfrak{b}_v^{\mathbf{M}_{n,u}}$ is sampled uniformly from 2-connected maps with size $k_v(\mathbf{T}_{n,u})/2$ respectively. Therefore, conditionally on $\mathbf{T}_{n,u}$, for each vertex v in $\mathbf{T}_{n,u}$ we have

$$\begin{aligned}
P\big(v \in \mathrm{A}(\mathbf{M}_{n,u}) \,|\, \mathbf{T}_{n,u}\big) &= \mathbb{1}_{\left\{k_v(\mathbf{T}_{n,u})/2 \geq n^{1/6}\right\}} P\left(\mathrm{diam}(B_{k/2}^{\mathrm{map}}) \geq k^{(1+\delta)/4}\right)\Big|_{k=k_v(\mathbf{T}_{n,u})} \\
&\leq \sup_{k/2 \geq n^{1/6}} P\left(\mathrm{diam}(B_{k/2}^{\mathrm{map}}) \geq k^{(1+\delta)/4}\right) \\
&\leq \sup_{k \geq n^{1/6}} P\left(\mathrm{diam}(B_k^{\mathrm{map}}) \geq k^{(1+\delta)/4}\right).
\end{aligned}$$

Since $\mathbf{T}_{n,u}$ has $2n + 1$ vertices, this yields by a union bound,

$$P(\mathrm{A}(\mathbf{M}_{n,u}) \neq \emptyset) \leq (2n+1) \sup_{k \geq n^{1/6}/2} P\left(\mathrm{diam}(B_k^{\mathrm{map}}) \geq k^{(1+\delta)/4}\right),$$

which is stretched-exponential as $n \to \infty$ by Proposition 5.2, as announced. A similar use of Proposition 5.2 proves that the probabilities (5.5) are stretched-exponential as $n \to \infty$. $\square$

5.1.2 The supercritical and critical cases

Statement of the result

For $1 < \theta \leq 2$, let us denote by $\mathcal{T}^{(\theta)}$ a θ-stable Lévy tree equipped with its mass measure. There are several equivalent constructions of these objects. A common way is to define them *via* excursions of θ-stable Lévy processes. Namely, $\mathcal{T}^{(\theta)}$ is the real tree encoded by the height process of an excursion of length one of a θ-stable Lévy process, see [Duq03]. To fix a normalization for $\mathcal{T}^{(\theta)}$, we consider in the construction an excursion obtained by a cyclic shift from a θ-stable Lévy Bridge with Laplace exponent $\lambda \mapsto \lambda^\theta$. Note that the measured metric space $\mathcal{T}^{(2)}$ corresponds to $\sqrt{2}$ times the *Brownian Continuum Random Tree*, which is encoded by an excursion of length 1 of the standard Brownian motion. The precise definition *via* excursions is not important for our statement and one can take Proposition 5.10 below as an alternative definition.

Theorem 5.4. *There exist positive constants $(\kappa_u^{\mathrm{map}}, \kappa_u^{\mathrm{quad}})_{u \geq u_C}$ such that we have the following joint convergences in distribution, in the Gromov-Hausdorff-Prokhorov sense:*

1. *If $u > u_C$, we have*

$$\frac{\sigma(u)}{\sqrt{2}}(2n)^{-1/2} \cdot \left(\underline{\mathbf{T}}_{n,u}, \underline{\mathbf{M}}_{n,u}, \underline{\mathbf{Q}}_{n,u}\right) \xrightarrow[n\to\infty]{\mathrm{GHP},\,(d)} \left(\mathcal{T}^{(2)}, \kappa_u^{\mathrm{map}} \cdot \mathcal{T}^{(2)}, \kappa_u^{\mathrm{quad}} \cdot \mathcal{T}^{(2)}\right),$$

where we set

$$\sigma(u)^2 = 1 + \frac{4u\,(y(u))^2\, B_\circ''(y(u))}{uB_\circ(y(u)) + 1 - u} = \frac{3u - 3 + 2\sqrt{u\,(u-1)}}{5u - 9}. \tag{5.6}$$

2. *If $u = u_C = 9/5$, we have*

$$\frac{2}{3}(2n)^{-1/3} \cdot \left(\underline{\mathbf{T}}_{n,u_C}, \underline{\mathbf{M}}_{n,u_C}, \underline{\mathbf{Q}}_{n,u_C}\right) \xrightarrow[n\to\infty]{\mathrm{GHP},\,(d)} \left(\mathcal{T}^{(3/2)}, \kappa_{u_C}^{\mathrm{map}} \cdot \mathcal{T}^{(3/2)}, \kappa_{u_C}^{\mathrm{quad}} \cdot \mathcal{T}^{(3/2)}\right).$$

Additionally, the constants $(\kappa_u^{\mathrm{map}}, \kappa_u^{\mathrm{quad}})_{u \geq u_C}$ can be expressed as follows.

$$\kappa_u^{\mathrm{map}} = \sum_{j\geq 1} 2j\mu^u(2j)\mathcal{D}_j^{\mathrm{map}} \quad \text{and} \quad \kappa_u^{\mathrm{quad}} = \sum_{j\geq 1} 2j\mu^u(2j)\mathcal{D}_j^{\mathrm{quad}}, \tag{5.7}$$

where $\mathcal{D}_j^{\mathrm{map}}$ (resp. $\mathcal{D}_j^{\mathrm{quad}}$) is the expectation of the distance, in a uniform 2-connected map with j edges (resp. simple quadrangulation with j faces) of the distance of the root vertex to the base vertex of a uniform corner (resp. to the closest endpoint of a uniform edge).

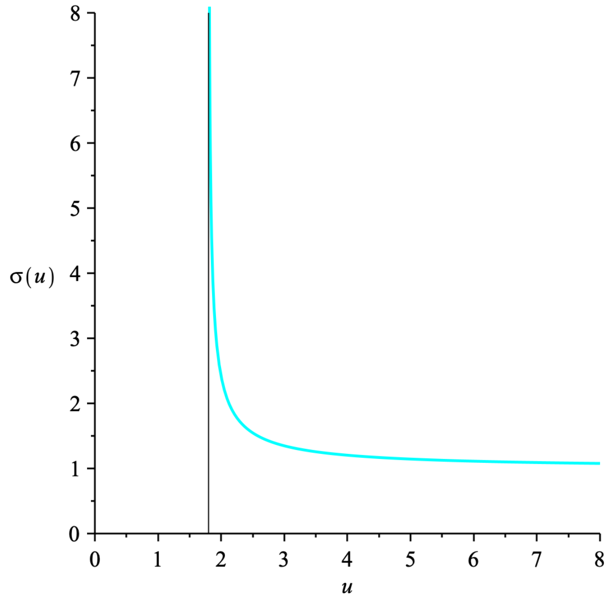

Figure 5.1: Plot of σ as a function of u. The vertical line corresponds to $u = u_C$.

Remark 5.5. Let us explain how one gets the second equality of (5.6), which allows to draw Fig. 5.1. From the proof of the convergence, one gets:

$$\sigma(u)^2 = 1 + \frac{4u\,(y(u))^2\,B_\circ''(y(u))}{uB_\circ(y(u)) + 1 - u}.$$

Differentiating the algebraic equation (2.16) satisfied by $B_\circ$ with respect to y and taking (2.13) as a fourth equation gives a polynomial system from which one can get $B_\circ(y(u))$ and $B_\circ''(y(u))$ as functions of u only (using the resultant or a Gröbner basis algorithm). Then, one can conclude from the expression of $\sigma(u)$ in (5.6).

Remark 5.6. The quantity κ_u^{quad} could in principle be obtained *via* the explicit formula obtained in [BG10] for the generating function g_ℓ of simple edge-rooted quadrangulations with a distinguished edge at prescribed distance ℓ from the root vertex.

Discussion and overview of the proof

Let $u \geq u_C$.

Consider a geodesic in either $\mathbf{M}_{n,u}$ or $\mathbf{Q}_{n,u}$ between two distant blocks $\mathfrak{b}$ and $\widetilde{\mathfrak{b}}$, respectively indexed by v and $\widetilde{v}$ in the block-tree. This geodesic must go through all the blocks whose index w in the block-tree $\mathbf{T}_{n,u}$ is on the path from v to $\widetilde{v}$, in the order induced by this path in the tree.

We have seen in Proposition 2.6 that under the law of $\mathbf{M}_{n,u}$ or $\mathbf{Q}_{n,u}$, the blocks are independent conditionally on the block-tree, and when $u \geq u_C$ they tend to all have non-macroscopic $o(n)$ size by Theorems 4.4 and 4.7. One therefore expects that when n is large, the distance between two distant blocks $\mathfrak{b}$ and $\widetilde{\mathfrak{b}}$ falls into a *law of large numbers* behaviour

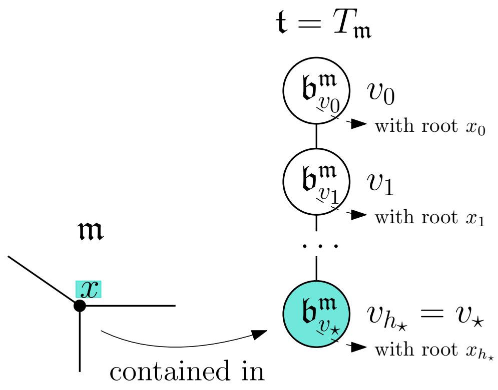

Figure 5.2: Situation of Lemma 5.7.

and is of the same order as $d_{\mathbf{T}_{n,u}}(v, \widetilde{v})$.

According to this heuristic, the macroscopic distances in $\mathbf{M}_{n,u}$ and $\mathbf{Q}_{n,u}$ should be concentrated around a deterministic scalar multiple of the distances in $\mathbf{T}_{n,u}$. But $\mathbf{T}_{n,u}$ is a critical Galton-Watson tree conditioned to have $2n+1$ vertices, with explicit tail asymptotic given by Theorem 2.8 for its offspring distribution, yielding that its scaling limit is a stable tree.

To make this heuristic work, one needs to understand the typical distribution of degrees on a typical path in the tree. It turns out that on a typical path in a size-n critical Galton-Watson tree, the degrees are asymptotically independent and identically distributed; and moreover they are distributed as the size-biased version of the offspring distribution. This will be obtained by a spine decomposition for trees, adapted to our context.

We bring the attention of the reader to the fact that a proof similar in spirit has been done for the Gromov-Hausdorff metric in the general abstract setup of enriched trees by Stufler [Stu20a, Theorem 6.60], and we could readily apply this result to deal with the case $u > u_C$, modulo a technical complication regarding the additivity of distances in the quadrangulation case. When $u = u_C$ however, the distances within blocks have fat tails, so we fall outside the scope of Stufler's result. To deal with this, our last technical ingredient is a suitable large deviation estimate: we show that after an adequate truncation of the variables depending on n, large (and moderate) deviation events still have very small probability.

We now proceed with the proof.

Additivity of the distances along consecutive blocks

In Lemmas 5.7 and 5.8, we justify that a macroscopic distance is indeed a sum of distances on "in-between" blocks, in the case of blocks lying on the same branch in the block-tree.

The map case. For $\mathfrak{b}$ a 2-connected map, and l an integer in $\{1, \dots, 2|\mathfrak{b}|\}$, let us denote by $D(\mathfrak{b}, l)$ the graph distance in $\mathfrak{b}$ between its root vertex and the vertex on which lies the l-th corner of $\mathfrak{b}$ in breadth-first order (or in whatever arbitrary ordering rule is chosen in the block-tree decomposition, see Section 2.1.1).

Fix a vertex x on $\mathfrak{m}$. Let $v_\star$ be the vertex v of the block-tree $\mathfrak{t}$, closest to the root of $\mathfrak{t}$, such that x is a vertex of $\mathfrak{b}^{\mathfrak{m}}_v$. Denote by $h_\star := h_{\mathfrak{t}}(v_\star)$, and by $(v_i)_{0 \le i \le h_\star}$ the ancestor line of

$v_\star$ in $\mathfrak{t}$, with v_0 the root and $v_{h_\star} = v_\star$. For $0 \leq i \leq h_\star$, let x_i be the root vertex of $\mathfrak{b}^{\mathfrak{m}}_{v_i}$. Finally, let $(l_i)_{0\leq i<h_\star}$ be the respective breadth-first index of the corner in $\mathfrak{b}^{\mathfrak{m}}_{v_i}$ in which the block $\mathfrak{b}^{\mathfrak{m}}_{v_{i+1}}$ is attached. The situation is illustrated on Fig. 5.2.

Lemma 5.7. *For $0 \leq i \leq h_\star$, we have*

$$d_{\mathfrak{m}}(x, x_i) = d_{\mathfrak{b}^{\mathfrak{m}}_{v_\star}}(x, x_{h_\star}) + \sum_{i\leq j<h_\star} D(\mathfrak{b}^{\mathfrak{m}}_{v_j}, l_j).$$

Proof. By definition, $D(\mathfrak{b}^{\mathfrak{m}}_{v_j}, l_j) = d_{\mathfrak{m}}(x_j, x_{j+1})$ for $i \leq j < h_\star$. We get by the triangle inequality that the left-hand-side is at most the right-hand side. Therefore it suffices to show that any geodesic path in $\mathfrak{m}$ from x to x_i visits each of the points $(x_j)_{i<j\leq h_\star}$, in decreasing order of j.

Let j be such that $i < j \leq h_\star$. Denote by $\mathfrak{t}_j$ the tree of descendants of v_j in $\mathfrak{t}$ (rooted in v_j) and also $\mathfrak{m}_j$ and $\widetilde{\mathfrak{m}}_j$ the submaps of $\mathfrak{m}$ made of the blocks $(b^{\mathfrak{m}}_v)_{v\in\mathfrak{t}_j}$ and $(b^{\mathfrak{m}}_v)_{v\in\mathfrak{t}\setminus\mathfrak{t}_j}$ respectively. By the recursive description of the block-tree, the submaps $\mathfrak{m}_j$ and $\widetilde{\mathfrak{m}}_j$ share only the vertex x_j. But x is a vertex of $\mathfrak{m}_j$ since $v_\star$ is a descendant of v_j, and x_i is a vertex of $\widetilde{\mathfrak{m}}_j$ since v_i is an ancestor of v_j. Hence any injective path between x and x_i must visit x_j in decreasing order of j; and in particular for a geodesic path.

Notice that it does not require the (x_j) to be mutually distinct. This concludes the proof. □

The quadrangulation case. A slight complication arises for quadrangulations because the "interface" between two blocks is a double edge, containing two vertices instead of a single vertex in the map case. At first sight it is thus unclear through which of these vertices a geodesic should go. We show that there is a canonical choice: the vertex between those two which is closest to the root vertex. This relies crucially on the fact that quadrangulations are bipartite.

Fix a quadrangulation $\mathfrak{q}$. For $\mathfrak{b}$ a simple block of $\mathfrak{q}$, and l an integer in $\{1, \ldots, 2|\mathfrak{b}|\}$, let us denote by $D_{\mathfrak{q}}(\mathfrak{b}, l)$ the graph distance in $\mathfrak{b}$ between the endpoint of the l-th edge of $\mathfrak{b}$ in the ordering described thereafter, and the endpoint of the root edge of $\mathfrak{b}$ which is closest to the root vertex of $\mathfrak{q}$. The order on the edges of $\mathfrak{b}$ that we use is the image of the lexicographic order on vertices of $\mathfrak{t}$ *via* the block-tree decomposition. This is consistent with the ordering of corners in the map case.

Fix a vertex x on $\mathfrak{q}$. Similarly to the map case, let $v_\star$ be the vertex v of the block-tree $\mathfrak{t}$, closest to the root of $\mathfrak{t}$, such that x is a vertex of $\mathfrak{b}^{\mathfrak{q}}_v$. Define accordingly $h_\star = h_{\mathfrak{t}}(v_\star)$ and the ancestor line $(v_i)_{0\leq i\leq h_\star}$ of $v_\star$ in the block-tree $\mathfrak{t}$, with v_0 the root and $v_{h_\star} = v_\star$. Let also $(x_i)_{0\leq i\leq h_\star}$ be the respective root vertex of $\mathfrak{b}_{v_i}$. Finally, let $(l_i)_{0\leq i<h_\star}$ be the respective breadth-first index of the edge in $\mathfrak{b}^{\mathfrak{q}}_{v_i}$ to which the root edge of $\mathfrak{b}^{\mathfrak{q}}_{v_{i+1}}$ is attached.

Lemma 5.8. *For all* $0 \leq i \leq h_\star$*, there exists* $\delta_{x,x_i} \in \{0, \pm 1, \pm 2\}$ *such that*

$$d_\mathfrak{q}(x, x_i) = \delta_{x,x_i} + d_{\mathfrak{b}^\mathfrak{q}_{v_\star}}(x, x_{h\star}) + \sum_{i \leq j < h_\star} D_\mathfrak{q}(\mathfrak{b}^\mathfrak{q}_{v_j}, l_j).$$

Proof. The idea is quite similar in principle as in the preceding lemma, except that consecutive blocks share two vertices in the quadrangulation case, instead of one.

For $0 \leq j \leq h_\star$, denote by y_j the endpoint of the root-edge of $\mathfrak{b}^\mathfrak{q}_{v_j}$ which is closest to the root vertex of $\mathfrak{q}$. In particular y_0 is the root vertex of $\mathfrak{q}$. Let $0 \leq i \leq h_\star$. Then, by construction, y_i and x_i are adjacent to the root edge of $\mathfrak{b}^\mathfrak{q}_{v_i}$. Notice that a geodesic from x to x_i must visit at least one of the endpoints of the root-edge of $\mathfrak{b}_{v_\star}$, which is at distance 0 or 1 of $y_{h_\star}$, and that x_i and y_i are at distance 0 or 1. Therefore, there exists some $\delta_{x,x_i} \in \{0, \pm 1, \pm 2\}$ such that

$$d_\mathfrak{q}(x, x_i) = \delta_{x,x_i} + d_{\mathfrak{b}^\mathfrak{q}_{v_\star}}(x, x_{h\star}) + d_\mathfrak{q}(y_{h_\star}, y_i).$$

We shall prove the following, which are sufficient to conclude:

1. For $0 \leq i \leq h_\star$, it holds that $d_\mathfrak{q}(y_{h_\star}, y_i) = \sum_{i \leq j < h_\star} d_\mathfrak{q}(y_{j+1}, y_j)$;
2. For $0 \leq j < h_\star$, it holds that $d_\mathfrak{q}(y_{j+1}, y_j) = D_\mathfrak{q}(\mathfrak{b}^\mathfrak{q}_{v_j}, l_j)$.

It is even sufficient to show the following:

$$\forall 0 \leq i \leq j \leq k \leq h_\star, \quad d_\mathfrak{q}(y_i, y_k) = d_\mathfrak{q}(y_i, y_j) + d_\mathfrak{q}(y_j, y_k). \tag{5.8}$$

Indeed, assuming (5.8) holds, by applying it iteratively, we directly get that $d_\mathfrak{q}(y_{h_\star}, y_i) = \sum_{i \leq j < h_\star} d_\mathfrak{q}(y_{j+1}, y_j)$. To verify the second set of identities, recall that y_j is defined as the endpoint of the root edge of $b^\mathfrak{q}_{v_j}$ which is closest to the root vertex y_0 of $\mathfrak{q}$. Denote by y'_j the other endpoint. Then, for $0 \leq j < h_\star$ we have

$$D_\mathfrak{q}(\mathfrak{b}^\mathfrak{q}_{v_j}, l_j) = \min\left(d_{\mathfrak{b}^\mathfrak{q}_{v_j}}(y_{j+1}, y_j), d_{\mathfrak{b}^\mathfrak{q}_{v_j}}(y'_{j+1}, y_j)\right) = \min\left(d_\mathfrak{q}(y_{j+1}, y_j), d_\mathfrak{q}(y'_{j+1}, y_j)\right).$$

The first equality comes from the definition of $D_\mathfrak{q}(\mathfrak{b}^\mathfrak{q}_{v_j}, l_j)$, and the second one from the fact that within a block $\mathfrak{b}$ of $\mathfrak{q}$, the graph distance respective to $\mathfrak{q}$ and the graph distance respective to $\mathfrak{b}$ coincide. Then, assuming (5.8), it holds that

$$d_\mathfrak{q}(y_{j+1}, y_j) = d_\mathfrak{q}(y_{j+1}, y_0) - d_\mathfrak{q}(y_j, y_0) \leq d_\mathfrak{q}(y'_{j+1}, y_0) - d_\mathfrak{q}(y_j, y_0) \leq d_\mathfrak{q}(y'_{j+1}, y_j),$$

where the first inequality comes from the definition of y_{j+1}, and the second inequality from triangle inequality. In particular, the above minimum is $d_\mathfrak{q}(y_{j+1}, y_j)$ and we have, as needed, $d_\mathfrak{q}(y_{j+1}, y_j) = D_\mathfrak{q}(\mathfrak{b}^\mathfrak{q}_{v_j}, l_j)$.

It still remains to prove (5.8). Let us first prove the case $i = 0$ and then deduce the general case. Let $0 \leq j \leq k \leq h_\star$, and let γ be a geodesic path from y_0 to y_k. If γ visits y_j, we readily have

$$d_\mathfrak{q}(y_0, y_k) = d_\mathfrak{q}(y_0, y_j) + d_\mathfrak{q}(y_j, y_k). \tag{5.9}$$

Otherwise it visits y'_j, and denote by γ_1, γ_2 the portions of γ form y_0 to y'_j, and from y'_j to y_k respectively. By definition of y_j, we have $d_{\mathfrak{q}}(y_0, y_j) \leq d_{\mathfrak{q}}(y_0, y'_j)$. But since $\mathfrak{q}$ is a quadrangulation, it is bipartite and the inequality is strict $d_{\mathfrak{q}}(y_0, y_j) < d_{\mathfrak{q}}(y_0, y'_j)$. Form $\widetilde{\gamma}_1$ the concatenation of a geodesic path from y_0 to y_j and of the oriented edge (y_j, y'_j). Then, from the strict inequality we mentioned, $\mathrm{len}(\widetilde{\gamma}_1) \leq \mathrm{len}(\gamma_1)$, and in particular the concatenation of $\widetilde{\gamma}_1$ and γ_2 is a geodesic path from y_0 to y_k which visits y_j. Therefore, the identity (5.9) also holds.

Finally, let us deduce the case $i \neq 0$. Let $0 \leq i \leq j \leq k \leq h_\star$. We have

$$\begin{aligned} d_{\mathfrak{q}}(y_i, y_k) = d_q(y_0, y_k) - d_{\mathfrak{q}}(y_0, y_i) &= (d_q(y_0, y_k) - d_{\mathfrak{q}}(y_0, y_j)) + (d_q(y_0, y_j) - d_{\mathfrak{q}}(y_0, y_i)) \\ &= d_{\mathfrak{q}}(y_j, y_k) + d_{\mathfrak{q}}(y_i, y_j). \end{aligned}$$

This proves (5.8) and concludes the proof of Lemma 5.8.

□

Scaling limit and largest degree of critical Galton-Watson trees

A slight technical complication that arises in our setting is that the block-tree has a *lattice* offspring distribution with span 2, in the sense of the following definition

Definition 5.9. A measure μ on $\mathbb{Z}$ is called *lattice* if its support is included in a subset $b + d\mathbb{Z}$ of $\mathbb{Z}$, with $d \geq 2$. The largest such d is called its *span*. If $d = 1$, μ is called *non-lattice*.

The results that we need [Kor13, Theorem 3] are stated for non-lattice offspring distributions. This turns out to be purely for convenience and we state the following more general result that is suited to our needs.

We recall that a probability distribution μ with mean m_μ is said to be in the domain of attraction of a stable law of index $\theta \in (1, 2]$ if there exist positive constants $(C_n)_{n\geq 0}$ such that we have the following convergence in distribution

$$\frac{U_1 + \cdots + U_n - n m_\mu}{C_n} \xrightarrow[n\to\infty]{(d)} X^{(\theta)}, \tag{5.10}$$

where $(U_1, \ldots U_n)$ are i.i.d. samples of the law μ, and $X^{(\theta)}$ is a random variable with Laplace transform $E\left[\exp(-\lambda X^{(\theta)})\right] = \exp(\lambda^\theta)$.

Proposition 5.10. *For all* $1 < \theta \leq 2$*, there exists a random measured metric space* $\mathcal{T}^{(\theta)} = \left(\mathcal{T}^{(\theta)}, d^{(\theta)}, \nu^{(\theta)}\right)$ *satisfying the following scaling limit result.*

Let μ *be a probability distribution on* $\mathbb{N}_0$*, with* $\mu(1) \neq 1$*, and which is assumed to be critical. Assume additionally that it is in the domain of attraction of a stable law of index* $\theta \in (1, 2]$*. Let* $d \geq 1$ *be the span of the measure* μ*. Then under those assumptions, we have*

1. *For all m large enough, the $\mathrm{GW}_\mu(\mathrm{d}T)$-probability that T has dm edges is positive. This probability is equivalent to $c_\theta/(C_{dm}dm)$ for some constant $c_\theta > 0$.*

2. *If we denote by T_n a GW_μ-tree conditioned to have n edges, then*

$$\left(\frac{C_{dm}}{dm}\right) \cdot \underline{T}_{dm} \xrightarrow[m\to\infty]{(d)} \mathcal{T}^{(\theta)},$$

in the Gromov-Hausdorff-Prokhorov *sense, with $(C_n)_{n\geq 0}$ the sequence in (5.10).*

3. *The largest degree in T_{dm} is of order at most C_{dm}, in the sense that for any $\varepsilon > 0$*

$$P\left(\exists v \in T_{dm}, k_v(T_{dm}) \geq (C_{dm})^{1+\varepsilon}\right) \xrightarrow[m\to\infty]{} 0.$$

Proof. The first statement can be obtained by a straightforward adaptation of the proof of [Kor13, Lemma 1], which relies on a local limit theorem and the cycle lemma. We specify below how this local limit theorem should be adapted. The cycle lemma adapts straightforwardly.

For the second statement, let us justify that [Kor13, Theorem 3] still applies when the *non-lattice* (or *aperiodic*) assumption is dropped, but with the number of vertices $n+1$ taken only along the subsequence $(dm+1)_{m\geq 0}$. This will prove functional convergence of the contour functions of the trees $(T_{dm})_m$ when properly rescaled, to the contour function of $\mathcal{T}^{(\theta)}$. This convergence of contour functions is sufficient to get the announced Gromov-Hausdorff-Prokhorov convergence.

The local limit theorem [Kor13, Theorem 2, (ii)] changes as follows

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left| \frac{a_n}{d} P\left(Y_n = k\right) - p_1\left(\frac{k}{a_n}\right)\right| = 0.$$

See for instance [Ibr71, Theorem 4.2.1]. Notice that the only difference with the *non-lattice* ($d = 1$) local limit theorem is the factor $1/d$ in the last display. Examining the details of Kortchemski's arguments, this extra $1/d$ factor would appear only in the discrete absolute continuity relations which are used in the proof. But in each instance, it would appear in both the numerator and denominator of some fraction. Hence the fraction simplifies and this factor has no impact on the proof, which carries without change, except that the integer n, which in the paper is the number of *vertices*, should now only be taken in $d\mathbb{Z}+1$.

Finally, in order to get the third statement, one can take as a basis the local limit theorem above. From this, one can get the functional convergence of the Łukasiewicz path of T_{dm}, when it is rescaled by dm in time and C_{dm} in space. In particular, $(C_{dm})^{-1}$ times the largest degree in T_{dm} is tight, and one obtains the claimed probabilistic bound. One could for instance use the same arguments as in the proof of [KM21, Proposition 3.4] □

Corollary 5.11 then just identifies the explicit scaling constants in specific instances of the above-mentioned scaling limit theorem.

Corollary 5.11. *Let μ be a critical probability distribution on $\mathbb{N}_0$ with span $d \geq 1$, and with $\mu(1) \neq 1$. Denote by T_n a GW_μ-tree conditioned to have n edges, for $n \in d\mathbb{Z}$ large enough. Then the following holds.*

1. *If μ has finite variance σ^2, then $P(|T| = dm) \sim cm^{-3/2}$ for some constant $c > 0$, and*

$$(dm)^{-1/2} \cdot \underline{T}_{dm} \xrightarrow[m\to\infty]{\mathrm{GHP},\,(d)} \frac{\sqrt{2}}{\sigma} \cdot \mathcal{T}^{(2)}.$$

Additionally for all $\varepsilon > 0$ the largest degree of T_{dm} is $o(m^{1/2+\varepsilon})$ in probability.

2. *If $\mu([x, +\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ for some $c > 0$ and $\theta \in (1, 2)$, then $P(|T| = dm) \sim c'_\theta m^{-(1+1/\theta)}$ for some constant $c'_\theta > 0$, and*

$$(dm)^{-(1-1/\theta)} \cdot \underline{T}_{dm} \xrightarrow[m\to\infty]{\mathrm{GHP},\,(d)} \left[\frac{\theta - 1}{c\,\Gamma(2-\theta)}\right]^{1/\theta} \cdot \mathcal{T}^{(\theta)}.$$

Additionally for all $\varepsilon > 0$ the largest degree of T_{dm} is $o(m^{1/\theta+\varepsilon})$ in probability.

Proof. Note that in the case where ν has finite exponential moments, [MM03] treats the case of lattice distributions. That would suffice for our applications when $u > u_C$. We still need the second statement to treat the case $u = u_C$. Let us apply the preceding proposition and identify the right constants, in these two cases.

Statement 1. If μ has finite variance σ^2, then by the Central Limit Theorem, for i.i.d. samples $(U_i)_i$ of the law μ, we have the convergence in distribution

$$\frac{U_1 + \cdots + U_n - n}{\sigma \cdot n^{1/2}} \xrightarrow[n\to\infty]{(d)} \mathcal{G},$$

where $\mathcal{G}$ is a standard normal variable. In particular, $\mathcal{G}$ has the same law as $\frac{1}{\sqrt{2}} \cdot X^{(2)}$. Therefore, the hypotheses of Proposition 5.10 are satisfied, with

$$C_n = \tfrac{\sigma}{\sqrt{2}} \cdot n^{1/2},$$

and the conclusion follows from this proposition.

Statement 2. We consider the case where $\mu([x, +\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ with $\theta \in (1, 2)$ and $c > 0$. Let $U := U_1$ and let us also introduce the notation

$$\begin{aligned} M_1(x) &= \int_x^\infty \mu(\mathrm{d}y) = \mu([x, +\infty)) \\ M_2(x) &= \int_x^\infty M_1(y)\,\mathrm{d}y \\ M_3(x) &= \int_0^x M_2(y)\mathrm{d}y. \end{aligned}$$

The function M_3 is non-decreasing and using the assumed tail asymptotic of μ, one has the asymptotic $M_3(x) \sim cx^{2-\theta}/(2-\theta)(\theta-1)$. We may therefore use the Karamata Tauberian theorem [BGT89, Theorem 1.7.1] to get

$$\widehat{M_3}(h) \sim \frac{c\,\Gamma(3-\theta)}{(2-\theta)(\theta-1)} h^{\theta-2} = \frac{c\,\Gamma(2-\theta)}{\theta-1} h^{\theta-2},$$

where $\widehat{M_3}$ is the *Laplace-Stieltjes transform* of M_3, defined — *e.g.* in [BGT89, Paragraph 1.7.0b] — as

$$\widehat{M_3}(h) = h \int_0^\infty e^{-hx} M_3(x) dx$$

for all h for which the integral converges absolutely. Then, if we integrate by parts three times, we obtain

$$\begin{aligned} E\left[\exp(-h(U-1))\right] = \int_0^\infty \mathrm{e}^{-h(x-1)}\,\mu(\mathrm{d}x) &= \mathrm{e}^h - h\mathrm{e}^h M_2(0) + h^3\mathrm{e}^h \int_0^\infty \mathrm{e}^{-hx} M_3(x)\,\mathrm{d}x \\ &= \mathrm{e}^h - h\mathrm{e}^h M_2(0) + h^2\mathrm{e}^h\,\widehat{M_3}(h), \end{aligned}$$

This, together with the fact that $M_2(0) = 1$ since it is the expectation of μ, yields the following expansion when $h \to 0^+$,

$$E\left[\exp(-h(U-1))\right] = 1 + \frac{c\,\Gamma(2-\theta)}{\theta-1} \cdot h^\theta \left(1 + o(1)\right). \tag{5.11}$$

Now, if we set

$$C_n = \left(\frac{c\,\Gamma(2-\theta)}{\theta-1}\right)^{1/\theta} n^{1/\theta},$$

and plug $h = \lambda/C_n$ into (5.11), we get for all $\lambda \geq 0$,

$$E\left[\exp\left(-\lambda\,\tfrac{U_1+\cdots+U_n-n}{C_n}\right)\right] = \left(E\left[\exp(-\tfrac{\lambda}{C_n}(U-1))\right]\right)^n \xrightarrow[n\to\infty]{} \exp\left(\lambda^\theta\right).$$

Hence there is convergence in distribution of $\frac{U_1+\cdots+U_n-n}{C_n}$ to $X^{(\theta)}$, as required in Proposition 5.10. So this proposition applies with the above-chosen sequence $(C_n)_n$, and the conclusion follows. □

Scaling limit of critical Galton-Watson trees equipped with a random measure

We shall need a version of the GHP scaling limits in Corollary 5.11, when the trees $\mathfrak{t}$ under consideration are equipped with some random measure on their vertices, instead of the uniform measure $\nu_{\mathfrak{t}}$. Let us describe more specifically our setting.

Let μ be a probability measure on $\mathbb{N}_0$ and $\eta = (\eta_k)_{k\geq 0}$ be a family of Borel probability measures on $\mathbb{R}_{\geq 0}$. We shall define an enriched version $\mathcal{L}(\mu,\eta)$ of the Galton-Watson law $\mathrm{GW}(\mu)$, defined on the set of pairs $(\mathfrak{t}, f)$ such that $\mathfrak{t}$ is a tree and f is a non-negative function $f\colon V(\mathfrak{t}) \to \mathbb{R}_{\geq 0}$. Namely, to sample a random pair $(\mathbf{T},\mathbf{f})$ with law $\mathcal{L}(\mu,\eta)$ first sample $\mathbf{T}$ according to $\mathrm{GW}(\mu)$, and then sample conditionally on $\mathbf{T}$ the variables $\mathbf{f}(v)$ for $v \in \mathbf{T}$, independently of each other, according to the laws $\eta_{k_v(\mathbf{T})}(\mathrm{d}x)$ respectively.

In particular, the random non-negative function $\mathbf{f}$ defines a random measure on $V(\mathbf{T})$ assigning weight $\mathbf{f}(v)$ to the vertex v. We shall use the same notation $\mathbf{f}$ for this measure, and denote by $|\mathbf{f}|$ its total weight.

Proposition 5.12. *Let μ be a critical offspring distribution with span $d \geq 1$ such that $\mu(1) \neq 1$. Let also $(\eta_k)_{k\geq 0}$ be Borel probability laws that are supported on $\mathbb{R}_{\geq 0}$. For $n \in d\mathbb{Z}$ large enough, denote by $(T_n, \mathbf{f}_n)$ a sample of the law $\mathcal{L}(\mu,\eta)$ conditioned to the event $\{|\mathbf{T}| = n\}$. If the annealed measure $\sum_k \mu(k)\eta_k(\mathrm{d}s)$ admits a positive and finite first moment, then the following holds.*

1. *If μ has finite variance σ^2, then*

$$(dm)^{-1/2} \cdot \left(V(T_{dm}), d_{T_{dm}}, \frac{\mathbf{f}_{dm}}{|\mathbf{f}_{dm}|}\right) \xrightarrow[m\to\infty]{\mathrm{GHP},\,(d)} \frac{\sqrt{2}}{\sigma} \cdot \mathcal{T}^{(2)}.$$

2. *If $\mu([x,+\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ for some $c > 0$ and $\theta \in (1,2)$, then*

$$(dm)^{-(1-1/\theta)} \cdot \left(V(T_{dm}), d_{T_{dm}}, \frac{\mathbf{f}_{dm}}{|\mathbf{f}_{dm}|}\right) \xrightarrow[m\to\infty]{\mathrm{GHP},\,(d)} \left[\frac{\theta-1}{c\,\Gamma(2-\theta)}\right]^{1/\theta} \cdot \mathcal{T}^{(\theta)}.$$

We shall first prove a rather general functional law of large numbers for the cumulative sum $s \mapsto \sum_{i\leq sn} \mathbf{f}_n(v_i)$, where $(v_i)_i$ are the vertices of T_n listed in depth-first order.

Lemma 5.13. *Let μ be a critical offspring distribution with span $d \geq 1$, and with $\mu(1) \neq 1$. Let also $(\eta_k)_{k\geq 0}$ be Borel probability laws on $\mathbb{R}_{\geq 0}$. For $n \in d\mathbb{Z}$ large enough, denote by $(T_n, \mathbf{f}_n)$ a sample of the law $\mathcal{L}(\mu,\eta)$ conditioned to the event $\{|\mathbf{T}| = n\}$. Assume that the annealed measure $\sum_k \mu(k)\eta_k(\mathrm{d}s)$ admits a positive and finite first moment and denote by $\overline{\eta} > 0$ its expectation. Assume also that μ is in the domain of attraction of a stable distribution of index α with $1 < \alpha \leq 2$. Then there holds the following convergence in probability*

$$\sup_{s\in[0,1]} \left| \frac{1}{n} \sum\nolimits_{i\leq sn} \mathbf{f}_n(v_i) - s\overline{\eta} \right| \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0,$$

where $(v_0,\ldots,v_n)$ are the vertices of T_n listed in depth-first order.

Proof. Let us denote by $(\widetilde{v}_i)_{0\leq i\leq n}$ a uniform cyclic shift of the sequence $(v_i)_{0\leq i\leq n}$, that is to say $\widetilde{v}_i = v_{i+\tau_n \bmod (n+1)}$ for all $0 \leq i \leq n$, where τ_n is a uniformly random element of $\{0,\ldots,n\}$, sampled independently from other variables. Then an elementary re-arranging of sums yields that

$$\sup_{s\in[0,1]} \left| \frac{1}{n} \sum_{i\leq sn} \mathbf{f}_n(v_i) - s\overline{\eta} \right| \leq 2 \sup_{0\leq s\leq t\leq 1} \left| \frac{1}{n} \sum_{sn\leq i\leq tn} \mathbf{f}_n(\widetilde{v}_i) - (t-s)\overline{\eta} \right|.$$

Distinguishing upon whether s and t are smaller than $1/2$, and cutting the sum at $1/2$ in the case $s < 1/2 < t$, we can bound further

$$\begin{aligned}
&\sup_{0\le s\le t\le 1}\left|\frac{1}{n}\sum_{sn\le i\le tn}\mathbf{f}_n(\widetilde{v}_i)-(t-s)\overline{\eta}\right| \\
&\le 2\sup_{0\le s\le t\le 1/2}\left|\frac{1}{n}\sum_{sn\le i\le tn}\mathbf{f}_n(\widetilde{v}_i)-(t-s)\overline{\eta}\right|+2\sup_{1/2\le s\le t\le 1}\left|\frac{1}{n}\sum_{sn\le i\le tn}\mathbf{f}_n(\widetilde{v}_i)-(t-s)\overline{\eta}\right| \\
&=2\sup_{0\le s\le t\le 1/2}\left|\frac{1}{n}\sum_{sn\le i\le tn}\mathbf{f}_n(\widetilde{v}_i)-(t-s)\overline{\eta}\right|+2\sup_{0\le s\le t\le 1/2}\left|\frac{1}{n}\sum_{sn\le i\le tn}\mathbf{f}_n(\widetilde{v}_{i+\lceil n/2\rceil})-(t-s)\overline{\eta}\right| \\
&\le 4\sup_{0\le t\le 1/2}\left|\frac{1}{n}\sum_{i\le tn}\mathbf{f}_n(\widetilde{v}_i)-t\overline{\eta}\right|+4\sup_{0\le t\le 1/2}\left|\frac{1}{n}\sum_{i\le tn}\mathbf{f}_n(\widetilde{v}_{i+\lceil n/2\rceil})-t\overline{\eta}\right|.
\end{aligned}$$

Now notice that $(\widetilde{v}_{i+\lceil n/2\rceil})_{0\le i\le n}$ is itself a uniform cyclic shift of the sequence $(v_i)_{0\le i\le n}$, so that the second term in the last display has the same law as the first one, and we only need to bound this one. We have reduced the problem to showing that the following convergence in probability holds

$$\sup_{0\le t\le 1/2}\left|\frac{1}{n}\sum_{i\le tn}\mathbf{f}_n(\widetilde{v}_i)-t\overline{\eta}\right|\xrightarrow[n\to\infty,\,n\in d\mathbb{Z}]{P}0. \tag{5.12}$$

We now appeal to the so-called cycle lemma, see [Pit06, Paragraph 6.1] and more precisely Lemma 6.1 for the cycle lemma and Lemma 6.3 for its application to trees. In our setting it implies that the cyclically shifted sequence of degrees $(k_{\widetilde{v}_i}(T_n))_{0\le i\le n}$ has the same law as that of an i.i.d. sequence $(\xi_i)_{0\le i\le n}$ of samples of the law μ conditioned to the event $\{\sum_{0,\le i\le n}(\xi_i-1)=-1\}$. Now recall that conditionally on T_n, each variable $\mathbf{f}(v_i)$ is sampled according to the law $\eta_{k_{v_i}(T_n)}$ and independently of the family $(\mathbf{f}(v_j))_{j\ne i}$. Therefore the identity in distribution obtained from the cycle lemma admits a straightforward generalization for the cyclically shifted sequence $\big(k_{\widetilde{v}_i}(T_n),\mathbf{f}_n(\widetilde{v}_i)\big)_{0\le i\le n}$. More precisely, let $(\xi_i,X_i)_{i\ge 0}$ be an i.i.d. sequence such that ξ_0 has law μ, and such that conditionally on ξ_0 the variable X_0 has law η_{ξ_0}. Then, there holds the following identity in distribution

$$\mathrm{Law}\Big(\big(k_{\widetilde{v}_i}(T_n),\mathbf{f}_n(\widetilde{v}_i)\big)_{0\le i\le n};P\Big)=\mathrm{Law}\Big(\big(\xi_i,X_i\big)_{0\le i\le n};P\Big(\cdot\ \Big|\ \textstyle\sum_{0\le i\le n}(\xi_i-1)=-1\Big)\Big).$$

Using the Markov property at time $\lfloor n/2\rfloor$ for the random walk $(\sum_{0\le i\le k}(\xi_i-1))_{k\ge 0}$, we get for every non-negative Borel function $F\colon(\mathbb{Z}\times\mathbb{R})^{\lfloor n/2\rfloor+1}\to\mathbb{R}_{\ge 0}$ the following

$$\begin{aligned}
E\Big[F\Big(\big(k_{\widetilde{v}_i}(T_n),\mathbf{f}_n(\widetilde{v}_i)\big)_{0\le i\le n/2}\Big)\Big]&=E\left[F\Big(\big(\xi_i,X_i\big)_{0\le i\le n/2}\Big)\frac{\mathbb{1}\{\sum_{0\le i\le n}(\xi_i-1)=-1\}}{P\left(\sum_{0\le i\le n}(\xi_i-1)=-1\right)}\right]\\
&=E\left[F\Big(\big(\xi_i,X_i\big)_{0\le i\le n/2}\Big)\frac{q_{n-\lfloor n/2\rfloor}(-1-\sum_{0\le i\le n/2}(\xi_i-1))}{q_{n+1}(-1)}\right],
\end{aligned}$$

where we used the notation $q_k(j)=P\big(\sum_{1\le i\le k}(\xi_i-1)=j\big)$. Let us remark that there exists n_0 such that $q_n(-1)\ne 0$ for all the integers $n\ge n_0$ which belong to $d\mathbb{Z}$, and that

$$\sup_{n\ge n_0,\,n\in d\mathbb{Z}}\ \sup_{j\in\mathbb{Z}}\frac{q_{n-\lfloor n/2\rfloor}(l)}{q_{n+1}(-1)}<+\infty. \tag{5.13}$$

Indeed, we may use the local limit theorem [Ibr71, Theorem 4.2.1] which covers the case of random walks on $\mathbb{Z}$ whose increments have law a (possibly non-aperiodic) distribution in the domain of attraction of a stable distribution with index $\alpha \in (1,2]$, such as the random walk $\left(\sum_{0\le i\le k}(\xi_i - 1)\right)_{k\ge 0}$. This gives us

$$\lim_{k\to\infty} \sup_{j\in\mathbb{Z}} \left| \frac{B_k}{d} q_k(j) - g\left(\frac{j}{B_k}\right) \right| = 0,$$

where g is the density function of some stable distribution with index α satisfying notably $g(0) \neq 0$, and where $(B_k)_k$ is a sequence of numbers such that $(k^{-1/\alpha} B_k)_k$ is slowly varying by [Ibr71, Paragraph 2.2]. We easily deduce (5.13) from the last display. Therefore there exists a constant $C > 0$ such that for every non-negative Borel function $F\colon (\mathbb{Z}\times\mathbb{R})^{\lfloor n/2\rfloor+1} \to \mathbb{R}_{\ge 0}$, we have for $n \ge n_0$,

$$E\left[F\Big(\big(k_{\widetilde{v}_i}(T_n), \mathbf{f}_n(\widetilde{v}_i)\big)_{0\le i\le n/2}\Big)\right] \le C \cdot E\left[F\Big(\big(\xi_i, X_i\big)_{0\le i\le n/2}\Big)\right].$$

We deduce for every $\varepsilon > 0$ and every $n \ge n_0$,

$$P\left(\sup_{0\le t\le 1/2}\left|\frac{1}{n}\sum_{i\le tn}\mathbf{f}_n(\widetilde{v}_i) - t\overline{\eta}\right| \ge \varepsilon\right) \quad\le\quad C \;\cdot\; P\left(\sup_{0\le t\le 1/2}\left|\frac{1}{n}\sum_{i\le tn} X_i - t\overline{\eta}\right| \ge \varepsilon\right). \tag{5.14}$$

Notice that the variables $(X_i)_{i\ge 0}$ are i.i.d. with mean $\overline{\eta}$ by definition. By the strong law of large numbers, it holds almost surely that for all $t \in [0,1]\cup\mathbb{Q}$,

$$\frac{1}{N}\sum_{i\le tN} X_i \xrightarrow[N\to\infty]{} t\overline{\eta}.$$

Since the variables (X_i) are non-negative, the left-hand-side is a (random) non-decreasing function of t for all $N \ge 1$. In particular, the pointwise almost sure convergence above yields by Dini's theorem almost sure convergence in the sup norm, namely

$$\sup_{0\le t\le 1/2}\left|\frac{1}{n}\sum_{i\le tn} X_i - t\overline{\eta}\right| \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{\text{a.s.}} 0.$$

Combining this with (5.14), we obtain the desired convergence in probability (5.12) and this concludes the proof. □

Proof of Proposition 5.12. Let U be sampled uniformly and independently of other variables and for $n \in d\mathbb{Z}$ large enough, let $(v_0,\ldots,v_n)$ be the vertices of T_n listed in depth-first order. We denote by $x_n(U)$ the vertex $v_{\lfloor (n+1)U\rfloor}$. Let also $y_n(U)$ be the vertex $v_{k_n(U)}$, where $k_n(U)$ is the smallest index $k \in \{0,\ldots,n\}$ such that $\sum_{i\le k}\mathbf{f}_n(v_i) \ge U|\mathbf{f}_n|$. By construction, conditionally on $(T_n,\mathbf{f}_n)$, the random vertex $x_n(U)$ has law ν_{T_n} and $y_n(U)$ has law $\mathbf{f}_n/|\mathbf{f}_n|$. Now by Lemma 5.13, the sequence of functions $(s \mapsto \frac{1}{|\mathbf{f}_n|}\sum_{i\le sn}\mathbf{f}_n(v_i))_n$ converges in probability for the uniform norm to the identity function $s\mapsto s$ when n tends to ∞ in $d\mathbb{Z}$. We deduce using the definition of $k_n(U)$ that $\left|\frac{k_n(U)}{n} - U\right|$ converges to 0 in probability, and in particular that the same goes for $\left|\frac{k_n(U)}{n+1} - \frac{\lfloor (n+1)U \rfloor}{n+1}\right|$.

Let $\theta = 2$ if μ has finite variance as in case 1. of the statement, or let θ be such that $\mu([x, +\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ for some $c > 0$ and $\theta \in (1,2)$ as in case 2. of the statement. Let us set $D_n = n^{-(\theta-1)/\theta}\, d_{T_n}$ the rescaled distance function on $V(T_n)$ and $h_n \colon s \in [0,1] \mapsto n^{-(\theta-1)/\theta}\, h_{T_n}(v_{\lfloor (n+1)s \rfloor})$ be the rescaled height process of T_n. Using the following well-known bound on distances in a tree

$$d_{T_n}(v_i, v_k) \leq h_{T_n}(v_i) + h_{T_n}(v_k) - 2 \inf_{j \in \{i,\ldots,k\}} h_{T_n}(v_j) + 2,$$

we get the bound

$$\begin{aligned} D_n(x_n(U), y_n(U)) &= \frac{1}{n^{(\theta-1)/\theta}}\, d_{T_n}(v_{k_n(U)}, v_{\lfloor (n+1)U \rfloor}) \\ &\leq 2\, \omega_{h_n}\left(\left|\tfrac{k_n(U)}{n+1} - \tfrac{\lfloor (n+1)U \rfloor}{n+1}\right|\right) + \frac{2}{n^{(\theta-1)/\theta}}, \end{aligned}$$

where $\omega_{h_n}(\delta) = \sup_{|x-y|\leq\delta} |h_n(x) - h_n(y)|$ is the modulus of continuity of h_n defined for all $\delta > 0$.

We justified in the proof of Corollary 5.11 that [Kor13, Theorem 3] applies, even if μ is not assumed to be aperiodic in our setting. This theorem tells us in particular that the rescaled height process h_n converges in distribution as n tends to infinity to some limit, in the Skorokhod topology. Since the limit is almost surely continuous, properties of the Skorokhod topology imply that the convergence actually holds in distribution with respect to the topology of uniform convergence. By characterization of tightness for this topology, we have for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{\delta\to 0}\, P\left(\omega_{h_n}(\delta) \geq \varepsilon\right) = 0,$$

from which we deduce that

$$D_n(x_n(U), y_n(U)) \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0. \tag{5.15}$$

Recall that conditionally on $(T_n, \mathbf{f}_n)$, the vertices $x_n(U)$ and $y_n(U)$ have law ν_{T_n} and $\mathbf{f}_n/|\mathbf{f}_n|$ respectively. This yields using the definition (5.1) a bound for the Prokhorov distance between these two measures

$$d_{\mathrm{P}}^{V(T_n), D_n}\left(\nu_{T_n}, \tfrac{\mathbf{f}_n}{|\mathbf{f}_n|}\right) \leq \inf\Big\{\varepsilon > 0 \colon P\left(D_n(x_n(U), y_n(U)) \geq \varepsilon \,\Big|\, T_n\right) \leq \varepsilon\Big\}.$$

In particular, we have for $\varepsilon > 0$,

$$\begin{aligned} P\left(d_{\mathrm{P}}^{V(T_n), D_n}\left(\nu_{T_n}, \tfrac{\mathbf{f}_n}{|\mathbf{f}_n|}\right) \geq \varepsilon\right) &\leq P\Big(P\left(D_n(x_n(U), y_n(U)) \geq \varepsilon \mid T_n\right) \geq \varepsilon\Big) \\ &\leq \varepsilon^{-1} P\big(D_n(x_n(U), y_n(U)) \geq \varepsilon\big), \end{aligned}$$

where we used Markov's inequality to get the last upper bound. By (5.15), we get the convergence in probability

$$d_{\mathrm{P}}^{V(T_n),D_n}\left(\nu_{T_n}, \frac{\mathbf{f}_n}{|\mathbf{f}_n|}\right) \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0.$$

By inequality (5.2), we deduce that

$$d_{\mathrm{GHP}}\left((V(T_n), D_n, \nu_{T_n}), \left(V(T_n), D_n, \frac{\mathbf{f}_n}{|\mathbf{f}_n|}\right)\right) \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0.$$

We conclude the proof by combining the last display with Corollary 5.11. □

The spine decomposition and size-biased laws

In this section we present a *size-biasing* relation for the block-tree, in the sense of [LPP95]. Actually, we extend in a straightforward way this size-biasing relation to our setting, where we have a Galton-Watson tree and some decorations, namely the blocks. More precisely, consider the following measure on maps with a distinguished vertex of their block tree $(\mathfrak{m}, v)$

$$\mathbb{P}_u(\mathrm{d}\mathbf{M}) \sum_{v\in\mathbf{T}} \delta_v(\mathrm{d}V_\star),$$

where δ_v is the Dirac measure $A \mapsto \delta_v(A) = \mathbb{1}_{\{v\in A\}}$. Then this σ-finite measure can be decomposed as a sum of probability measures $\sum_{h\geq 1} \widehat{\mathbb{P}}_{u,h}(\mathrm{d}\mathbf{M}, \mathrm{d}V_\star)$, where under $\widehat{\mathbb{P}}_u^h$ the vertex $V_\star$ has height h in $\mathbf{T}$, its ancestors' degrees having *size-biased* law as defined below. The present section makes that precise.

Description of $\widehat{\mathbb{P}}_{u,h}$.

Definition 5.14. Let ν be a probability distribution on $\mathbb{Z}_{\geq 0}$ with finite expectation m_ν. Then the *size-biased* distribution $\widehat{\nu}$ is defined by

$$\forall k \in \mathbb{Z}_{\geq 0}, \quad \widehat{\nu}(k) = \frac{k\,\nu(k)}{m_\nu}.$$

When ν is a (sub-)critical offspring distribution with $\nu(0), \nu(1) \neq 0$, denote by $\left(\widehat{\mathrm{GW}}_{\nu,h}\right)_{h\geq 0}$ the following family of laws, on the sets of discrete trees with a distinguished vertex at height h respectively. It may be described algorithmically:

- Each vertex will either be mutant or normal, and their number of offspring are sampled independently from each other;
- Normal vertices have only normal children, whose number is sampled according to ν;
- Mutant vertices of height less than h have a number of children sampled according to the size-biased distribution $\widehat{\nu}$, all of which are normal except one, chosen uniformly, which is mutant;

- The only mutant vertex at height h reproduces like a normal vertex and is the distinguished vertex $V_\star$.

This yields a pair $(T, V_\star)$, where T is a discrete tree and $V_\star$ is a distinguished vertex of T with height h. We denote by $(V_i)_{0\le i\le h_{\mathbf{T}}(V_\star)}$ the ancestor line of $V_\star$, and L_i the order of V_{i+1} in the children of V_i respectively. Observe that the construction gives that $(k_{V_i}(\mathbf{T}))_i$ are i.i.d. with law $\widehat{\nu}$, and conditionally on those variables, the variables $(L_i)_i$ are independent with uniform law on $\{1,\dots,k_{V_i}(\mathbf{T})\}$ respectively.

We may now define the family of probability measures $(\widehat{\mathbb{P}}_{u,h})_{h\ge 0}$ as follows. Let $h \ge 0$.

- Sample $(\mathbf{T}, V_\star)$ according to the law $\widehat{\mathrm{GW}}_{\mu^u,h}$.
- For each $v \in \mathbf{T}$, sample independently and uniformly a 2-connected map $\mathfrak{b}_v^{\mathbf{M}}$ with $k_v(\mathbf{T})/2$ edges.
- Build the map $\mathbf{M}$ whose block decomposition is $(b_v^{\mathbf{M}})_{v\in\mathbf{T}}$, and $\mathbf{Q} = \varphi(\mathbf{M})$ its image by Tutte's bijection.

We are now equipped to state the size-biasing relation.

Proposition 5.15. *For $u \ge u_C$, the σ-finite measure $\mathbb{P}_u(\mathrm{d}\mathbf{M})\sum_{v\in\mathbf{T}}\delta_v(\mathrm{d}V_\star)$ on maps with a distinguished vertex of their block-tree decomposes as the following sum of probability measures,*

$$\mathbb{P}_u(\mathrm{d}\mathbf{M})\sum_{v\in\mathbf{T}}\delta_v(\mathrm{d}V_\star) = \sum_{h\ge 0}\widehat{\mathbb{P}}_{u,h}(\mathrm{d}\mathbf{M}, \mathrm{d}V_\star).$$

Proof. The standard size-biasing relation for (sub-)critical Galton-Watson trees reads

$$\mathrm{GW}_\nu(\mathrm{d}\mathfrak{t})\sum_{v\in\mathfrak{t}}\delta_v(\mathrm{d}v_\star) = \sum_{h\ge 0}(m_\nu)^h \cdot \delta_h(h_{\mathfrak{t}}(v_\star)) \cdot \widehat{\mathrm{GW}}_{\nu,h}(\mathrm{d}\mathfrak{t}, \mathrm{d}v_\star).$$

When $u \ge u_C$, the offspring distribution μ^u is critical, so $m_{\mu^u} = 1$. Specializing the last display to $\nu = \mu^u$ and to the value of $\mathfrak{t}$ corresponding to the block-tree of some map $\mathfrak{m}$, this gives for all such $(\mathfrak{m}, v_\star)$,

$$\mathrm{GW}_{\mu^u}(\mathfrak{t}) = \sum_{h\ge 0}\delta_h(h_{\mathfrak{t}}(v_\star))\widehat{\mathrm{GW}}_{\mu^u,h}(\mathfrak{t}, v_\star).$$

Therefore, if we multiply both sides by $\prod_{v\in\mathfrak{t}}\frac{1}{b_{k_v(\mathfrak{t})/2}}$, we get the following by Proposition 2.6:

$$\mathbb{P}_u(\mathfrak{m}) = \sum_{h\ge 0}\delta_h(h_{\mathfrak{t}}(v_\star)) \cdot \widehat{\mathrm{GW}}_{\mu^u,h}(\mathfrak{t}, v_\star) \cdot \prod_{v\in\mathfrak{t}}\frac{1}{b_{k_v(\mathfrak{t})/2}} = \sum_{h\ge 0}\widehat{\mathbb{P}}_{u,h}(\mathfrak{m}, v_\star).$$

Since $\sum_{v\in\mathfrak{t}}\delta_v(v_\star) = 1$, the last display expresses the measure $\mathbb{P}_u(\mathrm{d}\mathbf{M})\sum_{v\in\mathfrak{t}}\delta_v(\mathrm{d}V_\star)$ as a sum of the probability measures $(\widehat{\mathbb{P}}_{u,h})_{h\ge 0}$. □

Probabilistic properties of $\widehat{\mathbb{P}}_{u,h}$. Since we need metric information on blocks whose size follows the size-biased law $\widehat{\mu}^u$, let us introduce adequate notation. Let $u \ge u_C$. Denote by

$\widehat{\xi}_u$ a sample of the distribution $\widehat{\mu}^u$ on some probability space (Ω, P). Then jointly define the random variables $\widehat{\mathbf{B}}_u^{\mathrm{map}}$ and $\widehat{\mathbf{B}}_u^{\mathrm{quad}}$ as sampled uniformly among respective blocks with size $\widehat{\xi}_u/2$, in such a way that they are linked by Tutte's bijection, i.e. their joint law satisfies

$$\left(\widehat{\mathbf{B}}_u^{\mathrm{map}}, \widehat{\mathbf{B}}_u^{\mathrm{quad}}\right) \overset{(d)}{=} \left(B_{\widehat{\xi}_u/2}^{\mathrm{map}}, B_{\widehat{\xi}_u/2}^{\mathrm{quad}}\right).$$

Furthermore, conditionally on $\widehat{\xi}_u$, sample independently U a uniform label in $\{1, \dots, \widehat{\xi}_u\}$. This yields the following 4-tuple

$$(\widehat{\xi}_u \, , \, \widehat{\mathbf{B}}_u^{\mathrm{map}} \, , \, \widehat{\mathbf{B}}_u^{\mathrm{quad}} \, , \, U).$$

Lemma 5.16. *For all $h \geq 1$, we have the identity in law*

$$\mathrm{Law}\left(\left(k_{V_i}(\mathbf{T}) \, , \, \mathfrak{b}_{V_i}^{\mathbf{M}} \, , \, \mathfrak{b}_{V_i}^{\mathbf{Q}} \, , \, L_i\right)_{0 \leq i < h} ; \, \widehat{\mathbb{P}}_{u,h}\right) = \left[\mathrm{Law}\left((\widehat{\xi}_u \, , \, \widehat{\mathbf{B}}_u^{\mathrm{map}} \, , \, \widehat{\mathbf{B}}_u^{\mathrm{quad}} \, , \, U) \, ; \, P\right)\right]^{\otimes h}$$

where $\mathrm{Law}(X; Q)$ is the law of X under Q.

Proof. Recall that under $\widehat{\mathbb{P}}_{u,h}$, the pair $(\mathbf{T}, V_\star)$ has law $\widehat{\mathrm{GW}}_{\mu^u,h}$. By definition of the law $\widehat{\mathrm{GW}}_{\mu^u,h}$, the ancestor line of the distinguished vertex $V_\star$ in $\mathbf{T}$ is made of mutant vertices. This means that the family $(k_{v_i}(\mathbf{T}))_{0,\leq i<h}$ is i.i.d. sampled from the size-biased distribution $\widehat{\mu}^u$, which is the law of $\widehat{\xi}_u$, and that independently of each other, each V_{i+1} has uniform rank L_i among the $k_{V_i}(\mathbf{T})$ children of V_i. Hence we have the identity in law

$$\mathrm{Law}\left(\left(k_{V_i}(\mathbf{T}) \, , \, L_i\right)_{0 \leq i < h} ; \, \widehat{\mathbb{P}}_{u,h}\right) = \left[\mathrm{Law}\left((\widehat{\xi}_u \, , \, U); P\right)\right]^{\otimes h}.$$

Now under $\widehat{\mathbb{P}}_{u,h}$ the conditional law of the blocks $(\mathfrak{b}_v^{\mathbf{M}})_{v \in \mathbf{T}}$ with respect to $\mathbf{T}$ is that of independent blocks, sampled uniformly from blocks with size $(k_V(\mathbf{T})/2)_{v \in \mathbf{T}}$ respectively. In particular, the blocks $(\mathfrak{b}_{V_i}^{\mathbf{M}})_{0 \leq i < h}$ are sampled independently, uniformly from blocks with size $(k_{V_i}(\mathbf{T})/2)_{0 \leq i < h}$ respectively. Therefore the preceding identity in law extends to the following one

$$\mathrm{Law}\left(\left(k_{V_i}(\mathbf{T}) \, , \, \mathfrak{b}_{V_i}^{\mathbf{M}} \, , \, L_i\right)_{0 \leq i < h} ; \, \widehat{\mathbb{P}}_{u,h}\right) = \left[\mathrm{Law}\left((\widehat{\xi}_u \, , \, \widehat{\mathbf{B}}_u^{\mathrm{map}} \, , \, U); P\right)\right]^{\otimes h}.$$

Finally, recall from Proposition 2.13 that $\mathfrak{b}_{V_i}^{\mathbf{Q}}$ is the image of $\mathfrak{b}_{V_i}^{\mathbf{M}}$ by Tutte's bijection. Since by definition $\widehat{\mathbf{B}}_u^{\mathrm{quad}}$ is also the image of $\widehat{\mathbf{B}}_u^{\mathrm{map}}$ by this bijection, the identity in law extends to the one in the proposition. □

We get in particular from Lemma 5.16 that the variables $\left(D(\mathfrak{b}_{V_i}^{\mathbf{M}}, L_i)\right)_{0 \leq i < h}$ are i.i.d. under $\widehat{\mathbb{P}}_{u,h}$. It is a bit less clear that the variables $\left(D_{\mathbf{Q}}(\mathfrak{b}_{V_i}^{\mathbf{Q}}, L_i)\right)_{0 \leq i < h}$ from Lemma 5.8 are also i.i.d., since they seem to simultaneously depend on global metric properties of $\mathbf{Q}$.

Lemma 5.17. *Denote by $D(\mathfrak{b}, l)$ the distance in a simple quadrangulation $\mathfrak{b}$ between its root vertex and the closest endpoint of the l-th edge in the order induced by the block-tree decomposition, the same order as the one introduced before Lemma 5.8. Then for all $h \geq 1$,* *there is the identity in law*

$$\mathrm{Law}\Big((D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_i}, L_i))_{0\le i<h}\,;\,\widehat{\mathbb{P}}_{u,h}\Big) = \Big[\mathrm{Law}\Big(D(\widehat{\mathbf{B}}^{\mathrm{quad}}_u, U)\,;\,P\Big)\Big]^{\otimes h}.$$

Proof. Recall from the notation introduced for Lemma 5.8 that for $\mathfrak{b}$ a simple block of a quadrangulation $\mathfrak{q}$, and l an integer in $\{1,\dots,2|\mathfrak{b}|\}$, $D_{\mathfrak{q}}(\mathfrak{b}, l)$ is the graph distance in $\mathfrak{b}$ between the endpoints of the l-th edge of $\mathfrak{b}$ in breadth-first order, and the endpoint of the root edge of $\mathfrak{b}$ which is closest to the root vertex of $\mathfrak{q}$.

Denote by $\mathfrak{b}\mapsto F(\mathfrak{b})$ the mapping which reverses the rooted oriented edge of a simple quadrangulation. Introduce also for $\mathfrak{b}$ a simple quadrangulation, $f_{\mathfrak{b}}$ the permutation of $\{1,\dots,2|\mathfrak{b}|\}$ which maps the breadth-first order on $\mathfrak{b}$ to the breadth-first-order on $F(\mathfrak{b})$. Finally, define the event $\mathcal{E}_i$ that the root vertex of $b^{\mathbf{Q}}_{V_i}$ is closer to the root vertex of $\mathbf{Q}$ than the other endpoint of the root edge of $b^{\mathbf{Q}}_{V_i}$. Then by definition, for all $0\le i<h$ we have that

$$D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_i}, L_i) = \mathbb{1}_{\mathcal{E}_i}\cdot D\Big(\mathfrak{b}^{\mathbf{Q}}_{V_i}, L_i\Big) + (1-\mathbb{1}_{\mathcal{E}_i})\cdot D\Big(F(\mathfrak{b}^{\mathbf{Q}}_{V_i}), f_{\mathfrak{b}^{\mathbf{Q}}_{V_i}}(L_i)\Big).$$

Let $\mathcal{F}_i$ denote the sigma-algebra of the variables $(k_{V_j}(\mathbf{T})\,,\,\mathfrak{b}^{\mathbf{M}}_{V_j}\,,\,\mathfrak{b}^{\mathbf{Q}}_{V_j}\,,\,L_j)_{0\le j<i}$. Then by Lemma 5.16, we have that the tuple $(k_{V_i}(\mathbf{T})\,,\,\mathfrak{b}^{\mathbf{M}}_{V_i}\,,\,\mathfrak{b}^{\mathbf{Q}}_{V_i}\,,\,L_i)$ is independent of $\mathcal{F}_i$, and has the same law as $(\widehat{\xi}_u\,,\,\widehat{\mathbf{B}}^{\mathrm{map}}_u\,,\,\widehat{\mathbf{B}}^{\mathrm{quad}}_u\,,\,U)$. Now the crucial point is that the event $\mathcal{E}_i$ is $\mathcal{F}_i$-measurable, since it can be decided whether or not it holds by looking only at the first i blocks on the spine. In particular it is independent of $(k_{V_j}(\mathbf{T})\,,\,\mathfrak{b}^{\mathbf{M}}_{V_j}\,,\,\mathfrak{b}^{\mathbf{Q}}_{V_j}\,,\,L_j)_{j\ge i}$. This implies the following

$$\begin{aligned}&\mathrm{Law}\left(\Big(D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_i}, L_i)\Big)_{0\le i<h}\,;\,\widehat{\mathbb{P}}_{u,h}\right)\\ &= \bigotimes_{0\le i<h}\Big[\widehat{\mathbb{P}}_{u,h}(\mathcal{E}_i)\cdot\mathrm{Law}\Big(D(\widehat{\mathbf{B}}^{\mathrm{quad}}_u, U)\Big) + (1-\widehat{\mathbb{P}}_{u,h}(\mathcal{E}_i))\cdot\mathrm{Law}\Big(D\big(F(\widehat{\mathbf{B}}^{\mathrm{quad}}_u), f_{\widehat{\mathbf{B}}^{\mathrm{quad}}_u}(U)\big)\Big)\Big].\end{aligned}$$

The proposition is therefore proved if we justify the identity in law

$$D(\widehat{\mathbf{B}}^{\mathrm{quad}}_u, U) \overset{(d)}{=} D\big(F(\widehat{\mathbf{B}}^{\mathrm{quad}}_u), f_{\widehat{\mathbf{B}}^{\mathrm{quad}}_u}(U)\big). \tag{5.16}$$

To check this, first notice that F is a bijection since it is involutive, so that in particular the uniform law on simple quadrangulations with k edges is invariant under F. By definition, for $\mathfrak{b}$ a simple quadrangulation, $f_{\mathfrak{b}}$ is also a bijection so that the uniform measure on $\{1,\dots,2|\mathfrak{b}|\}$ is invariant under it. Denoting U_k a uniform random variable on $\{1,\dots,2k\}$, this gives for each $k\ge 1$ the identity in law

$$D(B^{\mathrm{quad}}_k, U_k) \overset{(d)}{=} D\big(F(B^{\mathrm{quad}}_k), f_{B^{\mathrm{quad}}_k}(U_k)\big).$$

Since the pair $(\widehat{\mathbf{B}}^{\mathrm{quad}}_u, U)$ is the $\widehat{\xi}_u/2$-mixture of the laws $(B_k, U_k)_{k\ge1}$, the identity in law (5.16) also holds and this concludes the proof. □

Moments of typical distances in a size-biased block. We may now examine how fat are the tails of this i.i.d. family of distances along the spine, which we wish to sum.

Proposition 5.18. *Let D be either the variable $D(\widehat{\mathbf{B}}_u^{\mathrm{map}}, U)$ or $D(\widehat{\mathbf{B}}_u^{\mathrm{quad}}, U)$. Then for $u > u_C$, there exists $\varepsilon > 0$ such that $E[\exp(tD)] < \infty$ for all real $t < \varepsilon$. And for $u = u_C$, we have $E\left[D^\beta\right] < \infty$ for all $0 < \beta < 2$.*

Proof. The variable D is defined as a distance in $\widehat{\mathbf{B}}_u$, where $\widehat{\mathbf{B}}_u$ is either $\widehat{\mathbf{B}}_u^{\mathrm{map}}$ or $\widehat{\mathbf{B}}_u^{\mathrm{quad}}$. Hence it suffices to prove that the above moments are finite when we replace D by $\mathrm{diam}(\widehat{\mathbf{B}}_u)$.

Let $u > u_C$. Then $\mathrm{diam}(\widehat{\mathbf{B}}_u) \le \widehat{\xi}_u$, and the latter variable has finite exponential moments since $P(\widehat{\xi}_u \ge x) = \sum_{2j \ge x} 2j\mu^u(\{2j\})$, where μ^u has a tail decaying exponentially fast by Theorem 2.8.

Now take $u = u_C = 9/5$ and let $\delta \in (0, 2)$. Also let $\varepsilon > 0$ to be chosen later depending on δ. Using the notation B_k for B_k^{map} or B_k^{quad}, set

$$p_\varepsilon(k) = P\left(\mathrm{diam}(B_k) \ge k^{1/4+\varepsilon}\right).$$

By Proposition 5.2, we have that $p_\varepsilon(k)$ decays stretched exponentially as $k \to \infty$. Therefore we get a constant $C > 0$ such that $k^2 p_\varepsilon(k) \le C$ for all k. Recall that we have $\mathrm{diam}(\widehat{\mathbf{B}}_u) \le \widehat{\xi}_u$. Distinguishing upon whether $\mathrm{diam}(\widehat{\mathbf{B}}_u) \le (\widehat{\xi}_u)^{1/4+\varepsilon}$ or $\mathrm{diam}(\widehat{\mathbf{B}}_u) > (\widehat{\xi}_u)^{1/4+\varepsilon}$ and taking a conditional expectation with respect to $\widehat{\xi}_u$, we get

$$\begin{aligned} E\left[\left(\mathrm{diam}(\widehat{\mathbf{B}}_u)\right)^{2-\delta}\right] &\le E\left[\left((\widehat{\xi}_u)^{1/4+\varepsilon}\right)^{2-\delta} \mathbb{1}_{\left\{\mathrm{diam}(\widehat{\mathbf{B}}_u) \le (\widehat{\xi}_u)^{1/4+\varepsilon}\right\}}\right] + E\left[(\widehat{\xi}_u)^{2-\delta} p_\varepsilon\left(\widehat{\xi}_u\right)\right] \\ &\le E\left[(\widehat{\xi}_u)^{(1/4+\varepsilon)(2-\delta)}\right] + C \\ &= \sum_{2j \ge 0} (2j)^{(1/4+\varepsilon)(2-\delta)} \cdot 2j\mu^{u_C}(\{2j\}) + C. \end{aligned}$$

If ε is small enough so that $(1/4+\varepsilon)(2-\delta) < 1/2$, then the last sum is finite since by Theorem 2.8 we have $\mu^{u_C}(\{2j\}) = O(j^{-5/2})$. Therefore $E\left[\left(\mathrm{diam}(\widehat{\mathbf{B}}_u)\right)^{2-\delta}\right] < \infty$. □

Let us make a brief commentary, and justify that when $u = u_C$, Proposition 5.18 is optimal, in the sense that $D(\widehat{\mathbf{B}}_{u_C}^{\mathrm{quad}}, U)$ does not have moments of order β for $\beta \ge 2$. Firstly, one easily checks that functionals on pointed measured metric spaces of the form

$$(X, x_0, d_X, \nu_X) \mapsto \int_X \nu_X(\mathrm{d}x) \left(d_X(x_0, x)\right)^\beta$$

are continuous with respect to the Gromov-Hausdorff-Prokhorov topology. Addario-Berry and Albenque [ABA21] prove the GHP convergence of size k uniform simple quadrangulations, rescaled by $\mathrm{cst} \cdot k^{-1/4}$, to the measured Brownian sphere $(\mathcal{S}, D^*, \lambda)$. This holds when putting either the uniform measure on vertices of B_k^{quad} or the size-biased one by [ABW17]. In particular, by the aforementioned continuity, we have the convergence in distribution

$$E\left[\left(\mathrm{cst}\cdot k^{-1/4}D(B_k^{\mathrm{quad}},U_k)\right)^\beta \;\middle|\; B_k^{\mathrm{quad}}\right] \xrightarrow[k\to\infty]{(d)} \int_{\mathcal{S}} \lambda(\mathrm{d}x)\big(D^*(x_0,x)\big)^\beta,$$

where U_k is uniform on $\{1,\dots,2k\}$ and x_0 is the distinguished point on the Brownian sphere. Since the variable $\int_{\mathcal{S}}\lambda(\mathrm{d}x)\big(D^*(x_0,x)\big)^\beta$ is almost surely positive, the left-hand-side forms a tight sequence of $(0,\infty)$-valued random variables. Therefore it is bounded away from 0 with *uniform* positive probability. This implies a lower bound $E\left[D(B_k^{\mathrm{quad}},U_k)^\beta\right] \geq c(k^{1/4})^\beta$, for some $c = c(\beta) > 0$. In particular,

$$\begin{aligned} E\left[D(\widehat{\mathbf{B}}_{u_C}^{\mathrm{quad}},U)^\beta\right] = \sum_{2j\geq 0} E\left[D(B_j^{\mathrm{quad}},U_j)^\beta\right]\cdot \widehat{\mu}^{u_C}(\{2j\}) &\geq \sum_{2j\geq 0} c j^{\beta/4}\cdot 2j\mu^{u_C}(\{2j\}) \\ &= \sum_{2j\geq 0}\Theta(j^{\beta/4+1-5/2}). \end{aligned}$$

The latter sum is infinite when $\beta \geq 2$, which proves that $D(\widehat{\mathbf{B}}_{u_C}^{\mathrm{quad}},U)$ does not have moments of order β for $\beta\geq 2$. The same argument would hold for $D(\widehat{\mathbf{B}}_{u_C}^{\mathrm{map}},U)$, but we lack at the moment the GHP convergence of size-k uniform 2-connected maps.

Moderate deviations estimate

When increments of a random walk possess only a polynomial moment of order $\beta > 1$, as is the case of $D(\widehat{\mathbf{B}}_u^{\mathrm{map}},U)$ and $D(\widehat{\mathbf{B}}_u^{\mathrm{quad}},U)$ when $u = u_C$, moderate and large deviation events can possibly have probabilities which decay slowly, that is polynomially with n. In the case of heavy-tailed increments, this indeed happens since those moderate and large deviation events can be realised by taking one large increment. This *one-big-jump* behaviour is actually precisely how these large deviations events are realised. This phenomenon, which we have already encountered in Section 4.1.1 for $u < u_C$, is known as condensation. For a more precise statement, see [Jan12, AL09, AB19].

One could hope that if we prevent the variables from condensing, we could still get stretched-exponentially small probabilities for large deviation events. We make this precise in the following proposition, by stating that this is the case when we suitably truncate the increments. We were not able to find an instance of such an estimate in the literature, although it has certainly been encountered in some form. We thus include a short proof, which as usual relies on a Chernoff bound.

Proposition 5.19. *Let X be a real random variable with i.i.d. copies $(X_i)_{i\geq 1}$. Assume that there exists $\beta\in(1,2]$ such that $E\left[|X|^\beta\right] < \infty$ and that we have $E[X] = 0$.*

Then, for all $\delta > 0$, $\gamma\in(0,1/\beta+\delta)$, and $\nu\in\big(0,\delta\wedge(1/\beta+\delta-\gamma)\big)$, there exists a constant $C > 0$ such that for all $n\geq 1$,

$$P\left(\max_{1\leq k\leq n}\sum_{i=1}^k X_i \mathbb{1}_{\{X_i\leq n^\gamma\}} > n^{1/\beta+\delta}\right) \leq C\exp(-n^\nu).$$

Remark 5.20. A straightforward adaptation of the proof shows that the conclusion still holds if the only assumptions on the variables $(X_i)_i$ are $E\big[X_i \mid X_1, \dots, X_{i-1}\big] \le 0$ and $\sup_{i\ge 1} E\big[|X_i|^\beta \mid X_1, \dots, X_{i-1}\big] < \infty$.

Proof of Proposition 5.19. Fix an arbitrary θ such that $\max(\gamma, 1/\beta) < \theta < 1/\beta + \delta$. By Chernoff's bound, we get for all $1 \le k \le n$,

$$P\left(\sum_{i=1}^k X_i \mathbb{1}_{\{X_i \le n^\gamma\}} > n^{1/\beta+\delta}\right) \le \exp(-n^{1/\beta+\delta-\theta})\Big(E\left[\exp\big(n^{-\theta} X \mathbb{1}_{\{X\le n^\gamma\}}\big)\right]\Big)^k$$
$$\le \exp(-n^{1/\beta+\delta-\theta})\Big(1 \vee E\left[\exp\big(n^{-\theta} X \mathbb{1}_{\{X\le n^\gamma\}}\big)\right]\Big)^n.$$

Therefore we obtain by a union bound the estimate

$$P\left(\max_{1\le k\le n} \sum_{i=1}^k X_i \mathbb{1}_{\{X_i \le n^\gamma\}} > n^{1/\beta+\delta}\right) \le n \cdot \exp(-n^{1/\beta+\delta-\theta})\Big(1 \vee E\left[\exp\big(n^{-\theta} X \mathbb{1}_{\{X\le n^\gamma\}}\big)\right]\Big)^n.$$

Since θ is arbitrary in the interval $\big(\max(\gamma, 1/\beta)\,,\ 1/\beta+\delta\big)$, the exponent $\nu := 1/\beta + \delta - \theta$ is arbitrary in the interval $\big(0, \delta \wedge (1/\beta + \delta - \gamma)\big)$. As a consequence, to prove the proposition it is sufficient to show that

$$E\left[\exp\Big(n^{-\theta} X \mathbb{1}_{\{X\le n^\gamma\}}\Big)\right] \le 1 + O(n^{-1}). \tag{5.17}$$

Notice that since $\beta \in (1, 2]$, for all $M > 0$ the following inequality holds for t near 0 or $-\infty$:

$$\exp(t) \le 1 + t + M|t|^\beta.$$

Therefore, if one takes M large enough it holds for all $t \in (-\infty, 1]$. Fix such a constant M.

Given $\lambda, s \ge 0$, distinguishing upon whether $\lambda x \in (-\infty, 1]$ or not and using that $x\mathbb{1}_{\{x\le s\}} \le x$ and $x\mathbb{1}_{\{\lambda x \le 1\}} \le x$ (even when $x < 0$), we get for all $x \in \mathbb{R}$,

$$\exp\Big(\lambda x \mathbb{1}_{\{x\le s\}}\Big) \le \Big(1 + \lambda x \mathbb{1}_{\{x\le s\}} + M\lambda^\beta |x|^\beta \mathbb{1}_{\{x\le s\}}\Big) \cdot \mathbb{1}_{\{\lambda x\le 1\}} + \exp(\lambda x \mathbb{1}_{\{x\le s\}}) \cdot \mathbb{1}_{\{\lambda x > 1\}}$$
$$\le 1 + \lambda x + M\lambda^\beta |x|^\beta + \exp(\lambda s) \cdot \mathbb{1}_{\{\lambda x > 1\}}. \tag{5.18}$$

Applying this inequality with $x = X$, $\lambda = n^{-\theta}$, $s = n^\gamma$ and taking expectations we obtain

$$E\left[\exp\Big(n^{-\theta} X \mathbb{1}_{\{X\le n^\gamma\}}\Big)\right] \le 1 + n^{-\theta} E[X] + M n^{-\beta\theta} E\left[|X|^\beta\right] + \exp\big(n^{\gamma-\theta}\big) \cdot P\left(X \ge n^\theta\right).$$

Recall that $E[X] = 0$ by hypothesis, that $\gamma - \theta < 0$ by choice of θ, and use Markov's inequality. This yields

$$E\left[\exp\Big(n^{-\theta} X \mathbb{1}_{\{X\le n^\gamma\}}\Big)\right] \le 1 + 0 + M n^{-\beta\theta} E\left[|X|^\beta\right] + \exp(1) \cdot n^{-\beta\theta} E\left[|X|^\beta\right].$$

Since by hypothesis $E\left[|X|^\beta\right] < \infty$, we have

$$E\left[\exp\left(n^{-\theta}X\mathbb{1}_{\{X\le n^\gamma\}}\right)\right] \le 1 + O(n^{-\beta\theta}) \le 1 + O(n^{-1})$$

where the last inequality comes from the choice of θ, which is greater than $1/\beta$. Therefore (5.17) is satisfied and the proposition is proved. □

A lemma to compare $\mathfrak{m}$, $\mathfrak{q}$ and $\mathfrak{t}$

Let us state a lemma that elaborates on the additivity of distances on consecutive blocks, so that we can bound the GHP-distance between a map (resp. a quadrangulation) and its block-tree scaled by some constant. Let κ_1 and κ_2 be positive constants. Let $\mathfrak{m}$ be a map, $\mathfrak{q}$ its associated quadrangulation by Tutte's bijection, and $\mathfrak{t}$ their block-tree.

For x a vertex of either $\mathfrak{m}$ or $\mathfrak{q}$, denote as in Lemmas 5.7 and 5.8 by $v_\star$ the vertex v of $\mathfrak{t}$ closest to the root of $\mathfrak{t}$ such that x is a vertex of $\mathfrak{b}_v^{\mathfrak{m}}$ (resp. $\mathfrak{b}_v^{\mathfrak{q}}$). Set similarly $h_\star = h_{\mathfrak{t}}(v_\star)$ the height of $v_\star$ in $\mathfrak{t}$, and $(v_i)_{0\le i\le h_\star}$ the ancestor line of $v_\star$ in $\mathfrak{t}$, with v_0 the root of $\mathfrak{t}$ and $v_{h_\star} = v_\star$. Also denote by x_i the root vertex of $\mathfrak{b}_{v_i}^{\mathfrak{m}}$ (resp. $\mathfrak{b}_{v_i}^{\mathfrak{q}}$). Finally, let $(l_i)_{0\le i<h_\star}$ be the respective breadth-first index of the corner in $\mathfrak{b}_{v_i}^{\mathfrak{m}}$ (resp. the edge in $\mathfrak{b}_{v_i}^{\mathfrak{q}}$) to which the root corner of $\mathfrak{b}_{v_{i+1}}^{\mathfrak{m}}$ (resp. the root edge of $\mathfrak{b}_{v_{i+1}}^{\mathfrak{q}}$) is attached. Finally, denote by $\Delta(\mathfrak{m})$ (resp. $\Delta(\mathfrak{q})$) the largest diameter of a block of $\mathfrak{m}$ (resp. $\mathfrak{q}$). We set the quantities

$$R(\mathfrak{m}, v_\star, \kappa_1) = \max_{0\le i<h_\star}\left|\sum_{j=i}^{h_\star-1}\left(D(\mathfrak{b}_{v_j}^{\mathfrak{m}}, l_j) - \kappa_1\right)\right|,$$

$$\text{and}\quad R(\mathfrak{q}, v_\star, \kappa_2) = \max_{0\le i<h_\star}\left|\sum_{j=i}^{h_\star-1}\left(D_{\mathfrak{q}}(\mathfrak{b}_{v_j}^{\mathfrak{q}}, l_j) - \kappa_2\right)\right|.$$

Notice that the preceding quantities depend on x only through $v_\star$ and therefore make sense as functions of only $(\mathfrak{m}, v_\star, \kappa_1)$ and $(\mathfrak{q}, v_\star, \kappa_2)$ respectively.

Lemma 5.21. *Let $\mathbf{f}^{\mathfrak{m}}$ and $\mathbf{f}^{\mathfrak{q}}$ be the functions on $V(\mathfrak{t})$ defined by $\mathbf{f}^{\mathfrak{m}}(v) = |V(\mathfrak{b}_v^{\mathfrak{m}})| - 1$ and $\mathbf{f}^{\mathfrak{q}}(v) = |V(\mathfrak{b}_v^{\mathfrak{q}})| - 2$ respectively. With the above notation, we have for all $\varepsilon > 0$,*

$$\mathrm{d}_{\mathrm{GHP}}\left(\varepsilon\cdot\underline{\mathfrak{m}},\, \varepsilon\kappa_1\cdot\left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{m}}}{|\mathbf{f}^{\mathfrak{m}}|}\right)\right) \le \kappa_1\varepsilon + \frac{3\varepsilon}{2}\Delta(\mathfrak{m}) + \varepsilon\max_{v_\star\in V(\mathfrak{t})} R(\mathfrak{m}, v_\star, \kappa_1) + \frac{2}{|\mathbf{f}^{\mathfrak{m}}|},$$

and

$$\mathrm{d}_{\mathrm{GHP}}\left(\varepsilon\cdot\underline{\mathfrak{q}},\, \varepsilon\kappa_2\cdot\left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{q}}}{|\mathbf{f}^{\mathfrak{q}}|}\right)\right) \le (\kappa_2 + 3)\varepsilon + \frac{3\varepsilon}{2}\Delta(\mathfrak{q}) + \varepsilon\max_{v_\star\in V(\mathfrak{t})} R(\mathfrak{q}, v_\star, \kappa_2) + \frac{4}{|\mathbf{f}^{\mathfrak{q}}|}.$$

Proof. Let us first treat the inequality involving $\mathfrak{q}$, which is a bit more involved. Consider the correspondence C between $V(\mathfrak{q})$ and $V(\mathfrak{t})$ defined as follows. A vertex x of $\mathfrak{q}$ is set in correspondence with a vertex v of $\mathfrak{t}$ if and only if v is the vertex $v_\star$ defined as above from x; and let $\rho(x) := v_\star$. Put differently, a vertex v of $\mathfrak{t}$ is put in correspondence precisely with the

$\mathfrak{f}^{\mathfrak{q}}(v) = |V(\mathfrak{b}_v^{\mathfrak{q}})| - 2$ vertices of the block $\mathfrak{b}_v^{\mathfrak{q}}$ which are not incident to the root edge of $\mathfrak{b}_v^{\mathfrak{q}}$ (except when v is the root vertex in which case v is in correspondence with all the vertices of $\mathfrak{b}_v^{\mathfrak{q}}$).

Let γ be the uniform measure on the previously defined set $C = \{(x, \rho(x)) \colon x \in V(\mathfrak{q})\} \subset V(\mathfrak{q}) \times V(\mathfrak{t})$. Let the function $\pi \colon C \to V(\mathfrak{t})$ be the restriction of the projection $\pi_{\mathfrak{t}} : (x, y) \in V(\mathfrak{q}) \times V(\mathfrak{t}) \mapsto y \in V(\mathfrak{t})$. The preimages of π have cardinal $|\pi^{-1}(v)| = |\rho^{-1}(v)| = \mathfrak{f}^{\mathfrak{q}}(v) + 2\delta_{\mathrm{root}}(v)$, where $\delta_{\mathrm{root}}(v)$ is an indicator that x is incident to the root-edge of $\mathfrak{q}$. Tautologically, the measure γ defines a coupling between its images by the projections $\pi_{\mathfrak{q}} : (x, y) \in V(\mathfrak{q}) \times V(\mathfrak{t}) \mapsto x \in V(\mathfrak{q})$ and $\pi_{\mathfrak{t}}$. That is to say, γ is a coupling of the measures $\nu_{\mathfrak{q}}$ and $\frac{\mathfrak{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathfrak{f}^{\mathfrak{q}}|+2}$. It is also supported by C, *i.e.* $\gamma\big((V(\mathfrak{q}) \times V(\mathfrak{t})) \setminus C\big) = 0$. By the triangle inequality and the preceding observations, we have

$$\begin{aligned}
&d_{\mathrm{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{q}}\,, \varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathfrak{f}^{\mathfrak{q}}}{|\mathfrak{f}^{\mathfrak{q}}|}\right)\right)\\
&\le d_{\mathrm{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{q}}\,, \varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathfrak{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathfrak{f}^{\mathfrak{q}}|+2}\right)\right) + d_{\mathrm{GHP}}\left(\varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathfrak{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathfrak{f}^{\mathfrak{q}}|+2}\right), \varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathfrak{f}^{\mathfrak{q}}}{|\mathfrak{f}^{\mathfrak{q}}|}\right)\right)\\
&\le \frac{\varepsilon}{2}\operatorname{dis}(C; d_{\mathfrak{q}}, \kappa_2 d_{\mathfrak{t}}) + d_{\mathrm{P}}^{(V(\mathfrak{t}),\varepsilon\kappa_2 d_{\mathfrak{t}})}\left(\tfrac{\mathfrak{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathfrak{f}^{\mathfrak{q}}|+2}, \tfrac{\mathfrak{f}^{\mathfrak{q}}}{|\mathfrak{f}^{\mathfrak{q}}|}\right).
\end{aligned}$$

The last inequality uses (5.2) to bound the second GHP distance by a Prokhorov distance. Now, the Prokhorov distance between two measures is bounded by their total variation distance, and for measures μ and ν we have elementarily $d_{\mathrm{TV}}(\frac{\mu+\nu}{|\mu|+|\nu|}, \frac{\mu}{|\mu|}) \le \frac{2|\nu|}{|\mu|}$. Therefore, we have

$$d_{\mathrm{P}}^{(V(\mathfrak{t}),\varepsilon\kappa_2 d_{\mathfrak{t}})}\left(\frac{\mathfrak{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathfrak{f}^{\mathfrak{q}}|+2}, \frac{\mathfrak{f}^{\mathfrak{q}}}{|\mathfrak{f}^{\mathfrak{q}}|}\right) \le d_{\mathrm{TV}}\left(\frac{\mathfrak{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathfrak{f}^{\mathfrak{q}}|+2}, \frac{\mathfrak{f}^{\mathfrak{q}}}{|\mathfrak{f}^{\mathfrak{q}}|}\right) \le \frac{4}{|\mathfrak{f}^{\mathfrak{q}}|}.$$

It remains to bound the distortion $\operatorname{dis}(C; d_{\mathfrak{q}}, \kappa_2 d_{\mathfrak{t}})$. This amounts to bounding $|d_{\mathfrak{q}}(x, \widetilde{x}) - \kappa_2 d_{\mathfrak{t}}(\rho(x), \rho(\widetilde{x}))|$ uniformly for all pairs of vertices $(x, \widetilde{x})$ in $V(\mathfrak{q}) \times V(\mathfrak{q})$. Let x and $\widetilde{x}$ be vertices of $\mathfrak{q}$. As before, we define the vertices $v_\star = \rho(x)$ and $\widetilde{v}_\star = \rho(\widetilde{x})$ in $V(\mathfrak{t})$, their respective heights $h_\star$ and $\widetilde{h}_\star$, their ancestor lines $(v_i)_{0\le i\le h_\star}$ and $(\widetilde{v}_i)_{0\le i\le h_\star}$, the labels $(l_i)_{0\le i<h_\star}$ and $(\widetilde{l}_i)_{0\le i<h_\star}$, and the vertices $(x_i)_{0\le i\le h_\star}$ and $(\widetilde{x}_i)_{0\le i\le h_\star}$.

Let i be such that $v_i = \widetilde{v}_i$ is the last common ancestor of $v_\star$ and $\widetilde{v}_\star$ in $\mathfrak{t}$. First notice that there exists $\delta_0 \in \{0, \pm1, \pm2\}$ such that

$$d_{\mathfrak{q}}(x, \widetilde{x}) = \delta_0 + d_{\mathfrak{q}}(x, x_{i+1}) + d_{\mathfrak{b}_{v_i}^{\mathfrak{q}}}(x_{i+1}, \widetilde{x}_{i+1}) + d_{\mathfrak{q}}(\widetilde{x}_{i+1}, \widetilde{x}). \tag{5.19}$$

Indeed, similarly as in the proof of Lemma 5.8, a geodesic from x to $\widetilde{x}$ must visit, once and in that order,

- the vertex x,
- either x_{i+1}, or x'_{i+1} the other endpoint of the root-edge of $\mathfrak{b}_{v_{i+1}}^{\mathfrak{q}}$,
- either $\widetilde{x}_{i+1}$, or $\widetilde{x}'_{i+1}$ the other endpoint of the root-edge of $\mathfrak{b}_{\widetilde{v}_{i+1}}^{\mathfrak{q}}$,
- the vertex $\widetilde{x}$.

Since x_{i+1} and x'_{i+1}, as well as $\widetilde{x}_{i+1}$ and $\widetilde{x}'_{i+1}$, are at distance 1 respectively, and since a geodesic between points in $\mathfrak{b}_{v_i}$ must stay in $\mathfrak{b}_{v_i}$, we get that (5.19) holds, for some $\delta_0 \in \{0, \pm 1, \pm 2\}$. Then, Lemma 5.8 allows to decompose the distances $d_\mathfrak{q}(x, x_{i+1})$ and $d_\mathfrak{q}(\widetilde{x}_{i+1}, \widetilde{x})$, with some $\delta, \widetilde{\delta}$ in $\{0,1\}$. Combining this with (5.19), this gives

$$\begin{aligned} d_\mathfrak{q}(x, \widetilde{x}) &= \delta_0 + \left[\delta + d_{\mathfrak{b}^\mathfrak{q}_{v_\star}}(x, x_{h_\star}) + \sum_{i+1 \le j < h_\star - 1} D_\mathfrak{q}(v_j, l_j)\right] + d_{\mathfrak{b}^\mathfrak{q}_{v_i}}(x_{i+1}, \widetilde{x}_{i+1}) \\ &\quad + \left[\widetilde{\delta} + d_{\mathfrak{b}^\mathfrak{q}_{\widetilde{v}_\star}}(\widetilde{x}, \widetilde{x}_{\widetilde{h}_\star}) + \sum_{i+1 \le j < \widetilde{h}_\star - 1} D_\mathfrak{q}(\widetilde{v}_j, \widetilde{l}_j)\right] \\ &= \delta_0 + \delta + \widetilde{\delta} + d_{\mathfrak{b}^\mathfrak{q}_{v_\star}}(x, x_{h_\star}) + d_{\mathfrak{b}^\mathfrak{q}_{v_i}}(x_{i+1}, \widetilde{x}_{i+1}) + d_{\mathfrak{b}^\mathfrak{q}_{\widetilde{v}_\star}}(\widetilde{x}, \widetilde{x}_{\widetilde{h}_\star}) \\ &\quad + \sum_{i+1 \le j < h_\star - 1} (D_\mathfrak{q}(v_j, l_j) - \kappa_2) + \sum_{i+1 \le j < \widetilde{h}_\star - 1} \left(D_\mathfrak{q}(\widetilde{v}_j, \widetilde{l}_j) - \kappa_2\right) \\ &\quad + \kappa_2(h_\star - i - 1) + \kappa_2(\widetilde{h}_\star - i - 1). \end{aligned}$$

The sum of the first six terms has absolute value at most $6 + 3\Delta(\mathfrak{q})$, the two sums have absolute value at most $R(\mathfrak{q}, x, \kappa_2)$ and $R(\mathfrak{q}, \widetilde{x}, \kappa_2)$ respectively, and the two remaining terms sum to $\kappa_2(h_\star - i - 1) + \kappa_2(\widetilde{h}_\star - i - 1) = \kappa_2 d_\mathfrak{t}(v, \widetilde{v}) - 2\kappa_2$. Therefore by the triangle inequality,

$$|d_\mathfrak{q}(x, \widetilde{x}) - \kappa_2 d_\mathfrak{t}(v, \widetilde{v})| \le 2\kappa_2 + 6 + 3\Delta(\mathfrak{q}) + R(\mathfrak{q}, x, \kappa_2) + R(\mathfrak{q}, \widetilde{x}, \kappa_2).$$

Since this holds for every $(x, v) = (x, \rho(x)) \in C$ and $(\widetilde{x}, \widetilde{v}) = (\widetilde{x}, \rho(\widetilde{x})) \in C$, the max of the right-hand side over $x, \widetilde{x} \in V(\mathfrak{q})$ is actually a bound on the distortion $\mathrm{dis}(C; d_\mathfrak{q}, \kappa_2 d_\mathfrak{t})$, which is precisely what we needed to conclude.

For the inequality involving $\mathfrak{m}$, the reasoning is quite similar. Take C the correspondence such that $x \in V(\mathfrak{m})$ is in correspondence with $v \in V(\mathfrak{t})$ if and only if v is the vertex $\rho(x) = v_\star$. Equivalently, a vertex v of $\mathfrak{t}$ is put in correspondence with the $\mathbf{f}^\mathfrak{m}(v) = |V(\mathfrak{b}^\mathfrak{m}_v)| - 1$ non-root vertices of the block $\mathfrak{b}^\mathfrak{m}_v$, except when v is the root vertex of $\mathfrak{t}$ in which case v is in correspondence with all the vertices of $\mathfrak{b}^\mathfrak{m}_v$. Then, similarly, the uniform measure γ on C defines a coupling between the measures $\nu_\mathfrak{m}$ and $\frac{\mathbf{f}^\mathfrak{m} + \delta_{\mathrm{root}}}{|\mathbf{f}^\mathfrak{m}| + 1}$, where $\delta_{\mathrm{root}}(x)$ is the indicator that x is the root vertex of $\mathfrak{m}$. As in the quadrangulation case, we have

$$d_{\mathrm{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{m}}, \varepsilon\kappa_1 \cdot \left(\mathfrak{t}, d_\mathfrak{t}, \tfrac{\mathbf{f}^\mathfrak{m}}{|\mathbf{f}^\mathfrak{m}|}\right)\right) \le \frac{\varepsilon}{2} \mathrm{dis}(C; d_\mathfrak{m}, \kappa_1 d_\mathfrak{t}) + d_{\mathrm{P}}^{(V(\mathfrak{t}), \varepsilon\kappa_1 d_\mathfrak{t})}\left(\tfrac{\mathbf{f}^\mathfrak{m} + \delta_{\mathrm{root}}}{|\mathbf{f}^\mathfrak{m}| + 1}, \tfrac{\mathbf{f}^\mathfrak{m}}{|\mathbf{f}^\mathfrak{m}|}\right),$$

with the similar bound

$$d_{\mathrm{P}}^{(V(\mathfrak{t}), \varepsilon\kappa_1 d_\mathfrak{t})}\left(\frac{\mathbf{f}^\mathfrak{m} + \delta_{\mathrm{root}}}{|\mathbf{f}^\mathfrak{m}| + 1}, \frac{\mathbf{f}^\mathfrak{m}}{|\mathbf{f}^\mathfrak{m}|}\right) \le \frac{2}{|\mathbf{f}^\mathfrak{m}|}.$$

The distortion of C is bounded with a very similar argument as above involving Lemma 5.7 instead of Lemma 5.8, except that we do not need to introduce $\delta_0, \delta, \widetilde{\delta}$. We leave the details to the reader. This gives for all $(x, v) \in C$ and $(\widetilde{x}, \widetilde{v}) \in C$, the bound

$$|d_\mathfrak{m}(x, \widetilde{x}) - \kappa_1 d_\mathfrak{t}(v, \widetilde{v})| \le 2\kappa_1 + 3\Delta(\mathfrak{m}) + R(\mathfrak{m}, x, \kappa_1) + R(\mathfrak{m}, \widetilde{x}, \kappa_1),$$

which proves the inequality involving $\mathfrak{m}$ in the statement. $\square$

Proof of Theorem 5.4

Let $u \geq u_C$. Let us first prove the claimed scaling limit for the block-tree $\mathbf{T}_{n,u}$. By Proposition 2.6, $\mathbf{T}_{n,u}$ has law $\mathrm{GW}(\mu^u, 2n)$, where the distribution μ^u has span 2.

Scaling limit of $\mathbf{T}_{n,u}$ for $u > u_C$. If $u > u_C$, then by the third statement of Theorem 2.8, μ^u is critical and admits a variance $\sigma(u)^2 < \infty$. Corollary 5.11 thus gives the announced scaling limit for $\mathbf{T}_{n,u}$,

$$(2n)^{-1/2} \cdot \underline{\mathbf{T}}_{n,u} \xrightarrow[n\to\infty]{GHP,(d)} \frac{\sqrt{2}}{\sigma(u)} \cdot \mathcal{T}^{(2)}.$$

The expression for $\sigma(u)$ in terms of the generating function $B_\circ$ which is given in the statement comes from a straightforward computation from the generating function of μ^u, which by (2.7) is

$$\sum_{k\geq 0} x^k \mu^u(k) = \frac{uB_\circ(x^2 y(u)) + 1 - u}{uB_\circ(y(u)) + 1 - u}.$$

This expression admits the explicit form in terms of u which is given in the statement and explained in Remark 5.5.

Scaling limit of $\mathbf{T}_{n,u}$ for $u = u_C$. If $u = u_C$, then by the second statement of Proposition 2.6, μ^{u_C} is critical and satisfies $\mu^{u_C}(\{2j\}) \sim \frac{1}{4\sqrt{3\pi}} j^{-5/2}$. Therefore we get the equivalent

$$\mu^{u_C}([x,\infty)) = \sum_{2j\geq x} \mu^{u_C}(\{2j\}) \sim \int_{x/2}^{\infty} \frac{1}{4\sqrt{3\pi}} s^{-5/2}\,\mathrm{d}s = \frac{1}{3}\sqrt{\frac{2}{3\pi}}\, x^{-3/2}.$$

Therefore, using Corollary 5.11 with $\theta = 3/2$, we get

$$(2n)^{1-2/3} \cdot \underline{\mathbf{T}}_{n,u_C} \xrightarrow[n\to\infty]{GHP,(d)} \left[\frac{\frac{3}{2} - 1}{\frac{1}{3}\sqrt{\frac{2}{3\pi}}\Gamma(2-\frac{3}{2})}\right]^{2/3} \cdot \mathcal{T}^{(3/2)}.$$

Using that $\Gamma(1/2) = \sqrt{\pi}$, the constant on the right-hand side simplifies and this translates as announced to

$$\frac{2}{3}(2n)^{-1/3} \cdot \underline{\mathbf{T}}_{n,u_C} \xrightarrow[n\to\infty]{GHP,(d)} \mathcal{T}^{(3/2)}.$$

Restatement of the problem. We let $\alpha = 2$ when $u > u_C$, and $\alpha = 3/2$ when $u = u_C$. We have identified the GHP-limit of $n^{-(\alpha-1)/\alpha} \cdot \underline{\mathbf{T}}_{n,u}$. By Proposition 5.12, the measured metric spaces

$$n^{-(\alpha-1)/\alpha} \cdot \left(V(\mathbf{T}_{n,u}), d_{\mathbf{T}_{n,u}}, \frac{\mathfrak{f}^{\mathbf{M}_{n,u}}}{|\mathfrak{f}^{\mathbf{M}_{n,u}}|}\right) \quad \text{and} \quad n^{-(\alpha-1)/\alpha} \cdot \left(V(\mathbf{T}_{n,u}), d_{\mathbf{T}_{n,u}}, \frac{\mathfrak{f}^{\mathbf{Q}_{n,u}}}{|\mathfrak{f}^{\mathbf{Q}_{n,u}}|}\right),$$

also converge to the same limit. It remains to compare them in the GHP sense to the measured metric spaces $n^{-(\alpha-1)/\alpha}\cdot\underline{\mathbf{M}}_{n,u}$ and $n^{-(\alpha-1)/\alpha}\cdot\underline{\mathbf{Q}}_{n,u}$ respectively. Let $\kappa_1=\kappa_u^{\mathrm{map}}$ and $\kappa_2=\kappa_u^{\mathrm{quad}}$. For the ease of reading, we introduce for $\eta>0$ the following "bad" events,

$$B^{\mathbf{M}}_{n,\eta}=\left\{d_{\mathrm{GHP}}\left(n^{-\frac{\alpha-1}{\alpha}}\cdot\underline{\mathbf{M}}\,,\,n^{-\frac{\alpha-1}{\alpha}}\kappa_1\cdot\left(V(\mathbf{T}),d_{\mathbf{T}},\tfrac{\mathbf{f}^{\mathbf{M}}}{|\mathbf{f}^{\mathbf{M}}|}\right)\right)\geq 2\eta\right\},$$
$$B^{\mathbf{Q}}_{n,\eta}=\left\{d_{\mathrm{GHP}}\left(n^{-\frac{\alpha-1}{\alpha}}\cdot\underline{\mathbf{Q}}\,,\,n^{-\frac{\alpha-1}{\alpha}}\kappa_2\cdot\left(V(\mathbf{T}),d_{\mathbf{T}},\tfrac{\mathbf{f}^{\mathbf{Q}}}{|\mathbf{f}^{\mathbf{Q}}|}\right)\right)\geq 2\eta\right\},$$

as well as auxiliary events for $\eta,\delta>0$,

$$A^{\mathbf{M}}_{1;n,\eta}=\left\{\exists v\in V(\mathbf{T}),\ R(\mathbf{M},v,\kappa_1)\geq\eta n^{\frac{\alpha-1}{\alpha}}\right\},$$
$$A^{\mathbf{Q}}_{1;n,\eta}=\left\{\exists v\in V(\mathbf{T}),\ R(\mathbf{Q},v,\kappa_2)\geq\eta n^{\frac{\alpha-1}{\alpha}}\right\},$$
$$A^{\mathbf{M}}_{2;n,\delta}=\left\{\Delta(\mathbf{M})\leq n^{(1+\delta)^2(\alpha-1)/2\alpha}\right\},$$
$$A^{\mathbf{Q}}_{2;n,\delta}=\left\{\Delta(\mathbf{Q})\leq n^{(1+\delta)^2(\alpha-1)/2\alpha}\right\},$$
$$A^{\mathbf{M}}_{3;n,\eta}=\left\{\kappa_1 n^{-\frac{\alpha-1}{\alpha}}+\frac{3n^{-\frac{\alpha-1}{\alpha}}}{2}\Delta(\mathbf{M})+\frac{2}{|\mathbf{f}^{\mathbf{M}}|}\geq\eta\right\},$$
$$A^{\mathbf{Q}}_{3;n,\eta}=\left\{(\kappa_2+3)n^{-\frac{\alpha-1}{\alpha}}+\frac{3n^{-\frac{\alpha-1}{\alpha}}}{2}\Delta(\mathbf{Q})+\frac{4}{|\mathbf{f}^{\mathbf{Q}}|}\geq\eta\right\}.$$

With this notation, what we have to show is

$$\lim_{\eta\to0}\limsup_{n\to\infty}\mathbb{P}_{n,u}(B^{\mathbf{M}}_{n,\eta})=0\quad\text{and}\quad\lim_{\eta\to0}\limsup_{n\to\infty}\mathbb{P}_{n,u}(B^{\mathbf{Q}}_{n,\eta})=0.$$

Using Lemma 5.21. Thanks to the GHP upper bounds in Lemma 5.21, we have

$$\mathbb{P}_{n,u}(B^{\mathbf{M}}_{n,\eta})\leq\mathbb{P}_{n,u}(A^{\mathbf{M}}_{1;n,\eta})+\mathbb{P}_{n,u}(A^{\mathbf{M}}_{3;n,\eta})\quad\text{and}\quad\mathbb{P}_{n,u}(B^{\mathbf{Q}}_{n,\eta})\leq\mathbb{P}_{n,u}(A^{\mathbf{Q}}_{1;n,\eta})+\mathbb{P}_{n,u}(A^{\mathbf{Q}}_{3;n,\eta}).\tag{5.20}$$

Bounding the diameters of the blocks. By Corollary 5.3, for $\delta>0$, the maximum diameter of blocks of either $\mathbf{M}_{n,u}$ or $\mathbf{Q}_{n,u}$ is bounded with probability $1-o(1)$ by $\max(n^{1/6},W(\mathbf{T}_{n,u})^{(1+\delta)/4})$, where $W(\mathfrak{t})$ denotes the largest degree of $\mathfrak{t}$. By Corollary 5.11, $W(\mathbf{T}_{n,u})$ is $o\left(n^{(1+\delta)/\alpha}\right)$ in probability. Since $(1+\delta)^2/4\alpha\geq1/6$, what precedes gives that for all $\delta>0$,

$$\max\bigl(\Delta(\mathbf{M}_{n,u}),\Delta(\mathbf{Q}_{n,u})\bigr)=o\left(n^{(1+\delta)^2/4\alpha}\right)\qquad\text{in probability.}$$

Notice that for δ small enough, $(1+\delta)^2/4\alpha<(1+\delta)^2(\alpha-1)/2\alpha$ since $\alpha\geq3/2$. This implies that for all $\delta>0$ sufficiently small, we have

$$\limsup_{n\to\infty}\mathbb{P}_{n,u}\Bigl((A^{\mathbf{M}}_{2;n,\delta})^c\Bigr)=0\quad\text{and}\quad\limsup_{n\to\infty}\mathbb{P}_{n,u}\Bigl((A^{\mathbf{Q}}_{2;n,\delta})^c\Bigr)=0.$$

By Lemma 5.13, the quantities $|\mathbf{f}^{\mathbf{M}}|$ and $|\mathbf{f}^{\mathbf{Q}}|$ are $\Theta(n)$ in probability under $\mathbb{P}_{n,u}$. Therefore, the preceding bound on diameters also implies the following

$$\lim_{\eta\to 0}\limsup_{n\to\infty} \mathbb{P}_{n,u}(A^{\mathbf{M}}_{3;n,\eta}) = 0 \quad \text{and} \quad \lim_{\eta\to 0}\limsup_{n\to\infty} \mathbb{P}_{n,u}(A^{\mathbf{Q}}_{3;n,\eta}) = 0.$$

Thanks to (5.20), it suffices to show that for sufficiently small $\delta > 0$,

$$\lim_{\eta\to 0}\limsup_{n\to\infty} \mathbb{P}_{n,u}(A^{\mathbf{M}}_{1;n,\eta} \cap A^{\mathbf{M}}_{2;n,\delta}) = 0 \quad \text{and} \quad \lim_{\eta\to 0}\limsup_{n\to\infty} \mathbb{P}_{n,u}(A^{\mathbf{Q}}_{1;n,\eta} \cap A^{\mathbf{Q}}_{2;n,\delta}) = 0.$$

Bounding the height of $\mathbf{T}_{n,u}$. We have identified above the scaling limit of $\mathbf{T}_{n,u}$ and the appropriate normalization of distances. In particular, $n^{(\alpha-1)/\alpha}\cdot\mathbf{T}_{n,u}$ is tight in the GHP-topology. An immediate consequence is that $n^{-(\alpha-1)/\alpha}H(\mathbf{T}_{n,u})$ is tight, where $H(\mathbf{T}_{n,u})$ is the height of $\mathbf{T}_{n,u}$. In particular, our problem reduces once more to showing that for sufficiently small $\delta > 0$,

$$\lim_{\eta\to 0}\limsup_{n\to\infty} \mathbb{P}_{n,u}\Big(A^{\mathbf{M}}_{1;n,\eta} \cap A^{\mathbf{M}}_{2;n,\delta} \cap \Big\{H(\mathbf{T}) \le \eta^{-1} n^{\frac{(\alpha-1)}{\alpha}}\Big\}\Big) = 0$$
$$\text{and} \quad \lim_{\eta\to 0}\limsup_{n\to\infty} \mathbb{P}_{n,u}\Big(A^{\mathbf{Q}}_{1;n,\eta} \cap A^{\mathbf{Q}}_{2;n,\delta} \cap \Big\{H(\mathbf{T}) \le \eta^{-1} n^{\frac{(\alpha-1)}{\alpha}}\Big\}\Big) = 0. \tag{5.21}$$

Using the spine decomposition. Fix $\delta > 0$, as small as necessary. Let us only treat the term involving $\mathbf{Q}$ in (5.21), as the expression for $R(\mathfrak{q}, x, \kappa_2)$ we used to define the event $A^{\mathbf{Q}}_{1;n,\eta}$ carries more dependence than that of $R(\mathfrak{m}, x, \kappa_1)$. Indeed the summands $D_{\mathfrak{q}}(b^{\mathfrak{q}}_{v_i}, l_i)$ involve in their definition a global metric property of $\mathfrak{q}$. The case of the term involving $\mathbf{M}$ is similar and easier to deal with.

Recall that by definition, the law $\mathbb{P}_{n,u}$ is the law $\mathbb{P}_u$, conditioned on the event $\{|\mathbf{T}| = n\}$. Since $\mathbb{P}_u(|\mathbf{T}| = n)$ decays polynomially by Corollary 5.11, we may get rid of the conditioning if the unconditional versions of the probabilities we wish to bound decay sufficiently fast. Namely, it suffices to prove that for all $\eta > 0$, the following (unconditional) probability is stretched-exponential in n

$$\mathbb{P}_u\Big(A^{\mathbf{Q}}_{1;n,\eta} \cap A^{\mathbf{Q}}_{2;n,\delta} \cap \{H(\mathbf{T}) \le \eta^{-1} n^{\frac{(\alpha-1)}{\alpha}}\}\Big). \tag{5.22}$$

By a union bound and then by Proposition 5.15, one can bound this by

$$\begin{aligned}
&\mathbb{E}_u\left[\sum_{v\in V(\mathbf{T})}\mathbb{1}_{\left\{R(\mathbf{Q},v,\kappa_2)\geq \eta n^{\frac{\alpha-1}{\alpha}}\right\}}\mathbb{1}_{\left\{H(\mathbf{T})\leq \eta^{-1} n^{\frac{\alpha-1}{\alpha}}\right\}}\mathbb{1}_{\{A^{\mathbf{Q}}_{2;n,\delta}\}}\right]\\
&=\sum_{h\geq 1}\widehat{\mathbb{P}}_{u,h}\left(\{R(\mathbf{Q},V_\star,\kappa_2)\geq \eta n^{\frac{\alpha-1}{\alpha}}\}\cap\{H(\mathbf{T})\leq \eta^{-1}n^{\frac{\alpha-1}{\alpha}}\}\cap A^{\mathbf{Q}}_{2;n,\delta}\right)\\
&=\sum_{h=1}^{\lfloor \eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\widehat{\mathbb{P}}_{u,h}\left(\{R(\mathbf{Q},V_\star,\kappa_2)\geq \eta n^{\frac{\alpha-1}{\alpha}}\}\cap A^{\mathbf{Q}}_{2;n,\delta}\right)\\
&=\sum_{h=1}^{\lfloor \eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\widehat{\mathbb{P}}_{u,h}\left(\left\{\max_{0\leq i<h}\left|\sum_{j=i}^{h-1}\left(D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{v_j},L_j)-\kappa_2\right)\right|\geq \eta n^{\frac{\alpha-1}{\alpha}}\right\}\cap A^{\mathbf{Q}}_{2;n,\delta}\right)\\
&\leq\sum_{h=1}^{\lfloor \eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\left[\widehat{\mathbb{P}}_{u,h}\left(\max_{0\leq i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_j},L_j)-\kappa_2\right)\geq \eta n^{\frac{\alpha-1}{\alpha}}\right)\right.\\
&\qquad\qquad\left.+\widehat{\mathbb{P}}_{u,h}\left(\max_{0\leq i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(\kappa_2-D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_j},L_j)\right)\geq \eta n^{\frac{\alpha-1}{\alpha}}\right)\right],
\end{aligned}$$

where

$$\psi_{n,\delta}(x)=x\mathbb{1}_{\left\{x\leq \max(\kappa_2,n^{(1+\delta)2(\alpha-1)/2\alpha})\right\}}.$$

The last inequality may require some explanations. First we apply a union bound with respect to the sign of the expression under the absolute value. Then we use the control that $A^{\mathbf{Q}}_{2;n,\delta}$ offers on $\Delta(\mathbf{Q})$ the maximum diameter of blocks of $\mathbf{Q}$, and the positivity of the distances $D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{v_j})$, to insert an indicator function. Hence the appearance of $\psi_{n,\delta}$.

Reducing to a large deviations event with truncated variables. We let $\left(\widehat{\xi}_{u,j},\widehat{\mathbf{B}}^{\text{quad}}_{u,j},U_j\right)_{j\geq 0}$ be an i.i.d. sequence of copies of the triple $\left(\widehat{\xi}_u,\widehat{\mathbf{B}}^{\text{quad}}_u,U\right)$. We also let $X_j=D\left(\widehat{\mathbf{B}}^{\text{quad}}_{u,j},U_j\right)-\kappa_2$. Then by Lemma 5.17, the arguments of the function $\psi_{n,\delta}$ that appear in the last upper bound we obtained, are actually i.i.d. and have joint law under $\widehat{\mathbb{P}}_{u,h}$ the law of $(X_j)_{0\leq j<h}$. Therefore this last upper bound is equal to

$$\sum_{h=1}^{\lfloor \eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\left[P\left(\max_{0\leq i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(X_j\right)\geq \eta n^{\frac{\alpha-1}{\alpha}}\right)+P\left(\max_{0\leq i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(-X_j\right)\geq \eta n^{\frac{\alpha-1}{\alpha}}\right)\right].$$

Since the sequence $(X_j)_{0\leq j<h}$ is i.i.d., we re-order the terms of the two sums which appear inside the probabilities in the last display, so that they run on indices $j=1,\ldots,i$. Hence, if we set $h_n=n^{(\alpha-1)/\alpha}$, then we can bound the last display by

$$\eta^{-1}h_n\left[P\left(\max_{0\leq i<\eta^{-1}h_n}\sum_{j=1}^{i}\psi_{n,\delta}\left(X_j\right)\geq \eta h_n\right)+P\left(\max_{0\leq i<\eta^{-1}h_n}\sum_{j=1}^{i}\psi_{n,\delta}\left(-X_j\right)\geq \eta h_n\right)\right].$$

Using the moderate deviations estimate. Let $\gamma = \gamma(\delta) = (1+\delta)^2/2$. Then $(h_n)^\gamma \geq \kappa_2$ for n large, so

$$\psi_{n,\delta}(x) = x\mathbb{1}_{\{x \leq \max(\kappa_2,(h_n)^\gamma)\}} = x\mathbb{1}_{\{x \leq (h_n)^\gamma\}}.$$

Let us check that the choice of $\kappa_2 = \kappa_u^{\text{quad}}$, the latter quantity being defined in (5.7), makes the variables (X_j) centred. Conditionally on the event $\{|\widehat{\mathbf{B}}_u^{\text{quad}}| = k\}$, the variable $\widehat{\mathbf{B}}_u^{\text{quad}}$ is a uniform simple quadrangulation with k edges, and U is uniform in $\{1,\dots,2k\}$. Therefore it holds that

$$\begin{aligned} E\left[D(\widehat{\mathbf{B}}_{u,j}^{\text{quad}}, U_j)\right] &= \sum_{k\geq 1} P\left(|\widehat{\mathbf{B}}_u^{\text{quad}}| = k\right) E\left[D(\widehat{\mathbf{B}}_u^{\text{quad}}, U) \mid |\widehat{\mathbf{B}}_u^{\text{quad}}| = k\right] \\ &= \sum_{k\geq 1} \hat{\mu}^u(2k)\mathcal{D}_k^{\text{quad}} = \kappa_2, \end{aligned}$$

where we used successively the definition of $(\mathfrak{b}, l) \mapsto D(\mathfrak{b}, l)$ in Lemma 5.17, the definition of $\mathcal{D}_k^{\text{quad}}$ after (5.7), and the definition of $\kappa_2 = \kappa_u^{\text{quad}}$ in (5.7). Therefore, the i.i.d. variables (X_j) are indeed centred. Now by Proposition 5.18, they possess moments of order β for all $1 \leq \beta < 2$. Since for δ sufficiently small we have $\gamma < 1$, Proposition 5.19 yields that

$$P\left(\max_{0\leq i<\eta^{-1}h_n} \sum_{j=1}^{i} X_j \mathbb{1}_{\{X_j \leq (h_n)^\gamma\}} \geq \eta h_n\right)$$

is stretched-exponential as $n \to \infty$, and the same holds when replacing (X_j) by $(-X_j)$.

This proves that for each $\eta > 0$, the probability (5.22) is indeed stretched-exponential in n, and concludes the proof. $\square$

5.1.3 Scaling limit of the quadrangulations in the subcritical case

Let us finally identify the scaling limit of the quadrangulation $\underline{\mathbf{Q}}_{n,u}$ when $u < u_C$.

Statement of the result

Denote by $\mathcal{S} = (\mathcal{S}, D^*, \lambda)$ the *Brownian sphere*, also known as the *Brownian Map*. One may take Proposition 5.23 below as a definition.

Theorem 5.22. *Assume $u < u_C = 9/5$. We have the following convergence in distribution for the* Gromov-Hausdorff-Prokhorov *topology*

$$\left(\frac{9(3+u)}{8(9-5u)}\right)^{1/4} n^{-1/4} \cdot \underline{\mathbf{Q}}_{n,u} \xrightarrow[n\to\infty]{(d),\text{GHP}} \mathcal{S}.$$

In the case $u = 1$, one recovers the Brownian sphere as the scaling limit of uniform quadrangulations with n faces, which has been proven in [LG13] and [Mie13]. It is also the scaling limit of uniform *simple* quadrangulations with n faces, which was proven in [ABA17]. The latter corresponds informally to the case $u \to 0$.

We emphasize that those results, and especially the one of [ABA17], serve as an input in our proof and we do not provide a new proof of them. Accordingly, let us precisely state the latter result, so that we can use it in the subsequent proof.

Proposition 5.23 ([ABA17]). *Uniform simple quadrangulations with k faces admit the Brownian sphere as scaling limit, with the following normalization*

$$\left(\frac{3}{8k}\right)^{1/4} \cdot B_k^{\mathrm{quad}} \xrightarrow[k\to\infty]{(d),\mathrm{GHP}} \mathcal{S}.$$

This is precisely the result [ABA17, Theorem 1.1], restricted to the case of simple quadrangulations. Notice that in their result, the scaling limit is stated in terms of M_n, a uniform simple quadrangulation with n vertices, not faces. This is not a problem since by Euler's formula, a quadrangulation has n vertices if and only if it has $n-2$ faces. Therefore B_k^{quad} has the same law as M_{k+2}.

Note that Theorem 5.22 only deals with the quadrangulation $\underline{\mathbf{Q}}_{n,u}$, but not the map $\underline{\mathbf{M}}_{n,u}$. Let us detail what would be needed to obtain a similar statement for $\underline{\mathbf{M}}_{n,u}$.

- To obtain a Gromov-Hausdorff scaling limit, the missing ingredient is the equivalent for 2-connected maps of the result of [ABA17], that is to say GH(P) convergence of uniform 2-connected maps with n edges, rescaled by a constant times $n^{-1/4}$, to the Brownian sphere.

- In order to strengthen this to GHP convergence when the map is equipped with the uniform measure on vertices, one would need the above mentioned convergence of 2-connected maps, but in the GHP sense. It would also require a way to compare, in the Prokhorov sense, the degree-biased measure on vertices of $\underline{\mathbf{M}}_{n,u}$, and the uniform measure. For quadrangulations on the other hand, this comparison can be done using [ABW17, Lemma 5.1].

The paper [ABW17] makes precise the relationship between the convergence of **uniform** quadrangulations with n faces [LG13, Mie13], and the convergence of **simple uniform** quadrangulations with n faces [ABA17]. It is shown that a quadrangulation sampled uniformly among those which have size n and whose biggest block has size $k(n) \sim cn$ with an adequate $c > 0$, converges jointly with said biggest block to the Brownian sphere, in the GHP sense.

The proof of Gromov-Hausdorff convergence for these quadrangulations amounts to showing that pendant submaps that are grafted on the macroscopic block have negligible diameter, that is $o\left(n^{1/4}\right)$, which is done by [ABW17, Proposition 1.12]. The strategy of proof is not directly applicable here, since it uses an *a priori* diameter bound on the pendant submaps, which we do not have for general u. As explained in what follows, it is sufficient to have an *a priori* diameter bound on single blocks themselves, which is why we need Proposition 5.2. To strengthen GH convergence to GHP convergence however, we use the same arguments as those exposed in [ABW17] modulo some technical details.

Sketch of the proof

On the combinatorics side, Section 4.1.1 characterizes the phase $u < u_C$ by a condensation phenomenon: when n is large, there is precisely one block of linear size, while others have size $O(n^{2/3})$. This theorem is stated for a map with law $\mathbb{P}_{n,u}$, that is the law of $\mathbf{M}_{n,u}$, but by Section 2.1.4, Tutte's bijection commutes with the block decomposition, so that the same happens for $\mathbf{Q}_{n,u}$.

On the metric side, there is not much more going on. The block-tree is subcritical in this phase by Theorem 2.8 and therefore has small height. Combining this with the $O(n^{2/3})$ bound on the size of non-macroscopic blocks, and the deviation estimate of Proposition 5.2 on diameters of blocks, we get that $\mathbf{Q}_{n,u}$ is approximately equal to its largest block, in the Gromov-Hausdorff sense in the scale $n^{1/4}$. This argument is rather general and should be easy to adapt to other models of graphs or maps with a block-tree decomposition under a condensation regime.

In order to strengthen this convergence to one in the Gromov-Hausdorff-Prokhorov sense, we use the rather general result [ABW17, Corollary 7.2], by comparing the mass measure on vertices with a projection on the macroscopic block, which is modulo some technical details an exchangeable vector on the edges where the pendant submaps are attached. This corollary tells that this random measure is well-approximated by its expectation, which is uniform on the edges of the macroscopic block, or equivalently that it is degree-biased on its vertices. The last part of the argument is specific to quadrangulations, for which we can compare the degree-biased and the uniform measure on vertices by [ABW17, Lemma 6.1].

Comparison of a quadrangulation and its biggest block

Let us introduce some notation. Let v° be the vertex of $\mathfrak{t}$ with largest outdegree, choosing one arbitrarily if there are several, and let $\mathfrak{q}^\circ = \mathfrak{b}^{\mathfrak{q}}_{v^\circ}$. Denote by $\mathfrak{t}[v]$ the subtree of descendants of a node v in $\mathfrak{t}$, rooted at v. For an edge e of $\mathfrak{q}^\circ = \mathfrak{b}^{\mathfrak{q}}_{v^\circ}$, the block-tree decomposition associates to it a vertex v, so that we can denote by $\mathfrak{q}[e]$ the quadrangulation whose block-tree decomposition is $(\mathfrak{b}^{\mathfrak{q}}_w)_{w\in\mathfrak{t}[v]}$. Recall that by convention, if v is a leaf then $\mathfrak{q}[e]$ is the edge map, with 2 vertices and 1 edge, the edge e. Write also $\mathfrak{q}^+$ for the quadrangulation whose block decomposition is $(\mathfrak{b}^{\mathfrak{q}}_v)_{v\in\mathfrak{t}[v^\circ]}$. In particular, $\mathfrak{q}^\circ$ is the simple core of $\mathfrak{q}^+$, and its other blocks are the blocks of the pendant subquadrangulations $(\mathfrak{q}[e])_{e\in E(\mathfrak{q}^\circ)}$. Finally, let $\pi^{\mathfrak{q}^+}_{\mathfrak{q}^\circ}$ be the probability measure on vertices of $\mathfrak{q}^\circ$ obtained by projection of the contribution to $\nu_{\mathfrak{q}}$ of each pendant map $(\mathfrak{q}[e])_{e\in E(\mathfrak{q}^\circ)}$ to the biggest block $\mathfrak{q}^\circ$. More formally, for each edge e of $\mathfrak{q}^\circ$, let $\{e^+, e^-\}$ be its extremities. Then,

$$\pi^{\mathfrak{q}^+}_{\mathfrak{q}^\circ} = \frac{1}{|V(\mathfrak{q}^+)| - |V(\mathfrak{q}^\circ)|} \sum_{e\in E(\mathfrak{q}^\circ)} \left(\left|V(\mathfrak{q}[e])\right| - 2\right) \left(\tfrac{1}{2}\delta_{e^-} + \tfrac{1}{2}\delta_{e^+}\right).$$

Observe that $V(\mathfrak{q}^\circ)$ shares exactly two elements with each $(V(\mathfrak{q}[e]))_{e\in E(\mathfrak{q}^\circ)}$, when those vertex-sets are naturally embedded in $V(\mathfrak{q})$. Hence the last display indeed defines a probability measure.

Lemma 5.24. *For any $\varepsilon > 0$, it holds that*

$$d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}, \varepsilon \cdot \underline{\mathfrak{q}}^\circ) \leq 2r_{\mathrm{GH}} + r_{\mathrm{P}} + d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}\left(\pi_{\mathfrak{q}^\circ}^{\mathfrak{q}^+}, \nu_{\mathfrak{q}^\circ}\right),$$

where

$$r_{\mathrm{GH}} = 2\varepsilon H(\mathfrak{t}) \max_{v \neq v^\circ} \operatorname{diam}(\mathfrak{b}_v^{\mathfrak{q}}) \quad \textit{and} \quad r_{\mathrm{P}} = \frac{2\left|V(\mathfrak{q}) \setminus V(\mathfrak{q}^+)\right|}{|V(\mathfrak{q})|}.$$

Proof. There are successive comparisons to be made for the GHP distance.

Metric comparison. The term r_{GH} bounds how distant the spaces $\varepsilon \cdot \mathfrak{q}$, $\varepsilon \cdot \mathfrak{q}^+$ and $\varepsilon \cdot \mathfrak{q}^\circ$ are, from a metric point of view, *i.e.* in the GH sense. Recall that we can see $\mathfrak{q}$ and $\mathfrak{q}^+$ as their biggest block $\mathfrak{q}^\circ$, together with some maps attached to it. Therefore one needs to bound the maximal diameter of the attached maps. We use a brutal bound on the diameter of the non-macroscopic blocks by their maximal diameter, together with a bound on the number of consecutive blocks in the attached maps. This number is bounded by $\operatorname{diam}(\mathfrak{t}) \leq 2H(\mathfrak{t})$. Therefore the maximal diameter of attached maps in $\varepsilon \cdot \mathfrak{q}$ or $\varepsilon \cdot \mathfrak{q}^+$ is bounded by

$$r_{\mathrm{GH}} := 2\varepsilon H(\mathfrak{t}) \max_{v \neq v^\circ} \operatorname{diam}(\mathfrak{b}_v^{\mathfrak{q}}).$$

In particular, take the correspondence B_1 on $V(\mathfrak{q}) \times V(\mathfrak{q}^+)$ such that $x \in V(\mathfrak{q})$ is in correspondence with only itself if it belongs to $V(\mathfrak{q}^+)$, or otherwise with both endpoints of the root-edge of $\mathfrak{q}^+$ if it belongs to $V(\mathfrak{q}) \setminus V(\mathfrak{q}^+)$. The uniform measure on B_1 is a coupling between $\nu_{\mathfrak{q}}$ and some measure μ^+ on $V(\mathfrak{q}^+)$. One therefore gets, using the triangle inequality and (5.2),

$$d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}, \varepsilon \cdot \underline{\mathfrak{q}}^+) \leq r_{\mathrm{GH}} + d_{\mathrm{P}}^{(V(\mathfrak{q}^+), \varepsilon d_{\mathfrak{q}})}(\mu^+, \nu_{\mathfrak{q}^+}). \tag{5.23}$$

Similarly, take the correspondence B_2 on $V(\mathfrak{q}^+) \times V(\mathfrak{q}^\circ)$ such that $x \in V(\mathfrak{q}^+)$ is in correspondence with only itself if it belongs to $V(\mathfrak{q}^\circ)$, or otherwise with both endpoints $\{e^+, e^-\}$ of the root-edge of $\mathfrak{q}[e]$ if x belongs to $V(\mathfrak{q}[e]) \setminus \{e^+, e^-\}$ for some edge $e \in E(\mathfrak{q}^\circ)$. Then the uniform measure on B_2 is a coupling between $\nu_{\mathfrak{q}^+}$ and some measure μ° on $V(\mathfrak{q}^+)$. We get as above

$$d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}^+, \varepsilon \cdot \underline{\mathfrak{q}}^\circ) \leq r_{\mathrm{GH}} + d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}(\mu^\circ, \nu_{\mathfrak{q}^\circ}). \tag{5.24}$$

Comparing the uniform measures on vertices of $\mathfrak{q}$ and $\mathfrak{q}^+$. Observe that $\nu_{\mathfrak{q}^+}$ is the counting measure on $V(\mathfrak{q}^+)$ renormalized to a probability distribution, while μ^+ is the renormalized version of the same counting measure but with additional mass

$$m := |V(\mathfrak{q}) \setminus V(\mathfrak{q}^+)| - 2,$$

the latter being split equally on the endpoints of the root-edge of $\mathfrak{q}^+$. Elementarily, this yields a total variation bound, as follows

$$d_{\mathrm{TV}}(\mu^+, \nu_{\mathfrak{q}^+}) \leq \frac{2m}{V(\mathfrak{q})} \leq \frac{2|V(\mathfrak{q}) \setminus V(\mathfrak{q}^+)|}{V(\mathfrak{q})} =: r_{\mathrm{P}}.$$

Since the Prokhorov distance is bounded by the total variation distance, we have

$$d_{\mathrm{P}}^{(V(\mathfrak{q}^+), \varepsilon d_{\mathfrak{q}})}(\mu^+, \nu_{\mathfrak{q}^+}) \leq r_{\mathrm{P}}. \tag{5.25}$$

Comparing the uniform measures on vertices of $\mathfrak{q}^+$ and $\mathfrak{q}^\circ$. From the definition of $\pi_{\mathfrak{q}^\circ}^{\mathfrak{q}^+}$ and from the following partitioning, under the natural embedding of the vertex-sets in $V(\mathfrak{q})$,

$$V(\mathfrak{q}^+) = V(\mathfrak{q}^\circ) \bigsqcup_{e \in E(\mathfrak{q}^\circ)} V(\mathfrak{q}[e]) \setminus \{e^+, e^-\},$$

observe that the measure μ° obtained from the correspondence B_2 above decomposes as follows

$$\mu^\circ = \tfrac{|V(\mathfrak{q}^\circ)|}{|V(\mathfrak{q}^+)|} \nu_{\mathfrak{q}^\circ} + \tfrac{|V(\mathfrak{q}^+)| - |V(\mathfrak{q}^\circ)|}{|V(\mathfrak{q}^+)|} \pi_{\mathfrak{q}^\circ}^{\mathfrak{q}^+}.$$

In particular, we obtain from (5.3) that

$$d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}(\mu^\circ, \nu_{\mathfrak{q}^\circ}) \leq d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}\left(\pi_{\mathfrak{q}^\circ}^{\mathfrak{q}^+}, \nu_{\mathfrak{q}^\circ}\right). \tag{5.26}$$

Concluding the proof. By the triangle inequality, we have

$$d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}, \varepsilon \cdot \underline{\mathfrak{q}}^\circ) \leq d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}, \varepsilon \cdot \underline{\mathfrak{q}}^+) + d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}^+, \varepsilon \cdot \underline{\mathfrak{q}}^\circ).$$

Using (5.23) and (5.25) to bound the first term, and (5.24) and (5.26) to bound the second one, we get the claimed inequality.

□

Exchangeable decorations

We aim to use Addario-Berry & Wen's argument for [ABW17, Lemma 6.2] which tells that for exchangeable attachments of mass on edges of Q_n, a quadrangulation with n faces sampled uniformly, the resulting measure on Q_n is asymptotically close to the uniform measure on vertices, in the sense of the Prokhorov distance on $n^{-1/4} \cdot Q_n$. They use the following ingredients:

1. A concentration inequality [ABW17, Lemma 5.2] which compares the measure with exchangeable attachments of mass on edges, to the degree-biased measure on vertices.

2. A Prokhorov comparison [ABW17, Lemma 5.1] between the degree-biased and uniform measure on vertices of a quadrangulation.

3. GHP convergence of $n^{-1/4} \cdot \underline{Q}_n$ to the Brownian sphere.

4. Properties of the Brownian sphere such as compacity and re-rooting invariance.

The first ingredient is rather general and actually stated for any graph in [ABW17, Lemma 5.3]. We will ever-so-slightly adapt its proof since there is a double edge in their setting which we do not have, and the mass is not projected on vertices in the exact same way. The second ingredient is specific to quadrangulations and one may need different arguments to compare the degree-biased and uniform measures for other classes of maps.

Let us state which result we extract for our purpose from Addario-Berry & Wen's paper. For $\mathbf{n} = (\mathbf{n}(e))_{e\in E(G)}$ a family of nonnegative numbers indexed by edges of a graph G, we denote its p-norm for $p \geq 1$ by

$$|\mathbf{n}|_p := \left(\sum_{e\in E(G)} (\mathbf{n}(e))^p \right)^{1/p}.$$

Then define the following measure on $V(G)$:

$$\mu_G^{\mathbf{n}} := \frac{1}{|\mathbf{n}|_1} \sum_{e\in E(G)} \mathbf{n}(e)\left(\tfrac{1}{2}\delta_{e^+} + \tfrac{1}{2}\delta_{e^-}\right),$$

with $\{e^+, e^-\}$ the set of endpoints of the edge e. Notice that this definition is slightly different from that of $\nu_G^{\mathbf{n}}$ in [ABW17, Section 5], because the mass of an edge is projected uniformly and independently on either of its endpoints in their case, while we deterministically split this mass on both endpoints. This does not change much except that we find it easier to work with. One of their results translates as the following.

Proposition 5.25 ([ABW17, Corollary 6.2]). *Let $Q_k = B_k^{\mathrm{quad}}$, which is a simple quadrangulation with k faces, sampled uniformly. Consider for each $k \geq 1$, a random family $\mathbf{n}_k = (\mathbf{n}_k(e))_{e\in E(Q_k)}$ of nonnegative numbers, such that conditionally on Q_k it is an exchangeable family. Assume that $|\mathbf{n}_k|_2/|\mathbf{n}_k|_1 \to 0$ in probability as $k \to \infty$. Then there holds the convergence in probability*

$$d_{\mathrm{P}}^{(V(Q_k),\varepsilon_k d_{Q_k})}\left(\mu_{B_k}^{\mathbf{n}_k}, \nu_{Q_k}\right) \xrightarrow[k\to\infty]{\mathbb{P}} 0,$$

where ν_{Q_k} is the uniform measure on vertices of Q_k and $\varepsilon_k = k^{-1/4}$.

This is the statement of [ABW17, Corollary 6.2], adapted to our setting. The proof goes *mutatis mutandi*, except for an adjustment in the concentration inequality [ABW17, Lemma 5.3], which we adapt below in Lemma 5.26.

Lemma 5.26 ([ABW17, Lemma 5.3]). *Let G be a graph and $\mathbf{n} = (\mathbf{n}(e))_{e\in G}$ a random and exchangeable family of nonnegative numbers with $|\mathbf{n}|_2 > 0$ almost surely. Then for any*

$V \subset V(G)$, and any $t > 0$,

$$\mathbb{P}\left(\left| \mu_G^{\mathbf{n}}(V) - \nu_G(V) \right| > \frac{2t}{|\mathbf{n}|_1} \,\middle|\, |\mathbf{n}|_2 \right) \leq 2 \exp\left(-\frac{2t^2}{|\mathbf{n}|_2^2} \right).$$

The proof goes the same way as that of [ABW17, Lemma 5.3], except that we do not have a double edge here, and the mass on edges is projected deterministically on vertices in our case, instead of randomly. The reader may notice that there is an extra term inside the probability in their lemma. This term accounts for the double edge, which we do not have here. The same line of arguments still works though. Indeed, we have

$$\mu_G^{\mathbf{n}}(V) = \sum_{e \in E(G[V])} \frac{\mathbf{n}(e)}{|\mathbf{n}|_1} + \frac{1}{2} \sum_{e \in \partial_e V} \frac{\mathbf{n}(e)}{|\mathbf{n}|_1},$$

with $G[V]$ the induced-graph on V by G, and $\partial_e V$ the subset of the edges of V who have only one endpoint which belongs to V. By exchangeability, we have the expectation

$$\mathbb{E}\left[\sum_{e \in E(G[V])} \mathbf{n}(e) + \frac{1}{2} \sum_{e \in \partial_e V} \mathbf{n}(e) \,\middle|\, |\mathbf{n}|_1 \right] = |\mathbf{n}|_1 \frac{|E(G[V])|}{|E(G)|} + |\mathbf{n}|_1 \frac{\frac{1}{2}|\partial_e V|}{|E(G)|} = |\mathbf{n}|_1 \nu_G(V).$$

The last equality holds because the degree biased-measure counts twice each edge of $G[V]$, since this edge appears in the degree of both its endpoints, while the edges of $\partial_e V$ are only counted once, in the degree on the only one of its endpoints which is in V.

Then one concludes as in the proof of [ABW17, Lemma 5.3], by a Hoeffding-type bound for exchangeable vectors.

Proof of Theorem 5.22

Scaling limit of the biggest block. By Proposition 2.5 and Proposition 2.12, the biggest block of $\mathbf{Q}_{n,u}$, whose size we denote $C(n,u)$, is a simple quadrangulation sampled uniformly with size $C(n,u)$. Also by Section 4.1.1, this size is asymptotically in probability,

$$C(n,u) = (1 - E(u))n + O_{\mathbb{P}}(n^{2/3}) = \frac{9 - 5u}{3(3+u)} n + O_{\mathbb{P}}(n^{2/3}).$$

By conditioning on $C(n,u)$ and using Proposition 5.23, we therefore get the following GHP scaling limit for the biggest block

$$\left(\frac{3}{8C(n,u)} \right)^{1/4} \cdot \underline{\mathbf{Q}}_{n,u}^{\circ} \xrightarrow[n\to\infty]{(d),\mathrm{GHP}} \mathcal{S},$$

which by the preceding equivalent in probability for $C(n,u)$ reduces to

$$\left(\frac{9(3+u)}{8(9-5u)} \right)^{1/4} n^{-1/4} \cdot \underline{\mathbf{Q}}_{n,u}^{\circ} \xrightarrow[n\to\infty]{(d),\mathrm{GHP}} \mathcal{S}.$$

GHP comparison of $\mathbf{Q}_{n,u}$ with its biggest block. By the preceding scaling limit, and the use of Lemma 5.24 with $\mathfrak{q} = \mathbf{Q}_{n,u}$ and $\varepsilon = n^{-1/4}$, the proof of the theorem reduces to showing the convergence to 0 in probability of the following quantities

$$
\begin{aligned}
r_{\mathrm{GH}} &:= \frac{2}{n^{1/4}} H(\mathbf{T}_{n,u}) \max_{v \neq v^\circ} \operatorname{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) \\
r_{\mathrm{P}} &:= \frac{2\left|V(\mathbf{Q}_{n,u}) \setminus V(\mathbf{Q}_{n,u}^+)\right|}{|V(\mathbf{Q}_{n,u})|} \\
d_P &:= d_{\mathrm{P}}^{(V(\mathbf{Q}_{n,u}^\circ), \varepsilon_n d_{\mathbf{Q}_{n,u}})}\left(\pi_{\mathbf{Q}_{n,u}^\circ}^{\mathbf{Q}_{n,u}^+}, \nu_{\mathbf{Q}_{n,u}^\circ}\right),
\end{aligned}
$$

where $\varepsilon_n = n^{-1/4}$.

Bounding r_{GH}. By Section 4.1.1, the second-biggest block of $\mathbf{Q}_{n,u}$ has size $O(n^{2/3})$ in probability. Combining this with Corollary 5.3, one gets for all $\delta > 0$ the bound in probability

$$
\max_{v \neq v^\circ} \operatorname{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) = o\left(n^{(1+\delta)/6}\right).
$$

Also, by Theorem 2.8, $\mathbf{T}_{n,u}$ is a *non-generic subcritical* Galton-Watson tree conditioned to have $2n+1$ vertices, in the terminology of [Kor15]. We may therefore use [Kor15, Theorem 4] to get for all $\delta > 0$ the bound in probability

$$
H(\mathbf{T}_{n,u}) = o\left(n^{\delta}\right).
$$

Combining the two preceding estimates, we get in probability

$$
r_{\mathrm{GH}} = o\left(n^{-\frac{1}{4}+\delta+\frac{(1+\delta)}{6}}\right) \xrightarrow[n\to\infty]{} 0,
$$

provided that we chose $\delta > 0$ small enough so that $\delta + (1+\delta)/6 < 1/4$.

Bounding r_{P}. First, notice that since $\mathbf{Q}_{n,u}$ is a quadrangulation we have

$$
|V(\mathbf{Q}_{n,u})| = |E(\mathbf{Q}_{n,u})| = 2n,
$$

and by the block-tree decomposition which puts in correspondence edges of $\mathbf{Q}_{n,u}$ and edges of $\mathbf{T}_{n,u}$, we also have

$$
\left|V(\mathbf{Q}_{n,u}) \setminus V(\mathbf{Q}_{n,u}^+)\right| = \left|E(\mathbf{Q}_{n,u})\right| - \left|E(\mathbf{Q}_{n,u}^+)\right| = \left|E(\mathbf{T}_{n,u})\right| - \left|E(\mathbf{T}_{n,u}^+)\right| = \left|E(\mathbf{T}_{n,u} \setminus \mathbf{T}_{n,u}^+)\right|.
$$

Therefore we have to bound the size of the subtree $\mathbf{T}_{n,u} \setminus \mathbf{T}_{n,u}^+$. A moment of thought shows that it is bounded by

$$
U_{\rightarrow}(\mathbf{T}_{n,u}) + U_{\leftarrow}(\mathbf{T}_{n,u}),
$$

where $U_{\rightarrow}(\mathfrak{t})$ is the index in lexicographical order of the vertex with largest degree of the tree $\mathfrak{t}$, and $U_{\leftarrow}(\mathfrak{t})$ is the index in reverse lexicographical order of that same vertex. Now, [Kor15, Theorem 2] shows that $(U_{\to}(\mathbf{T}_{n,u}))_{n\geq 1}$ is a tight sequence. Since $U_{\leftarrow}(\mathbf{T}_{n,u})$ has the same law as $U_{\to}(\mathbf{T}_{n,u})$, the respective sequence is also tight. All in all, we get in probability

$$r_{\mathbf{P}} = O(1/n) \xrightarrow[n\to\infty]{} 0.$$

Bounding d_{P}. Notice that

$$d_P = d_{\mathrm{P}}^{(V(\mathbf{Q}^\circ_{n,u}),\varepsilon_n d_{\mathbf{Q}_{n,u}})}\left(\pi^{\mathbf{Q}^+_{n,u}}_{\mathbf{Q}^\circ_{n,u}}, \nu_{\mathbf{Q}^\circ_{n,u}}\right) = d_{\mathrm{P}}^{(V(\mathbf{Q}^\circ_{n,u}),\varepsilon_n d_{\mathbf{Q}_{n,u}})}\left(\mu^{\mathbf{n}}_{\mathbf{Q}_{n,u}}, \nu_{\mathbf{Q}^\circ_{n,u}}\right),$$

where $\mathbf{n} = \mathbf{n}_{n,u}$ is the family of nonnegative numbers defined by

$$\forall e \in E(\mathbf{Q}^\circ_{n,u}), \quad \mathbf{n}(e) = |V(\mathbf{Q}_{n,u}[e])| - 2.$$

Let us argue that conditionally on $\mathbf{Q}^\circ_{n,u}$, this family $\mathbf{n}$ is exchangeable. Recall that $\mathbf{Q}_{n,u}$ has the law of $\mathbf{Q}$ under $\mathbb{P}_u$, conditioned to the event $\{|\mathbf{Q}| = n\}$. By the symmetries of the Galton-Watson law and Proposition 2.6, the family

$$(|V(\mathbf{Q}[e])| - 2)_{e\in E(\mathbf{Q}^\circ)} = (|E(\mathbf{T}[v_e])| - 2)_{e\in E(\mathbf{Q}^\circ)}$$

is i.i.d. conditionally on $\mathbf{Q}^\circ$, where v_e is the child of v° that the block-tree decomposition associates to e. In particular, this family is exchangeable. Since the event $\{|\mathbf{Q}| = n\}$ is invariant by each permutation of the subtrees attached to the node v° with their respective blocks, the above family stays exchangeable when conditioning by this event. Therefore $\mathbf{n}$ is indeed exchangeable.

Now, [Kor15, Corollary 1] tells that the subtrees $(\mathbf{T}[v_e])_{e\in\mathbf{Q}^\circ_{n,u}}$ have size $O(n^{2/3})$ in probability, uniformly in the edge e. We thus get that

$$|\mathbf{n}|_2 = O\left(\sqrt{n^{5/3}}\right).$$

On the other hand, we have in probability

$$|\mathbf{n}|_1 = |V(\mathbf{Q}_{n,u}) \setminus V(\mathbf{Q}^+_{n,u})| \sim cn,$$

for some constant $c > 0$. Hence, in probability

$$\frac{|\mathbf{n}|_2}{|\mathbf{n}|_1} = O\left(n^{-1/6}\right) \xrightarrow[n\to\infty]{} 0.$$

All the hypotheses of Proposition 5.25 have been checked, so that we may apply it, after conditioning by the size of $\mathbf{Q}^\circ_{n,u}$, since conditionally on its size k it is a uniform simple quadrangulation of size k. We obtain in probability

$$d_{\mathrm{P}}^{(V(\mathbf{Q}^\circ_{n,u}),\varepsilon_n d_{\mathbf{Q}_{n,u}})}\left(\mu^{\mathbf{n}}_{\mathbf{Q}_{n,u}}, \nu_{\mathbf{Q}^\circ_{n,u}}\right) \xrightarrow[n\to\infty]{} 0.$$

Hence, d_{P} also tends to 0 in probability and this concludes the proof. □

5.2 Scaling limits for other decomposition schemes

For all models of Section 2.2, we expect to get similar regimes as in Section 5.1: the scaling limits should be the Brownian sphere when $u < u_C$, the Brownian tree when $u > u_C$ and the stable tree of parameter $3/2$ when $u = u_C$.

More precisely, in the subcritical case, the scaling limit should be that of a uniform block of the size of the largest block of the map, which is asymptotically equivalent to $(1 - E(u))n$ according to Theorem 4.2. Indeed, renormalisation by $n^{1/4}$ will make everything negligible except the macroscopic block (provided the diameter of blocks is adequately bounded). To conclude, it is then sufficient to know the scaling limit of the renormalised uniform blocks. We conjecture that for each family we consider, the uniform blocks converge to the Brownian sphere after renormalisation by $n^{1/4}$. Nevertheless, for the moment this has only been established for the decomposition scheme 7 whose blocks are the simple triangulations [ABA17]. For scheme 2, whose block are simple maps, the distance profile is known and suggests that the Brownian sphere is indeed the scaling limit [BCF14].

In a very broad survey, Stufler showed that the scaling limit in the GH sense for bipartite maps decomposed into (bipartite) 2-connected blocks (scheme 5) converge to the Brownian tree after rescaling by $n^{1/2}$ in the supercritical case [Stu20a, Section 6.1.5 and Theorem 6.60]. In general, the critical and supercritical phases do not require a precise scaling limit for the blocks, and only diameter estimates for the blocks are required. However, they are not always available either. For both scheme 2 and 7, we have an equivalent of Proposition 5.2: the probabilities

$$\left(\mathbb{P}\left(\mathrm{diam}(B_k) \geq k^{1/4+\varepsilon}\right)\right)_k$$

are stretched-exponential as $k \to \infty$ [BCF14, ABA17]. Then, if we show a property of additivity for the distances (as in Lemmas 5.7 and 5.8) then one can conclude that in both cases, the distances in the block-weighted map behave as the distances in the block tree, which converges to the stable tree of parameter $3/2$ after rescaling by $n^{1/3}$ in the critical case, and to the Brownian tree after rescaling by $n^{1/2}$ in the supercritical case. Additivity is not always easy to achieve: for example, simple triangulations decomposed into irreducible blocks (last scheme) are not bipartite (unlike quadrangulations), and, when following a geodesic, one has to consider all possible situations for the distances from each of the vertices of the face to the root (whereas for quadrangulations, there was only one possibility). This should not change the result, as when the geodesic is long enough, there is always an averaging phenomenon. The framework of *Markov modulated random walks* could be used to obtain this kind of result.

Chapter 6

Block decomposition of tree-rooted maps

In theoretical physics, decorated maps are instrumental to provide models of two-dimensional quantum gravity coupled with matter. They lead to new asymptotic behaviours, and the study of scaling limits in that context is currently a very challenging topic in random maps [GHS20]. Among decorated maps, tree-rooted maps, *i.e.* maps endowed with a spanning tree, are an emblematic family with very rich combinatorial properties and have been the focus of an extensive literature.

We now extend the work of the previous chapters to *tree-rooted maps*, which brings new difficulties, in particular for singularity analysis. This chapter is based on a joint work with Marie Albenque and Éric Fusy [AFS24].

The purpose of this chapter is to study the phase transition undergone by a random tree-rooted maps with a Boltzmann weight at 2-connected blocks. First, we describe tree-rooted maps and give their main properties (Section 6.1). Next, we provide an asymptotic estimate for 2-connected tree-rooted maps (Section 6.2). Following Addario-Berry [AB19] and what was done before in Chapter 2, this study is based on an interpretation of the block tree as a Bienaymé–Galton–Watson process (Section 6.3)

Then, we introduce a weight u on 2-connected blocks and show that the enumeration of block-weighted tree-rooted maps undergoes a phase transition at an explicit (transcendental) value u_C (Section 6.4). On the probabilistic side, we obtain asymptotic estimates and limit laws for the size of the largest blocks (Section 6.5), with the existence of a giant block if and only if $u < u_C$, mesoscopic blocks of order $\Theta(n^{1/2})$ if $u = u_C$ and "small" blocks of order $\Theta(\ln(n))$ if $u > u_C$.

Furthermore, we obtain scaling limit results in the critical and supercritical cases (Section 6.6). We show that the scaling limit is the CRT for all $u \geq u_C$, with a discontinuity at u_C for the order of magnitude of the rescaling: it is $\sqrt{n/\ln n}$ at u_C whereas it is $\sqrt{n}$ for $u > u_C$ (the scaling limit result for $u > u_C$ also follows from [Stu20a, Th 6.63]).

6.1 Tree-rooted maps

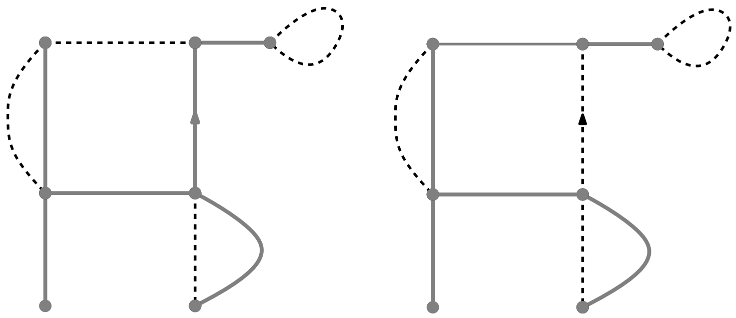

Figure 6.1: Two distinct tree-rooted maps with the same underlying map. Full grey (resp. dashed black) edges represent edges that are part (resp. not part) of the decorating spanning tree.

Definition 6.1. A (rooted planar) *tree-rooted* map is a pair $(\mathfrak{m}, \tau)$ where $\mathfrak{m}$ is a map and τ is a spanning tree of $\mathfrak{m}$ (see Fig. 6.1).

Let $\mathcal{M}^{\leaf}$ and $\mathcal{B}^{\leaf}$ denote, respectively, the set of tree-rooted maps and of 2-connected tree-rooted maps, by $\mathcal{M}^{\leaf}_n$ and $\mathcal{B}^{\leaf}_n$ the subset of $\mathcal{M}^{\leaf}$ and $\mathcal{B}^{\leaf}$ restricted to elements of size $n > 0$, and by $\widehat{M}_{\leaf}(z)$ and $B_{\leaf}(y)$ the associated generating series[1]. In the following, a tree-rooted map will be denoted by $\mathfrak{m}$ instead of being explicitly written as a pair, and, for $\mathfrak{m} \in \mathcal{M}^{\leaf}$, we write $\tau(\mathfrak{m})$ for its distinguished spanning tree.

The enumeration of tree-rooted maps goes back to Mullin, who obtained the following result, where the n-th Catalan number is denoted by Cat_n.

Proposition 6.2. *[Mul67] The number $m^{\leaf}_n$ of tree-rooted planar maps with $n \in \mathbb{N}_{>0}$ edges is*

$$m^{\leaf}_n = \sum_{k=0}^{n} \binom{2n}{2k} \mathrm{Cat}_k \mathrm{Cat}_{n-k} = \mathrm{Cat}_n \mathrm{Cat}_{n+1}. \tag{6.1}$$

Therefore, as $n \to \infty$, it holds that

$$m^{\leaf}_n \sim \frac{4}{\pi n^3} 16^n. \tag{6.2}$$

Mullin proved this by observing that a tree-rooted map is a shuffle of two plane trees (the spanning tree and its dual). A direct bijective proof that $m_n = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$ was later obtained by Bernardi [Ber07], who subsequently extended his bijection to maps endowed with a Potts model [Ber08].

The explicit expression for $m^{\leaf}_n$ shows that the radius of convergence $\rho_{\widehat{M}_{\leaf}}$ of $\widehat{M}_{\leaf}$ is equal to $1/16$ and one can compute

$$\widehat{M}_{\leaf}(\rho_{\widehat{M}_{\leaf}}) = 7 - \frac{64}{3\pi}.$$

This value is not algebraic whereas $1/16$ is, so $\widehat{M}_{\leaf}$ is not algebraic. In fact, $\widehat{M}_{\leaf}$ is D-finite

[1]Contrary to what is done in the corresponding article [AFS24], here we do not take into account the tree-rooted map of size 0. This simply creates a minor shift between the results here and in the article

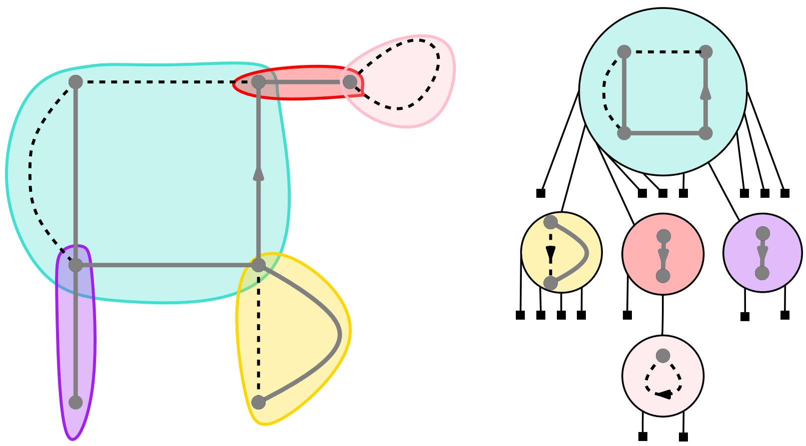

Figure 6.2: Block tree corresponding to a tree-rooted planar map. Full grey (resp. dashed black) edges represent edges that are part (resp. not part) of the decorating spanning tree.

and satisfies:

$$z^2(1-16z)\widehat{M}''_{\varnothing}(z)+4z(1-12z)\widehat{M}'_{\varnothing}(z)+2(1-6z)\widehat{M}_{\varnothing}(z)-12z=0. \tag{6.3}$$

This explains why the study of block-weighted tree-rooted maps is not as easy as for the decomposition schemes presented earlier in Section 2.2.

Tutte's decomposition of maps into 2-connected blocks, described in Section 2.1, extends readily to tree-rooted maps as follows. Fix $\mathfrak{m} \in \mathcal{M}^{\varnothing}$. Consider $(\mathfrak{b}, \tau_{\mathfrak{b}})$, where $\mathfrak{b}$ is, as before, the block of $\mathfrak{m}$ containing its root and $\tau_{\mathfrak{b}} = \tau(\mathfrak{m}) \cap \mathfrak{b}$. Let us show that $\tau_{\mathfrak{b}}$ is a spanning tree of $\mathfrak{b}$: clearly, $\tau_{\mathfrak{b}}$ is acyclic since $\tau(\mathfrak{m})$ is. Then, for any $u, v \in \mathfrak{b}$, since $\mathfrak{b}$ is 2-connected, any path between u and v that is not included in $\mathfrak{b}$ has to visit the same cut vertex at least twice, and in particular it is not simple. Any simple path between u and v in $\mathfrak{m}$ is then included in $\mathfrak{b}$, and so is the unique simple path between u and v in $\tau(\mathfrak{m})$. This proves that $\tau_{\mathfrak{b}}$ is connected. The same reasoning can be applied to all the pendant submaps $\mathfrak{m}_e$, which induces the following identity of generating series.

Proposition 6.3. *The generating series satisfy the following equality:*

$$\widehat{M}_{\varnothing}(z) = B_{\varnothing}(z(1+\widehat{M}_{\varnothing}(z))^2). \tag{6.4}$$

This relation is exactly the same as (2.1) obtained by Tutte in the non–tree-rooted case[2] [Tut63]. In the non–tree-rooted case, $M_\circ$ admits a Lagrangean parametrization, so, together with (6.4), one can obtain a Lagrangean parametrization for $B_\circ$ [Tut63]. However, such a parametrisation does not exist for $\widehat{M}_{\varnothing}$ as it is not algebraic so this cannot give a Lagrangean specification for the generating series of 2-connected tree-rooted map. However, it is still possible to use (6.4) and the knowledge of $\widehat{M}_{\varnothing}$ to obtain successively the coefficients of $B_{\varnothing}$ [WL75].

Moreover, the situation is still very similar to that of Chapter 2: the formula of (6.4), as

[2] Except that here we do not consider maps of size 0.

we have shown when it relates the series of general maps to the series of 2-connected maps, is amenable to a Lagrangean form and to a whole series of analyses for its block-weighted decomposition. In particular, writing

$$\widehat{M}_{\leaf}(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}^{\leaf}} z^{|\mathfrak{m}|}u^{b(\mathfrak{m})},$$

it holds that

$$\widehat{M}_{\leaf}(z,u) = uB_{\leaf}(z(1+\widehat{M}_{\leaf}(z,u))^2).$$

Now, setting

$$M_{\leaf}(z,u) = z(1+\widehat{M}_{\leaf}(z^2,u)) = \sum_{\mathfrak{m}\in\mathcal{M}^{\leaf}} z^{\mathrm{s}(\mathfrak{m})}u^{b(\mathfrak{m})}$$

with $\mathrm{s}(\mathfrak{m}) = 2|\mathfrak{m}|+1$, it holds that

$$M_{\leaf}(z,u) = z\left(1+uB_{\leaf}(M_{\leaf}(z,u)^2)\right)$$

so

$$M_{\leaf}(z,u) = z\Phi(M_{\leaf}(z,u),u) \qquad \text{where} \qquad \Phi(x,u) = 1+uB_{\leaf}(x^2), \tag{6.5}$$

which is equivalent to Theorem 2.15. Then, as in Section 2.2.1, one can construct a block tree for the tree-rooted case (see Fig. 6.2), for which Proposition 2.16 holds.

6.2 Asymptotic enumeration of 2-connected tree-rooted maps

Compared to Chapter 3, a difficulty here is that the singular behaviour of the generating series $B_{\leaf}$ of blocks is to be determined.

We obtain here an asymptotic estimate for the number $b_n^{\leaf} := [y^n]B_{\leaf}(y)$ of 2-connected tree-rooted maps of size n. The steps are as follows: we first lift (Lemma 6.4) the asymptotic estimate (6.2) for tree-rooted maps to a singular expansion for the generating function $\widehat{M}_{\leaf}(z)$. Then via (6.4), we get in Proposition 6.5 the radius of convergence $\rho_{B_{\leaf}}$ and the singular expansion of $B_{\leaf}(y)$ around $\rho_{B_{\leaf}}$. In order to transfer the singular expansion to an asymptotic estimate for $b_n^{\leaf}$, we also show that $\rho_{B_{\leaf}}$ is the unique dominant singularity of $B_{\leaf}(y)$, using a combinatorial argument.

Lemma 6.4. *When $z \to \rho_{\widehat{M}_{\leaf}} = \frac{1}{16}$ in $\mathbb{C}\setminus\{z \geq \rho_{\widehat{M}_{\leaf}}\}$, one has, with $Z = 1-16z$,*

$$\widehat{M}_{\leaf}(z) = 7 - \frac{64}{3\pi} - 8\left(\frac{10}{3\pi}-1\right)Z - \frac{2}{\pi}\ln(Z)\,Z^2 + O\left(Z^2\right). \tag{6.6}$$

Proof. The D-finite equation (6.3) satisfied by $\widehat{M}_{\leaf}$ can be derived to obtain a homogeneous

D-finite equation:

$$\widehat{M}'''_{\varnothing}(z) + \frac{5}{z}\widehat{M}''_{\varnothing}(z) + \frac{2(1-30z)}{z^2(1-16z)}\widehat{M}'_{\varnothing}(z) - \frac{2}{z^3(1-16z)}\widehat{M}_{\varnothing}(z) = 0. \tag{6.7}$$

D-finite equation theory [FS09, Sec. VII.9.1, p. 518] gives that the finite singularities of a solution $f(z)$ of (6.7) are among the zeroes of the denominators of the coefficients: $\mathcal{S} = \{0, 1/16\}$; and any solution of (6.7) is analytically continuable along any path avoiding $\mathcal{S}$. In particular, the solution $\widehat{M}_{\varnothing}(z) = \sum_{n\geq 0} \mathrm{Cat}_n \mathrm{Cat}_{n+1} z^n$, which is clearly analytic at 0, is continuable to the whole complex plane slit by the half-line $z \geq 1/16$.

Moreover, $1/16$ is a so-called *regular* singularity; and, using the `DEtools` package of the `Maple` computer algebra software, one can compute singular expansions for a basis of solutions of (6.7). The singular expansion of $\widehat{M}_{\varnothing}(z)$ is then a linear combination of the basis' singular expansions, which gives:

$$\widehat{M}_{\varnothing}(z) = \sum_{k=0}^{\infty} a_k Z^k - \ln(Z)\sum_{k=2}^{\infty} b_k Z^k, \quad \text{with} \quad Z = 1-16z,$$

holding in a slit neighbourhood of $1/16$. From the explicit expression (6.1) of $m_n^{\varnothing}$, it follows that

$$a_0 = \widehat{M}_{\varnothing}(1/16) = 7 - \frac{64}{3\pi} \qquad \text{and} \qquad a_1 = -\frac{1}{16}\widehat{M}'_{\varnothing}(1/16) = -8\left(\frac{10}{3\pi} - 1\right).$$

By Pringsheim's theorem, $\widehat{M}_{\varnothing}(z)$ is singular at its radius of convergence $1/16$ so there exists a smallest integer $k \geq 2$ such that $b_k \neq 0$. By applying transfer theorems [FS09, Chap. VI], one has

$$[z^n]\widehat{M}_{\varnothing}(z) \sim (-1)^k \frac{b_k k!}{n^{k+1}} 16^n.$$

By (6.1), one must have $k = 2$ and $b_2 = \frac{2}{\pi}$, which concludes the proof. ☐

Proposition 6.5. *The radius of convergence of $B_{\varnothing}(y)$ is*

$$\rho_{B_{\varnothing}} := \rho_{\widehat{M}_{\varnothing}} \widehat{M}^2_{\varnothing}(\rho_{\widehat{M}_{\varnothing}}) = \frac{4(3\pi-8)^2}{9\pi^2} \approx 0.091, \tag{6.8}$$

and, when $y \to \rho_{B_{\varnothing}}$ in a Δ-neighbourhood of $\rho_{B_{\varnothing}}$, one has, with $Y = 1 - y/\rho_{B_{\varnothing}}$,

$$B_{\varnothing}(y) = 7 - \frac{64}{3\pi} - \frac{8(10-3\pi)(3\pi-8)}{9\pi(4-\pi)} Y - \frac{2(3\pi-8)^3}{27\pi(4-\pi)^3}\ln(Y)Y^2 + O(Y^2). \tag{6.9}$$

Moreover, $\rho_{B_{\varnothing}}$ is the unique dominant singularity of $B_{\varnothing}(y)$.

Remark 6.6. The generating series $B_{\varnothing}(y)$ is not D-finite (having a transcendental radius of convergence); but, from Equations (6.4) and (6.7), it is D-algebraic and satisfies:

$$
\begin{aligned}
&y^2(B_{\leaf}(y)+1)\left(B_{\leaf}(y)^2+2B_{\leaf}(y)+1-16y\right)B_{\leaf}''(y) \\
&-2y^3\left(3B_{\leaf}(y)-5\right)B_{\leaf}'(y)^3+4y^2\left(3B_{\leaf}(y)^2-3-4y\right)B_{\leaf}'(y)^2 \\
&-4y(B_{\leaf}(y)+1)\left(2B_{\leaf}(y)^2+B_{\leaf}(y)-1-6y\right)B_{\leaf}'(y) \\
&+2B_{\leaf}(y)^4+6B_{\leaf}(y)^3-6(2y-1)B_{\leaf}(y)^2-2(12y-1)B_{\leaf}(y)-12y=0.
\end{aligned}
$$

Proof of Proposition 6.5. As for the proof of Theorem 3.2, we proceed by inversion of the Lagrangean equation. This gives the singular expansion of the analytic inverse $\Psi(M)$ of $M_{\leaf}(z) := M_{\leaf}(z,1)$. As before,

$$\Psi(M) = \frac{M}{\Phi(M,1)} = \frac{M}{1+B_{\leaf}(M^2)}$$

so one can deduce the expansion of $B_{\leaf}(x^2)$ from that of $M_{\leaf}$ and Ψ, as detailed now.

Let $\tau = M_{\leaf}(\rho_{M_{\leaf}})$. The function $M_{\leaf}$ is analytic on $D(0,\rho_{M_{\leaf}})$ and, since $M_{\leaf}'(z) > 0$ for any $z \in [0,\rho_{M_{\leaf}})$, one can apply the analytic local inversion theorem at any such value of z. Moreover, $M_{\leaf}$ maps the interval $[0,\rho_{M_{\leaf}}]$ to the interval $[0,\tau]$, so one can define a functional inverse $\Psi(.)$ of $M_{\leaf}$ on a neighbourhood of $[0,\tau)$, which is analytic on this domain. Furthermore, $M_{\leaf}'$ is continuous in a Δ-neighbourhood of $\rho_{M_{\leaf}}$, with positive value at $\rho_{M_{\leaf}}$, hence $M_{\leaf}(z)$ is injective on a Δ-neighbourhood U of $\rho_{M_{\leaf}}$, and maps U to an open region containing a Δ-neighbourhood V of τ.

Using bootstrapping, the singular expansion of $\Psi(x)$ at τ (valid in V) is easily obtained from the singular expansion of $M_{\leaf}(z)$ (which itself can be deduces from that of $\widehat{M}_{\leaf}(z)$ in Lemma 6.4). With $Y = 1 - x^2/\tau$, one gets, for explicit κ,

$$z = \Psi(x) = \rho_{M_{\leaf}} - \rho_{M_{\leaf}}\frac{\tau}{\kappa}Y - \rho_{M_{\leaf}}\frac{\tau^2}{\kappa^3}\ln(Y)Y^2 + O(Y^2).$$

With $Z = 1 - z/\rho_{\widehat{M}_{\leaf}}$, this gives

$$Z = \frac{\tau}{\kappa}Y + \frac{\tau^2}{\kappa^3}\ln(Y)Y^2 + O(Y^2).$$

Then, $B_{\leaf}(x^2) = \widehat{M}_{\leaf}(\Psi(x))$ is analytic at every point in $[0,\tau)$, and the claimed singular expansion of $B_{\leaf}(y)$ at $\rho_{B_{\leaf}} := \tau$ is obtained by composing the singular expansion of $\widehat{M}_{\leaf}(z)$ with the singular expansion of $\Psi(x)$.

By Pringsheim's theorem, $\rho_{B_{\leaf}}$ is the radius of convergence of $B_{\leaf}(y)$. It remains to prove that $\rho_{B_{\leaf}}$ is the unique dominant singularity of $B_{\leaf}(y)$. To do so, we use the trick of writing $B_{\leaf}(y)$ as a supercritical composition scheme (in the sense of [Gou98]), which we achieve thanks to a decomposition into series-parallel components. Doing so, we prove in Lemma 6.7 that $B_{\leaf}(y)$ can be written as

$$B_{\leaf}(y) = 2y + 2yA(y) + yA'(y)Q(A(y)), \tag{6.10}$$

for some generating functions $A(y)$ and $Q(w)$ with non-negative coefficients, such that $A(y)$ is non-periodic and has radius of convergence larger than $\rho_{B_{\varnothing}}$. This implies that the radius of convergence of $Q(w)$ is $A(\rho_{B_{\varnothing}})$. Moreover, by the Daffodil Lemma, see [FS09, Lem. IV.1, p. 266], for any $y \neq \rho_{B_{\varnothing}}$ such that $|y| = \rho_{B_{\varnothing}}$, we have that $|A(y)| < A(\rho_{B_{\varnothing}})$. Hence, $A(y)$ belongs to the disk of convergence of Q and y cannot be a singularity, which concludes the proof. □

Let $Q(w)$ be the generating function of 2-connected tree-rooted maps with no face of degree 2 nor vertex of degree 2, with w counting the number of non-root edges. A 2-connected map with at least 2 edges is called *series-parallel* if it has no K_4 minor. Let $A(y)$ (resp. $\bar{A}(y)$) be the generating function of 2-connected tree-rooted series-parallel maps such that the root-edge is not (resp. is) in the spanning tree, the variable y counting the number of non-root edges.

Lemma 6.7. *The generating series $B_{\varnothing}(y)$, $A(y)$ and $Q(w)$ satisfy the identity (6.10). Moreover, the radius ρ_A of convergence of A satisfies $\rho_A = 2 - 3 \cdot 2^{-2/3} \approx 0.11 > \rho_{B_{\varnothing}}$.*

Proof. A *series-parallel network* N is obtained by deleting the root-edge e of a series-parallel map, the two extremities of e being called the *poles* of N (which are distinguished as the *source*, the origin of e, and the *sink*, the end of e). Note that $A(y)$ is also the generating function of series-parallel networks endowed with a spanning tree, while $\bar{A}(y)$ is the generating function of series-parallel networks endowed with a spanning forest made of two trees containing each of the two poles. These two cases are respectively called *crossing* and *non-crossing*.

The *core* $\mathfrak{c}$ of a 2-connected tree-rooted map $\mathfrak{b}$ of size $|\mathfrak{b}| \geq 2$ is obtained by repeatedly collapsing faces of degree 2 and erasing vertices of degree 2 (turning the two incident edges into a single edge). This process is actually well-behaved only if $\mathfrak{b}$ is not series-parallel (otherwise it ends at a loop-edge with no vertex). Conversely, a 2-connected map $\mathfrak{b}$ is obtained from its core $\mathfrak{c}$ where every edge is replaced by a series-parallel network. By convention, the root-edge of $\mathfrak{c}$ is chosen as the one bearing the series-parallel network containing the root-edge of $\mathfrak{b}$. If $\mathfrak{b}$ is endowed with a spanning tree $\tau := \tau(\mathfrak{b})$, then for each edge e of $\mathfrak{c}$, letting N_e be the associated series-parallel network, on N_e the tree τ induces either a spanning tree, or a spanning forest with two trees containing each of the two poles. In the first case, e is declared a tree-edge of $\mathfrak{c}$, and thus τ induces a spanning tree on $\mathfrak{c}$. In terms of generating functions, $Q(w)$ is the counting series for the core, each non-root edge of the core then contributing either $A(y)$ if a tree-edge (crossing case) or contributing $\bar{A}(y)$ otherwise (non-crossing case). Since, by duality we have $A(y) = \bar{A}(y)$, every non-root edge turns out to have the same contribution $A(y)$. On the other hand, the root-edge of the core contributes $A'(y)$ because of the choice of the root-edge. This yields the claimed equation (6.10), where the added term $2yA(y)$ accounts for 2-connected tree-rooted series-parallel maps.

Now, to get the statement about ρ_A, it is well-known that a series-parallel network is either reduced to a single edge, or made of at least two series-parallel networks connected in series, or made of at least two series-parallel networks connected in parallel. The series-parallel decomposition then yields the following equation-system:

$$A(y) = y + S(y) + P(y), \qquad \bar{A}(y) = y + \bar{S}(y) + \bar{P}(y),$$

$$S(y) = \frac{(y + P(y))^2}{1 - y - P(y)}, \qquad \bar{S}(y) = (y + \bar{P}(y)) \left(\frac{1}{(1 - y - P(y))^2} - 1 \right),$$

$$P(y) = (y + S(y)) \left(\frac{1}{(1 - y - \bar{S}(y))^2} - 1 \right), \qquad \bar{P}(y) = \frac{(y + \bar{S}(y))^2}{1 - y - \bar{S}(y)}.$$

By symmetry, one has $S(y) = \bar{P}(y)$ and $P(y) = \bar{S}(y)$, which yields $A(y) = \bar{A}(y)$ (this is also clear by duality). Hence, the function $A(y)$ is algebraic and satisfies[3]

$$A(y)^3 + (y+1)\, A(y)^2 + (2y - 1)\, A(y) + y = 0$$

and its radius of convergence is the smallest positive root of $4y^3 - 24y^2 + 48y - 5$ which gives $\rho_A = 2 - 3 \cdot 2^{-2/3} \approx 0.11$ and is larger than $\rho_{B_{\varnothing}}$. □

From Proposition 6.5, by applying transfer theorems [FS09, Chap.VI] one directly obtains:

Corollary 6.8. *When $n \to \infty$,*

$$b_n^{\varnothing} \sim \frac{4\,(3\pi - 8)^3}{27\pi\,(4 - \pi)^3} \cdot \rho_{B_{\varnothing}}^{-n} \cdot n^{-3}. \tag{6.11}$$

6.3 Probabilistic framework for tree-rooted maps

One can define a probabilistic framework in the same way as in Section 2.2.2. Here are the main definitions: for $u \in \mathbb{R}_{>0}$, $n \in \mathbb{N}_{>0}$ and $\mathfrak{m} \in \mathcal{M}$, we set

$$\mathbb{P}_u(\mathfrak{m}) = \frac{\rho_{\varnothing}(u)^{\mathrm{s}(\mathfrak{m})} u^{b(\mathfrak{m})}}{M_{\varnothing}(\rho_{\varnothing}(u), u)} \quad \text{and} \quad \mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}}{[z^n] M_{\varnothing}(z, u)} \mathbb{1}_{\mathrm{s}(\mathfrak{m}) = n} \tag{6.12}$$

where $\rho_{\varnothing}(u)$ is the radius of convergence of $z \mapsto M_{\varnothing}(z, u)$. We introduce the following probability distribution: for all $j \in \mathbb{N}_0$,

$$\mu^{x,u}(j) = \frac{[X^j]\Phi(X, u)\, x^j}{\Phi(x, u)} \quad \text{so that} \quad \mu^{x,u}(2j) = \frac{\mathbb{1}_{j=0} + \mathbb{1}_{j \neq 0} u b_j^{\varnothing} x^{2j}}{1 + u B_{\varnothing}(x^2)}. \tag{6.13}$$

Moreover (see Remark 2.7 for a discussion, which still applies), we set for all $u > 0$:

$$x(u) = M_{\varnothing}(\rho_{\varnothing}(u), u) \quad \text{and} \quad \mu^u := \mu^{x(u),u}.$$

One could also define $x(u)$ as the only $x \in [0, \rho_{B_{\varnothing}}^{1/2}]$ such that $\mathbb{E}[\mu^{x,u}] = 1$ when such x exists, and $x = \rho_{B_{\varnothing}}^{1/2}$ otherwise. Then, Proposition 2.17 holds: the block-tree of $\mathbf{M}_{n,u}$ drawn according

[3] The series $A(y)$ is represented in Sloane's OEIS by the sequence A121873, which enumerates non-crossing plants in the $(n+1)$-sided regular polygon [Cha07].

to $\mathbb{P}_{n,u}$ is distributed as a μ^u-Bienaymé–Galton–Watson tree, and conditionally on the tree, the blocks are *i.i.d.* random variables uniform on the sets of blocks of the corresponding sizes. This enables to put into light a phase transition on the tree structure, using once again the usual phase transition for Bienaymé–Galton–Watson trees [Nev86], which depends on the mean of the reproduction law. Using Corollary 6.8, this gives the following equivalent of Theorem 2.19 and Proposition 2.22.

Theorem 6.9. *The law $\mathbb{P}_u$ undergoes the following phase transition, driven by the properties of μ^u, depending on the position of u with respect to*

$$u_C = \frac{1}{2\rho_{B_{\varnothing}} B'_{\varnothing}(\rho_{B_{\varnothing}}) - B_{\varnothing}(\rho_{B_{\varnothing}})} = \frac{9\pi\,(4-\pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02. \tag{6.14}$$

Moreover, for any $u > 0$, define

$$c(u) = \frac{4\,(3\pi-8)^3}{9\,(4-\pi)^3}\,\frac{u}{(21\pi-64)u+3\pi}. \tag{6.15}$$

Subcritical case. *For $u < u_C$, then there is no $x \in (0, \rho_\Phi(u)]$ such that $\mathbb{E}[\mu^{x,u}] = 1$,* i.e. *$\mu^{x,u}$ cannot be critical. In this case, $x(u) = M_{\varnothing}(\rho_{\varnothing}(u), u) = \rho_{B_{\varnothing}}^{1/2}$. Moreover,*

$$\mu^u(2j) \underset{j\to\infty}{\sim} c(u) j^{-3},$$

$$E(u) := \mathbb{E}[\mu^u] = \frac{2u\rho_{B_{\varnothing}} B_{\varnothing}'(\rho_{B_{\varnothing}})}{1 + uB_{\varnothing}(\rho_{B_{\varnothing}})} = \frac{16\,(3\pi-8)\,(10-3\pi)}{3\,(4-\pi)}\,\frac{u}{(21\pi-64)u+3\pi} < 1, \tag{6.16}$$

$$\text{and} \qquad \mathbb{V}(\mu^u) = \infty.$$

Critical case. *For $u = u_C$,*

$$\mu^{u_C}(2j) \underset{j\to\infty}{\sim} c(u_C) j^{-3} = \frac{(3\pi-8)^2}{12\,(10-3\pi)\,(4-\pi)^2} j^{-3} \simeq 0.40\, j^{-3},$$

$$\mathbb{E}[\mu^{u_C}] = 1 \qquad \text{and} \qquad \mathbb{V}(\mu^{u_C}) = \infty.$$

In this case, $x(u_C) = M_{\varnothing}(\rho_{\varnothing}(u_C), u_C) = \rho_{B_{\varnothing}}^{1/2}$.

Supercritical case. *For $u > u_C$, $x(u)$ satisfies*

$$2x(u)^2 B'_{\varnothing}(x(u)^2) - B_{\varnothing}(x(u)^2) - \frac{1}{u} = 0.$$

Moreover, μ^u has exponential moments and

$$\mu^u(2j) \underset{j\to\infty}{\sim} c(u) j^{-3} \left(\frac{x(u)^2}{\rho_{B_{\varnothing}}}\right)^j, \qquad \mathbb{E}[\mu^u] = 1 \qquad \text{and} \qquad \mathbb{V}(\mu^u) < \infty,$$

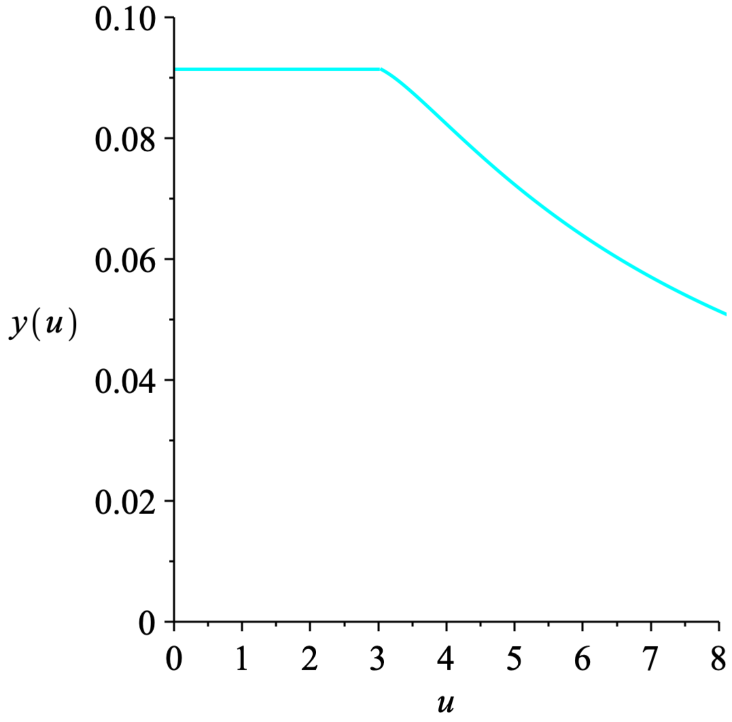

Figure 6.3: Plot of $y(u) = x(u)^2$. At u_C the tangent from the right side is also horizontal.

with

$$\sigma(u) := \mathbb{V}(\mu^u)^{1/2} = x(u)\sqrt{\frac{\Phi''(x(u),u)}{\Phi(x(u),u)}}. \tag{6.17}$$

Remark 6.10. The function $u \mapsto x(u)^2$ is non-increasing. It is plotted in Fig. 6.3.

6.4 Enumerative phase transition for block-weighted tree-rooted maps

Now, we investigate the singular expansion of $z \mapsto M_{\leaf}(z,u)$ around its radius of convergence. We prove that this expansion exhibits three possible behaviours depending on the value of u. As we explained in Section 3.1.2, the method used to obtain the singular expansion of $M(z,u)$ did not use either the algebraicity of M or that of B, which explains that we can use it again even though both $M_{\leaf}$ and $B_{\leaf}$ are not algebraic. Moreover, the radius of convergence $\rho_{\leaf}(u)$ of $z \mapsto M_{\leaf}(z,u)$ satisfies Proposition 3.1: when $u \le u_C$,

$$\rho_{\leaf}(u) = \frac{\rho_{B_{\leaf}}^{1/2}}{\Phi\left(\rho_{B_{\leaf}}^{1/2}, u\right)} = \frac{\rho_{B_{\leaf}}^{1/2}}{1 + uB_{\leaf}(\rho_{B_{\leaf}})} = \frac{6\pi - 16}{3\pi + (21\pi - 64)u}.$$

Theorem 6.11. *The series $M_{\leaf}(z,u)$ displays the following asymptotic behaviours when z*

is in a Δ-domain neighbourhood of $\rho_{\leaf}(u)$, where we let $Z = 1 - \frac{z}{\rho_{\leaf}(u)}$.

Subcritical regime. *When $u < u_C$, there exist $r(u), s(u) > 0$ such that*

$$M_{\leaf}(z,u) = M_{\leaf}(\rho_{\leaf}(u),u) + r(u)Z - s(u)\ln(Z)Z^2 + \Theta\left(Z^2\right),$$

and it holds that

$$\begin{cases} \rho_{\leaf}(u) & = \frac{\rho_{B_{\leaf}}^{1/2}}{1+uB_{\leaf}(\rho_{B_{\leaf}})} = \frac{6\pi-16}{3\pi+(21\pi-64)u} \\ M_{\leaf}(\rho_{\leaf}(u),u) & = \rho_{B_{\leaf}}^{1/2} = \frac{2(3\pi-8)}{3\pi} \\ r(u) & = \frac{\rho_{B_{\leaf}}^{1/2}}{1-E(u)} = \frac{2(3\pi-8)(3\pi+(21\pi-64)u)(4-\pi)}{-(207u+9)\pi^3+(1308u+36)\pi^2-2048\pi u} \\ s(u) & = \frac{8((21\pi-64)u+3\pi)^2(3\pi-8)^4u}{\pi((81u-9)\pi^2-(420u-36)\pi+512u)^3} \end{cases} \tag{6.18}$$

Critical regime. *There exists a constant s_C satisfying*

$$M_{\leaf}(z,u_C) = M_{\leaf}(\rho_{\leaf}(u_C),u_C) - s_C\frac{Z^{1/2}}{\ln(1/Z)^{1/2}} + \Theta\left(Z\right),$$

and it holds that

$$\begin{cases} \rho_{\leaf}(u_C) & = \frac{\rho_{B_{\leaf}}^{1/2}}{1+u_CB_{\leaf}(\rho_{B_{\leaf}})} = \frac{81\pi^2-420\pi+512}{72\pi^2-240\pi} \\ M_{\leaf}(\rho_{\leaf}(u_C),u_C) & = \rho_{B_{\leaf}}^{1/2} = \frac{2(3\pi-8)}{3\pi} \\ s_C & = \frac{4\sqrt{3}\,(4-\pi)\sqrt{10-3\pi}}{3\pi} \end{cases} \tag{6.19}$$

Supercritical regime. *When $u > u_C$, there exists $s(u) > 0$ such that*

$$M_{\leaf}(z,u) = M_{\leaf}(\rho_{\leaf}(u),u) - s(u)Z^{1/2} + \Theta\left(Z\right),$$

and it holds that

$$s(u) = \sqrt{\frac{2\Phi(x(u),u)}{\Phi''(x(u),u)}}.$$

Proof. As for Theorem 3.2, this singular expansion is obtained by looking at the function inverse $\Psi(M)$ of $\widehat{M}_{\leaf}(z)$. The supercritical case falls again the smooth-inverse function schema, and the subcritical case has once again a non-zero linear term which (informally) makes $\widehat{M}_{\leaf}(z)$ behave as $B_{\leaf}(y)$. The real difference is for the critical case, where one has

$$z - z_0 = \Psi(M) - \Psi(M_0) \sim -c_1(M-M_0)^2\ln(M-M_0)$$

which inverts into, *via* bootstrapping,

$$M - M_0 \sim -c_2\ln(z_0-z)^{-1/2}(z_0-z)^2.$$

We now detail this.

As in Proposition 6.5, let $\Psi(x,u)$ be the functional inverse of $M_{\varnothing}(z,u)$. The Lagrangean equation gives that

$$\Psi(x,u) = \frac{x}{\Phi(x,u)} = \frac{x}{1+uB_{\varnothing}(x^2)},$$

and the singular expansion of $M_{\varnothing}(z,u)$ depends on whether $\frac{\partial}{\partial x}\Psi(x,u) = 0$ – which is equivalent to $\mathbb{E}\left[\mu^{x,u}\right] = 1$ – admits a solution in $(0, \rho_{B_{\varnothing}})$.

Supercritical case. For $u > u_C$, $\widehat{M_{\varnothing}}(z,u)$ has a dominant singularity of square-root type at $\rho_{\varnothing}(u) = \Psi(x(u),u)$ (see [FS09, Theorem VII.3]).

Subcritical and critical cases. For $u \leq u_C$, one has for all $x \in (0, \rho_{B_{\varnothing}}^{1/2}), \mathbb{E}\left[\mu^{x,u}\right] < 1$, and there is no solution to $\frac{\partial}{\partial x}\Psi(x,u) = 0$ on $(0, \rho_{B_{\varnothing}}^{1/2})$. At $\rho_{B_{\varnothing}}$, by (6.9), one gets the expansion

$$\Phi(x,u) = 1 + uB_{\varnothing}(x^2) = 1 + \alpha u - u\beta Y(x) - u\gamma \ln(Y(x))Y(x)^2 + O(Y(x)^2),$$

where α, β and γ are explicit positive constants and $Y(x) = 1 - \frac{x^2}{\rho_{B_{\varnothing}}}$. Thus, one gets

$$\Psi(x,u) = \rho_{\varnothing}(u)\left(1 - \frac{1-u(2\beta-\alpha)}{2(1+u\alpha)}Y(x) + \frac{u\gamma}{1+u\alpha}\ln(Y(x))Y(x)^2 + O(Y(x)^2)\right),$$

and, it holds that

$$\frac{1-u(2\beta-\alpha)}{2(1+u\alpha)} = \frac{1-E(u)}{2}$$

Writing $\widehat{Y}(z) = Y(M_{\varnothing}(z,u)) = 1 - \frac{M_{\varnothing}(z,u)^2}{\rho_{B_{\varnothing}}}$ and using that $\Psi(M_{\varnothing}(z,u)) = z$,

$$Z := 1 - \frac{z}{\rho_{\varnothing}(u)} = \frac{1-E(u)}{2}\widehat{Y}(z) - \frac{u\gamma}{1+u\alpha}\ln(\widehat{Y}(z))\widehat{Y}(z)^2 + O(\widehat{Y}(z)^2). \tag{6.20}$$

Subcritical case This expression can now be bootstrapped: setting $\delta(u) = \left(\frac{1-E(u)}{2}\right)^{-1}$, one gets

$$\widehat{Y}(z) \sim \delta(u)Z \quad \text{and} \quad \ln(\widehat{Y}(z))\widehat{Y}(z)^2 \sim \delta(u)^2\ln(Z)Z^2$$

and this gives

$$M_{\varnothing}(z,u) = \rho_{B_{\varnothing}}^{1/2}\left(1 + \frac{1}{1-E(u)}Z - \frac{4u\gamma}{(1+\alpha u)(1-E(u))^3}Z^2\ln(Z) + O(Z^2)\right).$$

Critical case. In the limit case $u = u_C$, one has $\frac{\partial}{\partial x}\Psi(x,u) = 0$ at $x = \rho_{B_{\varnothing}}^{1/2}$, where one gets from (6.20) (with $\widehat{Y}(z) = 1 - \frac{M_{\varnothing}(z,u_C)^2}{\rho_{B_{\varnothing}}}$) the expansion

$$Z := 1 - \frac{z}{\rho_{\varnothing}(u_C)} = \frac{u_C\gamma}{1+u_C\alpha}\ln(1/\widehat{Y}(z))\widehat{Y}(z)^2 + O(\widehat{Y}(z)^2).$$

Bootstrapping, one gets

$$\widehat{Y}(z) \sim \sqrt{\frac{2(1+u_C\alpha)}{u_C\gamma}\frac{Z}{\ln(1/Z)}}$$

and, finally, since $u_C = \frac{1}{2\beta+\alpha}$,

$$M_{\varnothing}(z,u) = \rho_{B_{\varnothing}}^{1/2}\left(1 - \sqrt{\frac{\beta}{\gamma}}\sqrt{\frac{Z}{\ln(1/Z)}} + O(Z)\right).$$

□

As for Proposition 3.5, for every fixed $u > 0$, the radius of convergence $\rho_{\varnothing}(u)$ is not the only dominant singularity of the series $z \mapsto \widehat{M}_{\varnothing}(z,u)$, there is also $-\rho_{\varnothing}(u)$. This is due to the Lagrangean equation (6.5) being periodic, in the sense that Φ is periodic. Taking that into account, the singular developments allows to conclude with the transfer theorem to get the asymptotic expansion of $[z^n]\widehat{M}_{\varnothing}(z,u)$.

Corollary 6.12. *Let $u > 0$. Then, with the notation of Theorem 6.11, one has the following asymptotic estimates as $n \to \infty$ with $n \equiv 1[2]$.*

- *When $u < u_C$,*
$$[z^n]M_{\varnothing}(z,u) \sim 4s(u)\,\rho_{\varnothing}(u)^{-n}\,n^{-3}. \tag{6.21}$$
- *When $u = u_C$,*
$$[z^n]M_{\varnothing}(z,u_C) \sim \frac{s_C}{\sqrt{\pi}}\,\rho_{\varnothing}(u)^{-n}\,n^{-3/2}\ln(n)^{-1/2}. \tag{6.22}$$
- *When $u > u_C$,*
$$[z^n]M_{\varnothing}(z,u) \sim \frac{s(u)}{\sqrt{\pi}}\,\rho_{\varnothing}(u)^{-n}\,n^{-3/2}. \tag{6.23}$$

Remark 6.13. Instead of analytic combinatorics methods, one can also obtain these results using probabilistic methods, as was done in Section 3.2. In the subcritical case, they require having an estimate of the size of the largest block, which is provided in Section 6.5.

6.5 Phase transition for the sizes of the largest blocks

To be consistent with the way the article is written, we consider the probability distribution on $\mathcal{M}^{\varnothing}$,

$$\widehat{\mathbb{P}_{n,u}}(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}}{[z^n]\widehat{M}_{\varnothing}(z,u)}\mathbb{1}_{|\mathfrak{m}|=n} = \mathbb{P}_{2n+1,u}(\mathfrak{m})$$

and let $\widehat{\mathbf{M}_{n,u}}$ denote a tree-rooted map sampled from $\widehat{\mathbb{P}_{n,u}}$, by $\widehat{\mathbf{T}_{n,u}}$ the block tree associated to it, and by $(\mathbf{B}_v, v \in \widehat{\mathbf{T}_{n,u}})$ its corresponding decorations. This section puts into light a phase transition for the block sizes of random tree-rooted maps drawn according to $\widehat{\mathbb{P}_{n,u}}$. We use probabilistic techniques to obtain the results, but an analysis using a saddle-point method could also be carried out. As in Chapter 4, for $\mathfrak{m}$ a tree-rooted map, denote by $|\mathrm{LB}_1(\mathfrak{m})| \geq \cdots \geq |\mathrm{LB}_{b(\mathfrak{m})}(\mathfrak{m})|$ the *number of edges* of its blocks in decreasing order and set $|\mathrm{LB}_k(\mathfrak{m})| = 0$ if $k > b(\mathfrak{m})$.

Theorem 6.14. *The random tree-rooted map $\widehat{\mathbf{M}_{n,u}}$, drawn according to $\widehat{\mathbb{P}_{n,u}}$, exhibits the following behaviours when n tends to infinity, where $c(u)$ is defined in (6.15).*

Subcritical case. *For $u < u_C$, the largest bloc is macroscopic, and more precisely one has:*

$$\frac{|\mathrm{LB}_1(\widehat{\mathbf{M}_{n,u}})| - (1 - E(u))n}{\sqrt{c(u) n \ln(n)}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1). \tag{6.24}$$

Furthermore, for any fixed $j \geq 2$, it holds that $|\mathrm{LB}_j(\widehat{\mathbf{M}_{n,u}})| = \Theta_{\mathbb{P}}(n^{1/2})$ and for $x > 0$:

$$\mathbb{P}\left(|\mathrm{LB}_j(\widehat{\mathbf{M}_{n,u}})| \leq x\sqrt{n}\right) \xrightarrow[n\to\infty]{} e^{-\lambda(x)} \sum_{p=0}^{j-2} \frac{\lambda(x)^p}{p!}, \qquad \text{where } \lambda(x) := \frac{c(u)}{2x^2}. \tag{6.25}$$

Critical case. *For $u = u_C$, for any fixed $j \geq 1$, it holds that $|\mathrm{LB}_j(\widehat{\mathbf{M}_{n,u}})| = \Theta_{\mathbb{P}}(n^{1/2})$. More precisely, up to a shift of indices, the sizes of the blocks exhibit a similar behaviour as the sizes of non-macroscopic blocks in the subcritical regime, namely, for $x > 0$:*

$$\mathbb{P}\left(|\mathrm{LB}_j(\widehat{\mathbf{M}_{n,u_C}})| \leq x\sqrt{n}\right) \xrightarrow[n\to\infty]{} e^{-\lambda(x)} \sum_{p=0}^{j-1} \frac{\lambda(x)^p}{p!}, \qquad \text{where } \lambda(x) := \frac{c(u_C)}{2x^2}. \tag{6.26}$$

Supercritical case. *For $u > u_C$, for all fixed $j \geq 1$, it holds as $n \to \infty$ that*

$$|\mathrm{LB}_j(\widehat{\mathbf{M}_{n,u}})| = \frac{\ln(n)}{\ln\left(\frac{\rho_{B_{\varnothing}}}{x(u)^2}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_{B_{\varnothing}}}{x(u)^2}\right)} + O_{\mathbb{P}}(1).$$

Proof. In all three cases, as for the non–tree-rooted case of Chapter 4, we make extensive use of Janson's survey [Jan12]. In the supercritical case, one can proceed exactly as in the non–tree-rooted case and use the survey's Theorem 19.16. In the critical case, its Example 19.29 can be applied. The subcritical case is a bit more involved, Janson's Theorem 19.34 can be applied for the size of the j-th largest block for $j \geq 2$. Results from Kortchemski [Kor15, Theorem 1] allow to conclude for the largest block. □

Remark 6.15. One can get a local limit theorem for $|\mathrm{LB}_1(\widehat{\mathbf{M}_{n,u}})|$ in the subcritical case as in [Stu20b] (up to the technicality that nodes of the block tree have only even numbers of children). Furthermore, one can state a joint limit law for the sizes $|\mathrm{LB}_j(\widehat{\mathbf{M}_{n,u}})|$. For any fixed $r \geq 1$,

$$\left(\frac{c(u)}{2}\left(\frac{|\mathrm{LB}_j(\widehat{\mathbf{M}_{n,u}})|}{\sqrt{n}}\right)^{-2}, \ 2 \leq j \leq r+1\right) \xrightarrow[n\to\infty]{(d)} (A_1, \ldots, A_r),$$

where the A_i are the decreasingly ordered atoms of a Poisson Point Process of rate 1 on $\mathbb{R}_+$. The same joint limit law holds at u_C (with j from 1 to r).

Remark 6.16. In contrast to the case of non–tree-rooted maps of Chapter 4, here in the subcritical case $u < u_C$ the size of the second block is negligible compared to the order of fluctuation of the size of largest block. Moreover, for $u < u_C$ and fixed $j \geq 1$, the j-th largest critical block has the same limit law (up to constant rescaling) as the $j+1$-th largest block in the subcritical regime, which did not hold in the non–tree-rooted case. Informally, the conditioning that a random walk (subcritical case) is an excursion (critical case) has negligible effect on the law of the largest steps, so subcritical blocks (for $j \geq 2$) behave like critical blocks.

6.6 Scaling limit in the critical and supercritical cases

In the critical and the supercritical cases, we can establish the following convergence result.

Theorem 6.17. *For any fixed $u \geq u_C$, there exist some constants α_u, β_u and γ_u such that:*

- *If $u > u_C$, it holds that:*

$$\frac{\gamma_u}{\sqrt{n}} \cdot \left(\widehat{\mathbf{M}_{n,u}}, \tau(\widehat{\mathbf{M}_{n,u}}), \widehat{\mathbf{T}_{n,u}}\right) \xrightarrow[n\to\infty]{(d)} (\alpha_u \cdot \mathcal{T}^{(2)}, \beta_u \cdot \mathcal{T}^{(2)}, \mathcal{T}^{(2)}); \tag{6.27}$$

- *If $u = u_C$, it holds that:*

$$\frac{\gamma_{u_C}\sqrt{\ln(n)}}{\sqrt{n}} \left(\widehat{\mathbf{M}_{n,u_C}}, \tau(\widehat{\mathbf{M}_{n,u_C}}), \widehat{\mathbf{T}_{n,u_C}}\right) \xrightarrow[n\to\infty]{(d)} (\alpha_{u_C} \cdot \mathcal{T}^{(2)}, \beta_{u_C} \cdot \mathcal{T}^{(2)}, \mathcal{T}^{(2)}); \tag{6.28}$$

where, in both cases, each $\mathcal{T}^{(2)}$ is a copy of the same realization of the Brownian Continuum Random Tree (CRT), and the convergence holds in the Gromov-Hausdorff-Prokhorov sense.

Convergence towards the CRT in the supercritical case was previously obtained by Stufler [Stu20a, Theorem 6.63], who considers a framework where a block-weighted map is sampled, and *afterwards* one of its spanning trees is uniformly sampled. Block-weighted models of random tree-rooted maps fall in this model upon tweaking the weights [Stu20a, Rem 6.65]. Our contribution lies in showing scaling limit in the critical case (and having a unified proof for both the supercritical and critical cases), and finding the value of u_C.

Proof of Theorem 6.17. The convergence of the sequence of tree of blocks $(\widehat{\mathbf{T}_{n,u}})_n$ follows from classical results about the scaling limit of Bienaymé–Galton–Watson trees towards the CRT [LG05, DLG05], with an offspring distribution with a finite second moment in the $u > u_C$ case or with finite moments of order $2+\varepsilon$ in the critical case. The values of the constant γ_u follows from general results. In the case $u > u_C$, $\gamma_u = \sigma(u)/2$, where $\sigma(u)$ is defined in (6.17). For $u = u_C$, the variance of μ^{u_C} is infinite, and we get $\gamma_{u_C} = \sqrt{2c(u_C)}$ (see *e.g.* [Jan22, Ex. 7.10]).

To establish the scaling limit of $\widehat{\mathbf{M}_{n,u}}$ and of $\tau(\widehat{\mathbf{M}_{n,u}})$, we proceed as in [Stu20a] and Chapter 5 and prove that the distances in the map and in the spanning tree are – up to a linear factor – equivalent in the limit to the distances in the block tree. The proof extends effortlessly, and only requires (for the critical case) to have a control on the diameter of the 2-connected blocks and of their spanning trees, that we state in Lemma 6.18[4]. The Prokhorov part of the convergence can be established along the same lines as [FS24, Lemma 5.13]. □

Lemma 6.18 (Bound on the diameter of random 2-connected tree-rooted maps). *Let $\mathbf{B}_n$ be a uniformly random 2-connected tree-rooted map with n edges, and recall that $\tau(\mathbf{B}_n)$ denotes its distinguished spanning tree. Then, for any $\varepsilon > 0$, the sequences*

$$\left(\mathbb{P}(\operatorname{diam}(\mathbf{B}_n) \geq n^{1/2+\varepsilon})\right)_{n\geq 0} \quad \textit{and} \quad \left(\mathbb{P}(\operatorname{diam}(\tau(\mathbf{B}_n)) \geq n^{1/2+\varepsilon}\right)_{n\geq 0}$$

are stretched-exponential (see Definition 5.1).

Proof. Note first that it is enough to establish the result for $\operatorname{diam}(\tau(\mathbf{B}_n))$ since $\operatorname{diam}(\mathbf{B}_n) \leq \operatorname{diam}(\tau(\mathbf{B}_n))$ deterministically.

By Mullin's bijection [Mul67], for $\mathbf{M}_n$ a uniform tree-rooted map with n edges, the height of $\tau(\mathbf{M}_n)$ is distributed as the maximal abscissa X_n in a random walk (with steps in $\{W, E, S, N\}$) of length $2n$, ending at the origin and staying in the right-hand upper quadrant. It is easy to establish, *e.g.* using Chernoff's bound and a union bound, that the maximal abscissa $\widetilde{X_n}$ in a random walk of length $2n$ in $\mathbb{Z}^2$ is such that, for any $\varepsilon > 0$, the sequence $\left(\mathbb{P}(\widetilde{X_n} \geq n^{1/2+\varepsilon})\right)_{n\geq 0}$ is stretched-exponential.

Since the random walk has probability $\Theta(n^{-3})$ to end at the origin and to stay in the quadrant, the sequence $\left(\mathbb{P}(X_n \geq n^{1/2+\varepsilon})\right)_{n\geq 0}$ is also stretched-exponential. And so is the sequence $\left(\mathbb{P}(\operatorname{diam}(\tau(\mathbf{M}_n)) \geq n^{1/2+\varepsilon})\right)_{n\geq 0}$ since the diameter of a tree is at most twice its height. Let $\alpha = 1 - E(1)$. By the results of the previous section, one gets

$$\mathbb{P}(|\mathrm{LB}_1(\mathbf{M}_{\lfloor n/\alpha \rfloor})| = n) = \Theta(1/\sqrt{n \ln(n)}),$$

and in that case a block of maximal size is distributed as $\mathbf{B}_n$.

Hence, the sequence $\left(\mathbb{P}(\operatorname{diam}(\tau(\mathbf{B}_n)) \geq n^{1/2+\varepsilon})\right)_{n\geq 0}$ is also stretched-exponential. □

[4] To prove Theorem 6.17, it is actually enough to have the result for an exponent strictly smaller than 1.

Chapter 7

Conclusions and perspectives

The main contribution of this thesis is showing how block-weighting techniques can be used to shed light on phase transitions for families of maps and to obtain new behaviours. I show how a unified approach relying on tools from analytic combinatorics and probability allows to obtain probabilistic and enumerative results for various characteristics of the models. There are two natural paths for improvement: extending these studies to more involved models, and finding unified techniques to obtain new results for models that have already been studied.

7.1 Extension to other models

7.1.1 Models with "coreless" blocks

In their list of decomposition of families of maps, Banderier, Flajolet, Schaeffer and Soria considered models with so-called *coreless* components, namely:

- General maps decomposed into loopless (or bridgeless[1]) components;
- Bipartite maps decomposed into bipartite bridgeless components;
- 2-connected maps decomposed into 3-connected components.

For these schemes, there exist *coreless* maps: maps of positive size from which it is not possible to extract a block containing the root. For the first decomposition scheme, the smallest example is the map consisting of a loop (which is the root): there is no loopless component but the map still has positive size. In these three cases, the generating series equation can be written

$$M(z,u) = uB(H(z,M(z,u))) + D(z,M(z,u))$$

for explicit (simple) functions H and D, with D corresponding to coreless maps.

In all cases considered in this thesis, the decorated block tree is in bijection with the original map. This works because all coreless maps are of size 0 (*e.g.*, in the case of maps decomposed into 2-connected blocks, it is the vertex map). One can change the block tree by introducing 2

[1] A *bridge* is an edge whose removal disconnects the map.

types of vertices (*e.g.* black and white), which allows to recover a bijective encoding. Black vertices would represent "usual blocks", and coreless blocks would be represented by white vertices. Notice that black and white vertices may have both black and white children. For example, in the case of general maps decomposed into loopless components, whenever a component has a loop as its root, one adds a white vertex with two children corresponding to the maps in each corner of the loop.

Then, if one can describe the law of the block trees of a random block-weighted map as the law of a Bienaymé–Galton–Watson bi-type tree, then one can use results of degrees in these trees to get results on the block sizes. However, one needs to know the size of a block tree of a map of size n, which might not be immediate (nor uniquely determined by the size of the map).

The decomposition of 2-connected maps into 3-connected blocks follows a series-parallel scheme (as described in Section 6.2 and in particular in the proof of Lemma 6.7), see *e.g.* [Sch15]. This is also the case for the decomposition of simple quadrangulations into irreducible blocks, for which we expect that results could be obtained using the same techniques as for simple triangulations decomposed into irreducible blocks.

7.1.2 Models of decorated maps

In Chapter 6, we studied maps decorated with a spanning tree decomposed into 2-connected tree-rooted blocks. This subsection is a verbatim reproduction of [AFS24, §5].

It has been recently shown [GHS20, DG20, GP19] that for $\mathbf{M}_n$ a random tree-rooted map of size n, the volume-growth exponent (whose inverse should give the exponent for the order of magnitude of the diameter) is with high probability in the interval $[3.550408, 3.63299]$. It would be interesting to verify whether these bounds also hold for the random 2-connected tree-rooted map $\mathbf{B}_n$, and more generally for the random tree-rooted map $\mathbf{M}_{n,u}$ in the subcritical regime.

Regarding extensions of the model, one could consider maps endowed with a spanning forest, with weight v per tree in the forest, which were studied by Bousquet-Mélou and Courtiel [BMC15], and one could additionally have a weight $u > 0$ per 2-connected block. They showed that, for $v > 0$, one gets the asymptotic behaviour $n^{-5/2}$ as in pure maps [BMC15]. The phase transition should thus be of the same nature than for the non–tree-rooted case [FS24], and we expect the scaling limit to be a stable tree of parameter $3/2$ at the critical weight $u_C(v)$. The case $v = 0$ corresponds to tree-rooted maps as studied here. Interestingly, their model still has a combinatorial interpretation for $v \in [-1, 0)$, with asymptotic behaviour $n^{-3}\ln(n)^{-2}$ [BMC15]. From this behaviour it can be expected that, at the critical weight $u_C(v)$, the asymptotic enumeration has a correcting term $n^{-3/2}$ and the scaling limit is the CRT with distances rescaled by $n^{1/2}$ (same order of magnitude as in the supercritical case). To have a continuous range of asymptotic exponents, one could more generally consider random maps weighted by a Potts model, and additionally weighted at blocks (a method to derive the singular exponents of general maps weighted by a Potts model has been developed in [BBG12]; see also [EB99, BBM11]).

Finally, one could also consider other kinds of block-decompositions in the context of decorated maps, such as 3-oriented triangulations or 2-oriented quadrangulations decomposed into irreducible components, having a weight u per such component. The asymptotic exponents at $u = 1$ are n^{-5} and n^{-4} [FFN10], respectively. This suggests that, as in the above mentioned model, at the critical weight u_C, the model exhibits a tree-behaviour: the asymptotic enumeration has polynomial correction $n^{-3/2}$ and rescaling the distances by $n^{1/2}$ gives convergence towards the CRT.

7.1.3 Models of higher genera

In this thesis, we have only discussed *planar* maps, *i.e.* maps embedded into a surface of genus 0. The notion of map extends to embeddings onto surfaces of higher genus. I would like to extend these block-decompositions into higher genera and investigate whether one block "concentrates" all the holes of the surface or whether, on the contrary, they are distributed among the different components, and whether this depends on the phase. A difficulty of this study is that the generating series of 2-connected maps of genus g diverges in its radius of convergence as soon as g is positive. In particular, this makes it more difficult to describe the law of block trees as a Bienaymé–Galton–Watson law. However, this can be solved by using a subcritical Boltzmann generator (*i.e.*, evaluating the series just below its radius of convergence). We expect that in the subcritical case, it can be shown that the dominant case is when one block concentrates the genus (*i.e.*, all its decorations are planar). The supercritical and critical cases are probably difficult, but it should be possible to obtain results that give at least an idea of the behaviour.

7.2 More results on block-weighted decompositions

In this section we focus on the main example of this thesis: general maps decomposed into 2-connected blocks, with the idea of first obtaining new results for this case (for which many results are already available) and then extending them to other decomposition schemes.

7.2.1 Limit law for the number of blocks

A natural question is to ask how the number of blocks of $\mathbf{M}_{n,u}$ evolves as a function of u. The case $u = 1$ was already studied by Banderier, Flajolet, Schaeffer and Soria [BFSS01].

A first approach is to use the Lagrangean parametrisation of the decomposition scheme to get explicit values for the $[z^n u^k]M_\circ(z, u)$ for all k and n, then study for which k the contribution is the highest. Unfortunately, this is no easy task as the explicit expression is rather involved, see below (and this method is likely difficult to generalise to other models).

Theorem 7.1. *The coefficients of $M_\circ(z, u)$ are the following:*

$$[z^0]M_\circ(z, u) = 1$$

and, for $n \geq 1$,

$$[z^n]M_\circ(z,u) = \frac{2u}{n}\Bigg[\left(\sum_{i=0}^{n-2}(-1)^i(3u)^i\binom{2n}{i}\sum_{p=max(0,-n+2+2i)}^{i}(-1)^{p+1}\left(\frac{2}{3}\right)^p\frac{2p-4i-1}{n+p-2i-1}\binom{i}{p}\binom{3n+p-2i-3}{n+p-2i-2}\right)$$
$$+(-1)^{n+1}\left(\frac{3}{2}\right)^{n-1}\sum_{i=(n-1)/2}^{n-2}(-1)^i\left(\frac{4u}{3}\right)^i\binom{2n}{2n-i,2i+1-n,n-1-i}\Bigg]$$
$$+\frac{(2u)^n}{n+1}\binom{2n}{n}.$$

Therefore, for $n \geq 1$,

- *For $k = 0$ or $k > n$ (notice how this condition comes naturally from the preceding formula),*

$$[z^n u^k]M_\circ(z,u) = 0.$$

- *For $1 \leq k \leq n-1$,*

$$[z^n u^k]M_\circ(z,u) = \frac{2}{n}(-1)^{k-1}\Bigg[\mathbb{1}_{k\in[(n+1)/2,n-1]}(-1)^{n+1}\left(\frac{3}{2}\right)^{n-1}\left(\frac{4}{3}\right)^{k-1}\binom{2n}{2n-k+1,2k-1-n,n-k}$$
$$+3^{k-1}\binom{2n}{k-1}\sum_{p=max(0,-n+2k)}^{k-1}(-1)^{p+1}\left(\frac{2}{3}\right)^p\frac{2p-4k+3}{n+p-2k+1}\binom{k-1}{p}\binom{3n+p-2k-1}{n+p-2k}\Bigg].$$

- *For $k = n$,*

$$[z^n u^n]M_\circ(z,u) = \frac{2^n}{n+1}\binom{2n}{n}.$$

Fortunately, there are more direct ways to obtain the typical number of blocks of $\mathbf{M}_{n,u}$. Indeed, Hwang's quasi-power theorem (see [FS09, §IX.5]) makes us expect that for any fixed $u \neq u_C$, the number of blocks $b(\mathbf{M}_{n,u})$ follows a normal distribution. Formally, we expect that for all $u > 0$, $u \neq u_C$, it holds that

$$\frac{b(\mathbf{M}_{n,u}) - m(u)n}{s(u)\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1),$$

where

$$m(u) := -\frac{u}{2}\frac{\rho'(u)}{\rho(u)},$$

which is not analytic in u_C, and $s(u) = m'(u)$. However, the explicit expressions for ρ in both the subcritical case and supercritical case show that m admits a continuation in u_C which is $\mathcal{C}^1$, which is why we hope to obtain a similar Central Limit Theorem for $b(\mathbf{M}_{n,u_C})$.

In particular, we expect a map $\mathbf{M}_{n,k,u}$ with $k = m(u)n + o(n)$ being the (fixed) number of blocks to have the same properties as $\mathbf{M}_{n,u}$.

7.2.2 Critical window

In continuation of my current work on combinatorial and probabilistic aspects of block-weighted planar maps, I would also like to further investigate what happens at the critical point for the phase transitions I have studied: preliminary results suggest the presence of a critical window of width $n^{-1/3}$ and that the enumerative polynomial factor has an exponent going linearly from $-5/2$ ("map" case) to $-5/3$ (critical case), then linearly again to $-3/2$ ("tree" case). Indeed, let $\alpha > 0, \kappa \neq 0$ and set $u_n = u_C + \kappa n^{-\alpha}$. The results of Theorem 3.2 for $u < u_C$ and $u > u_C$ still hold for $M(z, u_n)$. Then, informally applying the transfer theorem in $\rho(u_n)$ gives, if $\kappa < 0$

$$[z^n]M(z, u_n) \sim d\frac{3s(u_n)}{4\sqrt{\pi}}\rho(u_n)^{-n}n^{-5/2},$$

where $s(u)$ is defined in (3.9); and, for $\kappa > 0$,

$$[z^n]M(z, u_n) \sim d\,\frac{s(u_n)}{2\sqrt{\pi}}n^{-3/2}\rho(u_n)^{-n},$$

where $s(u)$ can be found in Lemma 3.3. In the subcritical case, for all decomposition schemes, one gets

$$s(u_n) \underset{n\to\infty}{\sim} C_1 n^{5/2\alpha}$$

for some constant $C_1 > 0$ which gives

$$[z^n]M(z, u_n) \sim C_2\rho(u_n)^{-n}n^{-5/2+5\alpha/2}$$

for some constant $C_2 > 0$. The supercritical case gives similarly

$$[z^n]M(z, u_n) \sim C_3\rho(u_n)^{-n}n^{-3/2-\alpha/2}$$

for some constant $C_3 > 0$. In all cases, setting $\alpha = 1/3$ gives a polynomial correction in $n^{-5/3}$, which is the correction for the critical case. Saddle-point analysis could be used to make these asymptotic expansions rigorous for $\alpha \in [0, 1/3)$. This suggests that the width of the critical window (should it exist) is the same on both sides, which is not obvious *a priori*.

As explained above, for the sake of simplicity we focus here on general maps decomposed into 2-connected components. In this case, one can immediately get an equivalent of Theorem 2.8 for the case $u = u_n$ and show that this changes only multiplicative constants in the asymptotic behaviour of μ^{u_n}. Then, the proof of Theorem 4.2 still applies as long as Janson's result in Proposition 4.1 applies. Janson's proof can be extended to the u_n case with minimal tweaking: instead of showing that probabilities are $o(n^{-3/2})$, it is sufficient to show that they are $o(n^{-3/2+5\alpha/2})$ and everything works when $\alpha < 1/3$.

Theorem 7.2. *Consider general maps decomposed into 2-connected components. Let $\alpha \in (0, 1/3)$, $\kappa > 0$ and $u_n = u_C - \kappa n^{-\alpha}$. Recall that $E(u)$ and $c(u)$ are defined in*

Equations (2.31) and (2.32). Then, as $n \to \infty$ with $n \equiv 1[d]$

$$\mathrm{s}(\mathrm{LB}_1(\mathbf{M}_{n,u_n})) = (1 - E(u_n))n + O_{\mathbb{P}}(n^{2/3}) \quad \textit{and} \quad \mathrm{s}(\mathrm{LB}_2(\mathbf{M}_{n,u_n})) = O_{\mathbb{P}}(n^{2/3}).$$

Moreover, the following joint convergence holds:

$$\left(\frac{1}{2nc(u_n)}\right)^{2/3} ((1 - E(u_n))n - (\mathrm{s}(\mathrm{LB}_1(\mathbf{M}_{n,u_n})) - 1), (\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u_n})) - 1, j \geq 2))$$
$$\xrightarrow[n\to\infty]{(d)} \left(L_1, \left(\Delta L_{(j-1)}, j \geq 2\right)\right)$$

where $(L_t)_{t\in[0,1]}$ is a stable process of parameter $3/2$ such that $\mathbb{E}\left[e^{-sL_1}\right] = e^{\Gamma(-3/2)s^{3/2}}$ and $\Delta L_{(1)} \geq \Delta L_{(2)} \geq \dots$ is the ranked sequence of its jumps.

For the supercritical case, preliminary computations suggest that polynomial-sized blocks appear. Indeed, let $u_n = u_C + \varepsilon_n$, where $\varepsilon_n = \kappa n^{-\alpha}$. The bijective correspondence described by (2.13) implies that the value y_n such that $\mathbb{E}(\mu^{y_n,u_n}) = 1$ can be written $y_n = \rho_B - r_n$ for some sequence r_n tending to 0. Then, injecting y_n in (2.13) and using the development in series of $B_\circ$ gives $r_n \underset{n\to\infty}{\sim} \frac{625}{11664}\varepsilon_n^2$. Taking inspiration from Section 4.1.3 (which uses [Jan12, Theorem 19.16]), one looks at $k(n) = \max\{k \in \mathbb{N}_0 \mid \mu^{y_n,u_n}(k) \geq 1/n\}$. Replacing u_n and y_n by their asymptotic behaviours and using results about the Lambert W function gives the following first-order approximation for $k(n)$:

$$\frac{5184}{625}(1/3 - \alpha)\frac{n^{2\alpha}}{\kappa^2}\ln(n) \sim \frac{1728}{625}\varepsilon_n^{-2}\ln(\varepsilon_n^3 n) = 2,7648 \times \varepsilon_n^{-2}\ln(\varepsilon_n^3 n);$$

which allows to write the following conjecture.

Conjecture 7.3. *Consider general maps decomposed into 2-connected components. Let $\alpha \in (0, 1/3)$, $\kappa > 0$ and $u_n = u_C + \kappa n^{-\alpha}$. Then, for all fixed $j \geq 1$, it holds as $n \to \infty$ with $n \equiv 1[d]$ that*

$$\mathrm{s}(\mathrm{LB}_j(\mathbf{M}_{n,u_n})) = (1/3 - \alpha)\frac{5184}{625}\frac{n^{2\alpha}}{\kappa^2}\ln(n) + \Theta_{\mathbb{P}}(\ln(\ln(n))).$$

It is very interesting to compare this result to results about the Erdős-Rényi model for graphs. When drawing a random graph with n vertices and, for (u, v) a pair of vertices, a probability $p = (1 - \varepsilon)/n$ to have an edge between u and v ; then, with high probability, all blocks have size equivalent to $2\varepsilon^{-2}\ln(\varepsilon^3 n)$ when $\varepsilon^3 n \to \infty$, (*e.g.* $\varepsilon = n^{-\alpha}$, $\alpha < 1/3$) [Łuc90, Theorem 6].

Moreover, there are results for the scaling limit of Erdős-Rényi graphs in the critical window [ABBG12, ABBG10], hinting that we could do the same for block-weighted maps. My goal would be to prove the existence of the critical window, and, for each of its zones, enumerate the block-weighted maps, compute the sizes of the largest blocks and the scaling limit (as well as the rescaling of the distances). In particular, in the supercritical case, since the behaviour of μ^{u_n} can be precisely determined, one could deduce the appropriate renormalisation and the scaling limit of the renormalised block-tree and hopefully the scaling limit of the maps in this case.

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